Journal of Computational and Applied Mathematics 236 (2012) 4547–4560

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Computing the eigenvalues of the generalized Sturm–Liouville problems based on the Lie-group $SL(2, \mathbb{R})$

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ARTICLE INFO

Article history:
Received 8 April 2011
Received in revised form 16 April 2012

Keywords:
Generalized Sturm–Liouville problem
Eigenvalue
Eigenfunction
Eigen-parameter dependent boundary conditions
$SL(2, \mathbb{R})$ Lie-group shooting method
Characteristic equation

ABSTRACT

For the generalized Sturm–Liouville problems we can construct an $SL(2, \mathbb{R})$ Lie-group shooting method to find eigenvalues. By using the closure property of the Lie-group, a one-step Lie-group transformation between the boundary values at two ends of the considered interval is established. Hence, we can theoretically derive an analytical characteristic equation to determine the eigenvalues for the generalized Sturm–Liouville problems. Because the closed-form formulas are derived to calculate the unknown left-boundary values in terms of λ, the present method provides an easy numerical implementation and has a cheap computational cost. Numerical examples are examined to show that the present $SL(2, \mathbb{R})$ Lie-group shooting method is effective.

1. Introduction

The Sturm–Liouville problem has a variety of applications in partial differential equations, vibration of continuum mechanics, and quantum mechanics. There is a continued interest in the numerical solutions of Sturm–Liouville problems with the aim to improve the convergence speed and ease of numerical implementation. In order to obtain more efficient numerical results, several numerical methods have been developed [1–11].

Although Ghelardoni et al. [12] have discussed a shooting technique for computing the eigenvalues, to our best knowledge there is no study on the Lie-group $SL(2, \mathbb{R})$ shooting method and apply it to the generalized Sturm–Liouville problem. In this paper we propose a new shooting method based on the Lie-group $SL(2, \mathbb{R})$ for computing the eigenvalues and eigenfunctions of the following *generalized Sturm–Liouville problem*:

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + q(x,\lambda)y(x) = 0, \quad x_0 < x < x_f, \tag{1}$$

$$a_1(\lambda)y(x_0) + a_2(\lambda)p(x_0)y'(x_0) + a_3(\lambda)y(x_f) + a_4(\lambda)p(x_f)y'(x_f) = a_0(\lambda), \tag{2}$$

$$b_1(\lambda)y(x_0) + b_2(\lambda)p(x_0)y'(x_0) + b_3(\lambda)y(x_f) + b_4(\lambda)p(x_f)y'(x_f) = b_0(\lambda). \tag{3}$$

Here we suppose that $y^2(x_0) + [p(x_0)y'(x_0)]^2 > 0$ in order to avoid the solution of y to be zero. The present problem is that for the given functions of $p(x)$ and $q(x, \lambda)$ and all the coefficients $a_i, b_i, i = 0, \ldots, 4$ we need to calculate the eigenvalue λ and the eigenfunction $y(x)$. In the above we suppose that $p(x) > 0$, and $q(x, \lambda) > 0$ can be an arbitrary nonlinear function of x and λ. In the latter sense, Eqs. (1)–(3) constitute a *nonlinear Sturm–Liouville problem*.

The group-preserving scheme (GPS) has been developed in [13] for the integration of initial value problems (IVPs). Liu [14–16] has extended and modified the GPS for ordinary differential equations (ODEs) to solve the boundary value

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2012.05.006

problems (BVPs). Liu [17] could solve an inverse Sturm–Liouville problem by using the Lie-group method to find the potential function $q(x)$ with a high accuracy. In the construction of the Lie-group method for the numerical solutions of BVPs, Liu [14] has introduced the idea of one-step GPS by utilizing the closure property of the Lie-group, and hence, the new shooting method has been labeled the Lie-group shooting method (LGSM). However, this method needs to be modified for the Sturm–Liouville problem [18]. Recently, Liu [19] has developed a more powerful Lie-group adaptive method for computing a leading coefficient in the Sturm–Liouville operator by using the boundary data.

When the coefficient $q(x, \lambda)$ depends nonlinearly on the eigen-parameter λ, we have a generalized Sturm–Liouville problem. This also concerns the problems with eigen-parameter dependent boundary conditions [20–26]. The problem in Eqs. (1)–(3) essentially differs from the classical one and is more difficult to solve, and so far only a few methods have been proposed for solving the generalized Sturm–Liouville problems. Liu [26] has solved the generalized Sturm–Liouville problem by using a Lie-group shooting method to find the eigenvalues and eigenfunctions. Numerical methods for computing the eigenvalues of the generalized Sturm–Liouville problems with eigen-parameter dependent boundary conditions have been developed [21–28]. The shooting method presented in this paper is based on the Lie-group $SL(2, \mathbb{R})$, which is an extension and simpler than the previous works of Liu [18,26], which are based on the Lie-group $SO_o(2, 1)$. Here we develop a more powerful Lie-group shooting method (LGSM) directly based on the Lie-group $SL(2, \mathbb{R})$ for solving the generalized Sturm–Liouville problems. In most cases, it is not possible to obtain the eigenvalues of the generalized Sturm–Liouville problem analytically. However, in the present paper we can derive a closed-form algebraic equation to compute these eigenvalues analytically.

The remaining parts of this paper are arranged as follows. In Section 2 we propose a self-adjoint formulation of the second-order linear ODE, and develop an $SL(2, \mathbb{R})$ Lie-group shooting method for solving the boundary value problem (BVP) of second-order ODE. In Section 3, two numerical examples are given for solving the BVPs by using the newly developed $SL(2, \mathbb{R})$ Lie-group shooting method. Based on the results in Section 3, we derive a rather simple *characteristic equation* to compute the eigenvalues in Section 4. Numerical examples for the generalized Sturm–Liouville problems are given in Section 5. Finally, we draw some conclusions in Section 6.

2. An $SL(2, \mathbb{R})$ Lie-group shooting method

2.1. A group-preserving scheme (GPS)

Usually, we may write

$$u''(x) + a(x)u'(x) + b(x)u(x) = 0, \quad x > x_0 \tag{4}$$

as a system of two first-order ODEs by

$$\frac{d}{dx}\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b(x) & -a(x) \end{bmatrix}\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix}. \tag{5}$$

However, upon letting

$$p(x) = \exp\left[\int_{x_0}^{x} a(\xi)d\xi\right] \tag{6}$$

be the *integrating factor* of Eq. (4) we have

$$\frac{d}{dx}(p(x)u'(x)) = -p(x)b(x)u(x). \tag{7}$$

It shows that the left-hand side can be managed to a total differential form by using the concept of integrating factor. Furthermore, from Eq. (7) we have a self-adjoint system:

$$\frac{d}{dx}\begin{bmatrix} u(x) \\ p(x)u'(x) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{p(x)} \\ -p(x)b(x) & 0 \end{bmatrix}\begin{bmatrix} u(x) \\ p(x)u'(x) \end{bmatrix}. \tag{8}$$

This system is better than system (5), because it allows a Lie-group symmetry of $SL(2, \mathbb{R})$:

$$\frac{d}{dx}\mathbf{G} = \mathbf{A}\mathbf{G}, \qquad \mathbf{G}(x_0) = \mathbf{I}_2, \tag{9}$$

$$\mathbf{A}(x) = \begin{bmatrix} 0 & \dfrac{1}{p(x)} \\ -p(x)b(x) & 0 \end{bmatrix}, \tag{10}$$

$$\det \mathbf{G} = 1. \tag{11}$$

Here we prove $\det\,\mathbf{G} = 1$. For this purpose we take

$$\mathbf{G}(x) = \begin{bmatrix} G_{11}(x) & G_{12}(x) \\ G_{21}(x) & G_{22}(x) \end{bmatrix}, \tag{12}$$

such that

$$\det\,\mathbf{G}(x) = G_{11}(x)G_{22}(x) - G_{21}(x)G_{12}(x). \tag{13}$$

Taking the differential with respect to x and using Eqs. (9) and (10) we have

$$\begin{aligned}\frac{d}{dx}[\det\,\mathbf{G}(x)] &= G'_{11}(x)G_{22}(x) + G_{11}(x)G'_{22}(x) - G'_{21}(x)G_{12}(x) - G_{21}(x)G'_{12}(x) \\ &= \frac{1}{p(x)}G_{21}(x)G_{22}(x) - p(x)b(x)G_{11}(x)G_{12}(x) + p(x)b(x)G_{11}(x)G_{12}(x) - \frac{1}{p(x)}G_{21}(x)G_{22}(x) = 0.\end{aligned}$$

Thus $\det\,\mathbf{G}(x)$ is a constant, and by $\mathbf{G}(x_0) = \mathbf{I}_2$ we have $\det\,\mathbf{G}(x) = 1$.

Accordingly, we can develop a group-preserving scheme (GPS) to solve Eq. (8):

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k, \tag{14}$$

where $\mathbf{X}_k$ denotes the numerical value of $\mathbf{X} = (u, pu')^{\mathrm{T}}$ at the discrete space x_k, and $\mathbf{G}(k) \in SL(2, \mathbb{R})$. This Lie-group symmetry is known as a *two-dimensional real-valued special linear group*, denoted by $SL(2, \mathbb{R})$. Liu and Atluri [29] have taken the advantage of the Lie-group $SL(2, \mathbb{R})$ for effectively solving a discretized inverse Sturm–Liouville problem under specified eigenvalues.

2.2. A generalized mid-point rule

Applying the GPS in Eq. (14) to Eq. (8) with initial condition $\mathbf{X}(x_0) = \mathbf{X}_0$ we can find $\mathbf{X}(x)$. Assuming that the stepsize used in the GPS is $\Delta x = (x_f - x_0)/K$, we can calculate the value of $\mathbf{X}$ at $x = x_f$ by

$$\mathbf{X}_f = \mathbf{G}_K(\Delta x) \cdots \mathbf{G}_1(\Delta x)\mathbf{X}_0. \tag{15}$$

Now we prove the following closure property of the Lie-group $SL(2, \mathbb{R})$:

$$\mathbf{G}_1(x), \quad \mathbf{G}_2(x) \in SL(2, \mathbb{R}) \Rightarrow \mathbf{G}_2(x)\mathbf{G}_1(x) \in SL(2, \mathbb{R}). \tag{16}$$

By the assumptions of $\mathbf{G}_1(x) \in SL(2, \mathbb{R})$ and $\mathbf{G}_2(x) \in SL(2, \mathbb{R})$ we have $\det\,\mathbf{G}_1(x) = 1$ and $\det\,\mathbf{G}_2(x) = 1$. Then by using the following equation:

$$\det\,[\mathbf{G}_2(x)\mathbf{G}_1(x)] = \det\,\mathbf{G}_2(x)\,\det\,\mathbf{G}_1(x),$$

it is straightforward to verify that $\det\,[\mathbf{G}_2(x)\mathbf{G}_1(x)] = 1$, which means that $\mathbf{G}_2(x)\mathbf{G}_1(x) \in SL(2, \mathbb{R})$. Thus we have proven Eq. (16).

Because each $\mathbf{G}_i$, $i = 1, \ldots, K$ is an element of the Lie-group $SL(2, \mathbb{R})$, and by the above closure property of the Lie-group $SL(2, \mathbb{R})$, $\mathbf{G}_K(\Delta x) \cdots \mathbf{G}_1(\Delta x)$ is also a Lie-group element of $SL(2, \mathbb{R})$, denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}_f = \mathbf{G}\mathbf{X}_0. \tag{17}$$

This is a *one-step Lie-group transformation* from $\mathbf{X}_0$ to $\mathbf{X}_f$ by $\mathbf{G} \in SL(2, \mathbb{R})$.

Usually it is very hard to obtain an exact solution of $\mathbf{G}$. Before the derivation of a suitable $\mathbf{G}$, let us recall the mean value theorem for a continuous function $f(x)$, which is defined in an interval of $x \in [a, b]$. The mean value theorem asserts that there exists at least one $c \in [a, b]$, such that the following equality holds:

$$\int_a^b f(x)dx = f(c)[b - a], \tag{18}$$

where the value of c depends on the function $f(x)$. In terms of the weighting factor $r \in [0, 1]$, we can write $c = ra + (1 - r)b$. Therefore, it means that there exists at least one $r \in [0, 1]$, such that Eq. (18) is satisfied. The above theorem enables us to evaluate the value of the integral in Eq. (18) by an area of a rectangle with a width $b - a$ and a height $f(c)$, where $f(c)$ is calculated by a mid-point rule with a suitable $c \in [a, b]$.

Because $\mathbf{G}$ is a solution of Eq. (9), we can formally write it by an exponential mapping:

$$\mathbf{G}(x) = \exp\left[\int_{x_0}^{x} \mathbf{A}(\xi)d\xi\right]. \tag{19}$$

When $\mathbf{A}(x)$ is not a constant matrix, in general we do not have a closed-form solution of $\mathbf{G}(x)$. However, motivated by the above mean value theorem and to be a reasonable approximation, we can calculate $\mathbf{G}$ at $x = x_f$ by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the value of the variable x in $\mathbf{A}$ at a suitable mid-point:

$\hat{x} = rx_0 + (1-r)x_f$, where $r \in [0, 1]$ is an unknown constant to be determined, which is dependent on the problem. So we can compute this **G**, which is corresponding to a constant matrix **A**:

$$\mathbf{A}(\hat{x}) =: \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{p(\hat{x})} \\ -p(\hat{x})b(\hat{x}) & 0 \end{bmatrix}, \tag{20}$$

where c_1 and c_2 are two constants.

This Lie-group element generated from such a constant $\mathbf{A} \in sl(2, \mathbb{R})$ permits the closed-form solutions (see the Appendix):

$$\mathbf{G}(r) = \begin{bmatrix} \cos[(x_f - x_0)\hat{c}] & \frac{1}{p(\hat{x})\hat{c}} \sin[(x_f - x_0)\hat{c}] \\ -p(\hat{x})\hat{c} \sin[(x_f - x_0)\hat{c}] & \cos[(x_f - x_0)\hat{c}] \end{bmatrix}, \quad \text{if } b(\hat{x}) > 0, \tag{21}$$

$$\mathbf{G}(r) = \begin{bmatrix} \cosh[(x_f - x_0)\hat{c}] & \frac{1}{p(\hat{x})\hat{c}} \sinh[(x_f - x_0)\hat{c}] \\ p(\hat{x})\hat{c} \sinh[(x_f - x_0)\hat{c}] & \cosh[(x_f - x_0)\hat{c}] \end{bmatrix}, \quad \text{if } b(\hat{x}) < 0, \tag{22}$$

where

$$\hat{x} = rx_0 + (1-r)x_f, \tag{23}$$

$$\hat{c} = \sqrt{b(\hat{x})}, \quad \text{if } b(\hat{x}) > 0, \tag{24}$$

$$\hat{c} = \sqrt{-b(\hat{x})}, \quad \text{if } b(\hat{x}) < 0. \tag{25}$$

The above **G** is a single-parameter Lie-group element, denoted by $\mathbf{G}(r)$, $r \in [0, 1]$.

2.3. An $SL(2, \mathbb{R})$ shooting method

When Eq. (4) is subjected to the Dirichlet boundary values at the two ends of an interval of $x \in [x_0, x_f]$, we have a boundary-value problem (BVP). The stepping technique developed for solving the initial-value-problem (IVP) requires both the initial conditions of $w_1 = u$ and $w_2 = pu'$ for the second-order ODEs. Starting from the initial values of w_1 and w_2, we can numerically integrate the following IVP step-by-step from $x = x_0$ to $x = x_f$:

$$\frac{d}{dx} \begin{bmatrix} u(x) \\ p(x)u'(x) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{p(x)} \\ -p(x)b(x) & 0 \end{bmatrix} \begin{bmatrix} u(x) \\ p(x)u'(x) \end{bmatrix}, \tag{26}$$

$$u(x_0) = u_0, \tag{27}$$

$$p(x_0)u'(x_0) = A, \tag{28}$$

where A is an unknown constant.

When u_0 and u_f are given, by Eqs. (17), (21) and (22)we can obtain

$$\begin{bmatrix} u_f \\ p(x_f)u'(x_f) \end{bmatrix} = \begin{bmatrix} \cos[(x_f - x_0)\hat{c}] & \frac{1}{p(\hat{x})\hat{c}} \sin[(x_f - x_0)\hat{c}] \\ -p(\hat{x})\hat{c} \sin[(x_f - x_0)\hat{c}] & \cos[(x_f - x_0)\hat{c}] \end{bmatrix} \begin{bmatrix} u_0 \\ A \end{bmatrix}, \quad \text{if } b(\hat{x}) > 0, \tag{29}$$

$$\begin{bmatrix} u_f \\ p(x_f)u'(x_f) \end{bmatrix} = \begin{bmatrix} \cosh[(x_f - x_0)\hat{c}] & \frac{1}{p(\hat{x})\hat{c}} \sinh[(x_f - x_0)\hat{c}] \\ p(\hat{x})\hat{c} \sinh[(x_f - x_0)\hat{c}] & \cosh[(x_f - x_0)\hat{c}] \end{bmatrix} \begin{bmatrix} u_0 \\ A \end{bmatrix}, \quad \text{if } b(\hat{x}) < 0. \tag{30}$$

Thus we can solve the unknown A by

$$A = \frac{p(\hat{x})\hat{c}}{\sin[(x_f - x_0)\hat{c}]} \left(u_f - u_0 \cos[(x_f - x_0)\hat{c}]\right), \quad \text{if } b(\hat{x}) > 0, \tag{31}$$

$$A = \frac{p(\hat{x})\hat{c}}{\sinh[(x_f - x_0)\hat{c}]} \left(u_f - u_0 \cosh[(x_f - x_0)\hat{c}]\right), \quad \text{if } b(\hat{x}) < 0. \tag{32}$$

For the special case with $b = 0$ ($\hat{c} = 0$) we can use

$$A = \frac{p(\hat{x})}{x_f - x_0}[u_f - u_0]. \tag{33}$$

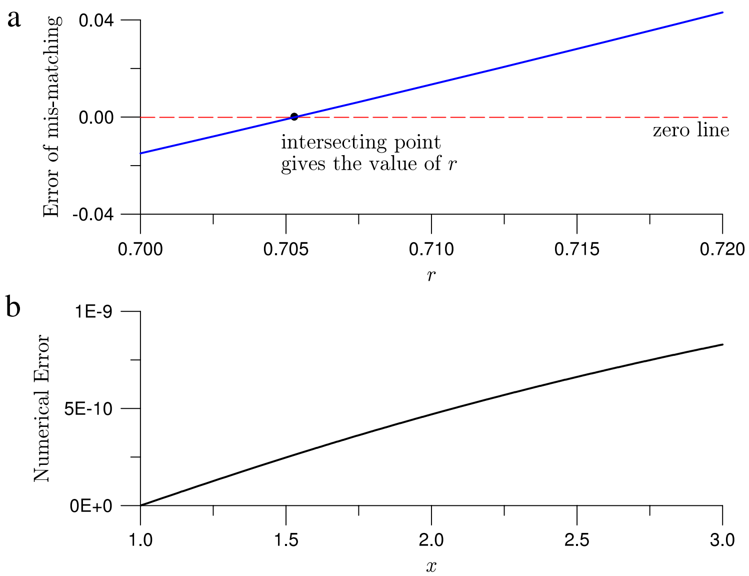

Fig. 1. For Example 1: (a) plotting the error of mis-matching, and (b) plotting the numerical error.

For a trial r, we can calculate A (and thus $p(x_0)u'(x_0)$) from the above equations and then numerically integrate Eq. (26) by the fourth-order Runge–Kutta method (RK4) from x_0 to x_f, and compare the end value of u_f with the exact one $u(x_f)$, which is a target to be matched. If the absolute value of the difference between the computed result u_f with the exact one $u(x_f)$ is smaller than a given error tolerance ε (say, $\varepsilon = 10^{-10}$), i.e., $|u(x_f) - u_f| < \varepsilon$, then the process of finding the solution $u(x)$ is finished. Indeed, we need to find the root of equation $u(x_f) - u_f = 0$, which can be done in practice by adjusting the value of r to a point such that the curve of mis-matching error is intersected with the zero line at that point. In Fig. 1(a) we plot the zero line and the curve of mis-matching error, whose intersecting point gives the desired value of r.

3. Numerical examples of BVP

3.1. A non-singular case

Let us consider a special case

$$u''(x) + \frac{u(x)}{x^2} = 0, \quad 1 \le x \le 3, \tag{34}$$

with $u(x)$ having a closed-form solution:

$$u(x) = \sqrt{x}\left[u(1)\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + \frac{2u'(1) - u(1)}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]. \tag{35}$$

By setting $u(1) = 1$ and $u'(1) = 1$ we can obtain $u_f = u(3)$ from the above equation.

When we apply the RK4 to integrate Eq. (34) we fix $\Delta x = 0.001$. In Fig. 1(a) we plot the curve of mis-matching to the target $u_f - u(3) = 0$ in a range of $r \in [0.7, 0.72]$. It can be seen that there exists a value of r near 0.705, of which the curve of mis-matching error is intersected with the zero line, and at which the mis-matching is zero. Then we give a finer tuning of r to $r = 0.705339222$, and the numerical solution and exact solution are almost coincident with the numerical error being shown in Fig. 1(b). It can be seen that the numerical error is very small in the order of 10^{-10}.

3.2. A singular case

We solve the second example given by

$$\epsilon u'' + u' = 1 + 2x, \tag{36}$$

$$u(0) = 0, \qquad u(1) = 1. \tag{37}$$

For $\epsilon = 0.01$, this example has been calculated in [30,31] with high-order finite difference methods.

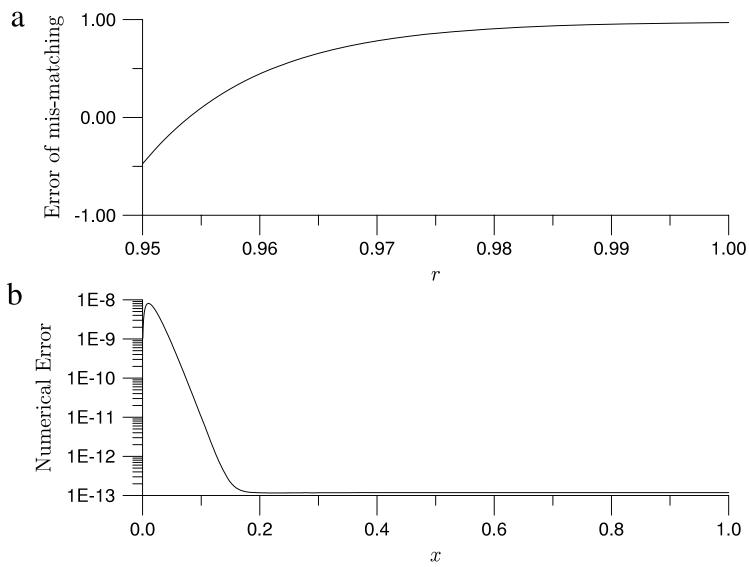

Fig. 2. For Example 2: (a) plotting the error of mis-matching, and (b) plotting the numerical error.

Table 1
For Example 2 comparing the numerical errors with that of [30,16].

x	[30]	[16]	Present
0.002	3.6×10^{-5}	8.7×10^{-9}	3.5×10^{-9}
0.004	5.0×10^{-5}	1.4×10^{-7}	5.9×10^{-9}
0.006	7.3×10^{-5}	1.8×10^{-7}	7.2×10^{-9}
0.008	8.1×10^{-5}	1.9×10^{-7}	7.9×10^{-9}
0.010	8.4×10^{-5}	2.0×10^{-7}	8.1×10^{-9}
0.012	8.4×10^{-5}	1.9×10^{-7}	7.9×10^{-9}
0.014	8.1×10^{-5}	1.8×10^{-7}	7.6×10^{-9}
0.016	7.8×10^{-5}	1.7×10^{-7}	7.1×10^{-9}
0.018	7.3×10^{-5}	1.6×10^{-7}	6.5×10^{-9}
0.020	6.9×10^{-5}	1.4×10^{-7}	5.9×10^{-9}
0.022	6.3×10^{-5}	1.3×10^{-7}	5.3×10^{-9}
0.024	6.0×10^{-5}	1.2×10^{-7}	4.8×10^{-9}
0.026	4.4×10^{-5}	1.0×10^{-7}	4.2×10^{-9}
0.028	5.0×10^{-5}	9.1×10^{-8}	3.7×10^{-9}
0.030	4.6×10^{-5}	8.0×10^{-8}	3.3×10^{-9}

Because of $\epsilon = 0.01$, the system is singular, and we can directly apply the shooting method in Section 2 to this problem. When we apply the RK4 we fix $\Delta x = 0.0005$. In Fig. 2(a) we plot the curve of mis-matching to the target $u(1) - 1 = 0$ in a range of $r \in [0.95, 1]$. It can be seen that there exists a value of r near 0.95, of which the mis-matching is zero. Then we give a finer tuning of r to $r = 0.95394829816182$. By comparing with the closed-form solution

$$u(x) = x(x + 1 - 2\epsilon) + (2\epsilon - 1)\frac{1 - \exp(-x/\epsilon)}{1 - \exp(-1/\epsilon)}, \tag{38}$$

the final numerical value of u can match very well with the exact one with an error 1.18×10^{-13}. The numerical error between the numerical solution and the exact solution is shown in Fig. 2(b). It can be seen that the numerical error is very small in the order of 10^{-8}.

In Table 1 we compare the numerical errors of the numerical results obtained in [30,16]. Obviously, our errors are much smaller than that of Varner and Choudhury [30], and is also better than that of Liu [16].

Ilicasu and Schultz [31] have calculated this problem by using the high-order finite-difference method. Our maximum error is 8.1×10^{-9}, which is slightly larger than that of 6.3×10^{-9} obtained in [31]. However, the present $SL(2, \mathbb{R})$ Lie-group shooting method is easily numerically implemented and can be readily extended to the generalized boundary value problem.

4. An analytical solution of eigenvalues

In Section 2 we have derived the closed-form formulas to calculate the slope A for each $r \in [0, 1]$. If A is available, then we can apply the RK4 to integrate Eq. (8). Up to this point we have developed an $SL(2, \mathbb{R})$ Lie-group shooting method for solving the BVP of second-order ODE. Now, we turn our attention to the generalized Sturm–Liouville problem in Eqs. (1)–(3).

In principle, if there exists one solution y of Eqs. (1)–(3), there are many solutions of the type αy, $\alpha \in \mathbb{R}$, $\alpha \neq 0$. It means that the slope A can be an arbitrary value. So the factor r in Eq. (29) can be any value in the interval of $r \in [0, 1]$. Now, we can fix $r = 1/2$, and then by solving the simultaneous equations (2), (3) and (29) or (30) we can derive the following equations for y_0 and A:

$$y_0 = \begin{vmatrix} a_0 & a_2 + a_3 G_{12} + a_4 G_{22} \\ b_0 & b_2 + b_3 G_{12} + b_4 G_{22} \end{vmatrix} \begin{vmatrix} a_1 + a_3 G_{11} + a_4 G_{21} & a_2 + a_3 G_{12} + a_4 G_{22} \\ b_1 + b_3 G_{11} + b_4 G_{21} & b_2 + b_3 G_{12} + b_4 G_{22} \end{vmatrix}^{-1}, \tag{39}$$

$$A = \begin{vmatrix} a_1 + a_3 G_{11} + a_4 G_{21} & a_0 \\ b_1 + b_3 G_{11} + b_4 G_{21} & b_0 \end{vmatrix} \begin{vmatrix} a_1 + a_3 G_{11} + a_4 G_{21} & a_2 + a_3 G_{12} + a_4 G_{22} \\ b_1 + b_3 G_{11} + b_4 G_{21} & b_2 + b_3 G_{12} + b_4 G_{22} \end{vmatrix}^{-1}, \tag{40}$$

where y_0 and A only depend on λ, G_{ij} denotes the components of $\mathbf{G}$ in Eq. (21) or Eq. (22), and $p(\hat{x}) = p((x_0 + x_f)/2)$ and $\hat{c} = \sqrt{q(\hat{x}, \lambda)/p(\hat{x})}$ with $\hat{x} = (x_0 + x_f)/2$.

In order to calculate the eigenvalues we let λ run in a selected interval we interest, and then insert λ into Eqs. (39) and (40) to obtain y_0 and A, and calculate y_f and $p(x_f)y'(x_f)$ by integrating Eq. (1). Therefore, we can plot a curve revealing the variation of

$$Z := b_1(\lambda)y(x_0) + b_2(\lambda)p(x_0)y'(x_0) + b_3(\lambda)y(x_f) + b_4(\lambda)p(x_f)y'(x_f) - b_0(\lambda) \tag{41}$$

with respect to λ, namely the eigenvalue curve, of which the intersecting points between the eigenvalue curve with the zero line give the values of the required eigenvalues.

Remark. If both a_0 and b_0 in Eqs. (2) and (3) are zeros, we have to impose the following condition:

$$\begin{vmatrix} a_1 + a_3 G_{11} + a_4 G_{21} & a_2 + a_3 G_{12} + a_4 G_{22} \\ b_1 + b_3 G_{11} + b_4 G_{21} & b_2 + b_3 G_{12} + b_4 G_{22} \end{vmatrix} = 0; \tag{42}$$

hence, y_0 and A are not both being zeros. The most of the following examples will use the above *characteristic equation* to compute the eigenvalues.

5. Numerical examples

5.1. Example 3

For a first and simple test example of the generalized Sturm–Liouville problem we consider

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < \pi, \tag{43}$$

$$y(0) = 1, \qquad y(\pi) = \cos\left(\sqrt{\lambda}\pi\right). \tag{44}$$

Here, $\lambda = k^2$, $k \in \mathbb{N}$ are eigenvalues and $\cos(kx)$, $k \in \mathbb{N}$ are eigenfunctions.

We apply the $SL(2, \mathbb{R})$ Lie-group shooting method in Section 4 to calculate the eigenvalues in a range of $0 < \lambda < 50$. From Fig. 3 it can be seen that the curve of $\operatorname{sgn}(Z) := Z/|Z|$ is intersected with the zero line at $\lambda = 1, 4, 9, 16, 25, 36, 49$, which are corresponding to $k = 1, 2, 3, 4, 5, 6, 7$. In this calculation the stepsize used in the RK4 is $\Delta x = 0.01$.

5.2. Example 4

Inspired by [21] we consider an eigen-parameter dependent boundary condition of a generalized Sturm–Liouville eigenvalue problem with

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < 1, \tag{45}$$

$$y'(0) = 0, \qquad \left(\lambda - \frac{\lambda^2}{\pi^2}\right) y(1) + y'(1) = 0. \tag{46}$$

As demonstrated in [21], $\lambda = 0$ is a double eigenvalue, $\lambda = \pi^2$ is a simple eigenvalue and all other simple eigenvalues are the roots of the following equation:

$$\tan\sqrt{\lambda} = \sqrt{\lambda}\left(1 - \frac{\lambda}{\pi^2}\right). \tag{47}$$

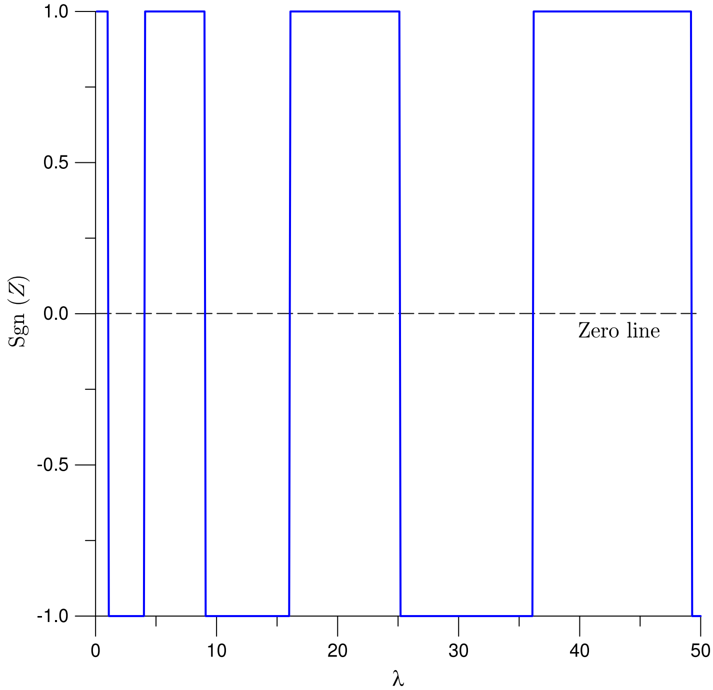

Fig. 3. For Example 3 displaying the eigenvalue curve.

Similarly, by using Eq. (42) we can derive the above same result as that obtained in [21]. Here, we however give a theoretical proof from the Lie-group shooting method based on $SL(2, \mathbb{R})$.

With the initial conditions of $y_0 = 0.5$ and $A = y_0' = 0$ which is obtained from the first equation in Eq. (46) while $y_0 = 0.5$ is an arbitrarily given nonzero value, we apply the $SL(2, \mathbb{R})$ Lie-group shooting method in Section 4 to calculate the eigenvalues in a range of $0 < \lambda < 25$. From Fig. 4(a) it can be seen that the curve of $\mathrm{sgn}(Z)$ is intersected with the zero line at two points. Therefore, for the purpose of comparison we also plot the curves of the above two functions in Fig. 4(b) by the dotted points, whose intersecting points are the roots. The results are coincident with the zero points in Fig. 4(a).

5.3. Example 5

The following test example is taken from [20,22]:

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < 1, \tag{48}$$

$$y(0) + (\lambda - 4\pi^2)y'(0) = 0, \qquad y(1) - \lambda y'(1) = 0. \tag{49}$$

Inserting these coefficient functions into Eq. (42) and using $\hat{c} = \sqrt{\lambda}$ we can derive

$$f(\lambda) = (1 + 4\pi^2\lambda^2 - \lambda^3)\sin\sqrt{\lambda} - \sqrt{\lambda}(2\lambda - 4\pi^2)\cos\sqrt{\lambda} = 0, \tag{50}$$

which can be used to determine λ for the generalized Sturm–Liouville eigenvalue problem in Eqs. (48) and (49).

With the initial conditions of $y_0 = A(4\pi^2 - \lambda)$ and $A = y_0' = 1$ we apply the $SL(2, \mathbb{R})$ Lie-group shooting method in Section 4 to calculate the eigenvalues in a range of $0 < \lambda < 160$. From Fig. 5(a) it can be seen that the curve of $\mathrm{sgn}(Z)$ is intersected with the zero line at three points. For the purpose of comparison we also plot the curve of the above function $f(\lambda)$ given in Eq. (50) in Fig. 5(b) by the solid line, whose intersecting points with the zero line are the roots. The results are coincident with the zero points in Fig. 5(a). Then when the eigenvalues are available we use the Lie-group shooting method to compute the corresponding first three eigenfunctions in Fig. 6.

Chanane [22] by using the sampling method has derived the above characteristic equation (50) and also found three eigenvalues in the range of $[0, 160]$. Here, we give a theoretical proof from the Lie-group shooting method based on $SL(2, \mathbb{R})$.

5.4. Example 6

The following example is taken from [22,24]:

$$y''(x) + (\lambda - e^x)y(x) = 0, \quad 0 < x < 1, \tag{51}$$

$$y(0) = 0, \qquad -\sqrt{\lambda}\sin\sqrt{\lambda}y(1) + \cos\sqrt{\lambda}y'(1) = 0. \tag{52}$$

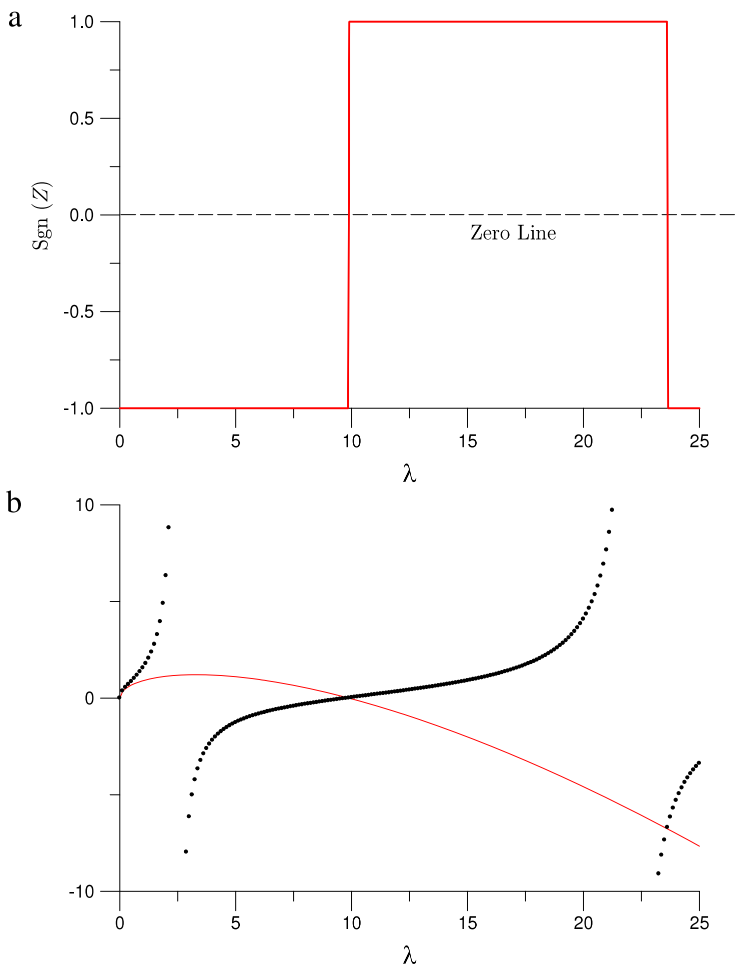

Fig. 4. For Example 4: (a) plotting the eigenvalue curve, and (b) the two curves determine the eigenvalues.

Inserting these coefficient functions into Eq. (42) and using $\hat{c} = \sqrt{|\lambda - \sqrt{e}|}$ we can derive

$$f(\lambda) = \cos\sqrt{\lambda}\cosh\hat{c} - \frac{\sqrt{\lambda}}{\hat{c}}\sin\sqrt{\lambda}\sinh\hat{c}, \quad \text{if } 0 < \lambda < \sqrt{e}, \tag{53}$$

$$f(\lambda) = \cos\sqrt{\lambda}\cos\hat{c} - \frac{\sqrt{\lambda}}{\hat{c}}\sin\sqrt{\lambda}\sin\hat{c}, \quad \text{if } \lambda > \sqrt{e}. \tag{54}$$

The zero points of $f(\lambda) = 0$ can be used to determine λ for the generalized Sturm–Liouville eigenvalue problem in Eqs. (51) and (52). Based on the regularized sampling method, Chanane [22] has derived a characteristic equation in terms of modified Bessel functions of the first kind, which is much complex than the above two equations.

With the initial conditions of $y_0 = 0$ and $A = y'_0 = 1$ we apply the $SL(2, \mathbb{R})$ Lie-group shooting method in Section 4 to calculate the eigenvalues in a range of $0 < \lambda < 80$. From Fig. 7(a) it can be seen that the curve of $\text{sgn}(Z)$ is intersected with the zero line at six points. For comparison purpose we also plot the curves of the above function $f(\lambda)$ given in Eqs. (53) and (54) in Fig. 7(b), whose intersecting points with the zero line are the roots. The results are coincident with the zero points in Fig. 7(a). In Table 3.4 of [22] and Table 5 of [24], there were also six eigenvalues being reported. Here, we give a theoretical result obtained from the Lie-group shooting method based on $SL(2, \mathbb{R})$, and the formulas (53) and (54) can be derived explicitly.

5.5. Example 7

The following example is taken from [24,25]:

$$y''(x) + \frac{y(x)}{(\lambda + x^2)^2} = 0, \quad 0 < x < 1, \tag{55}$$

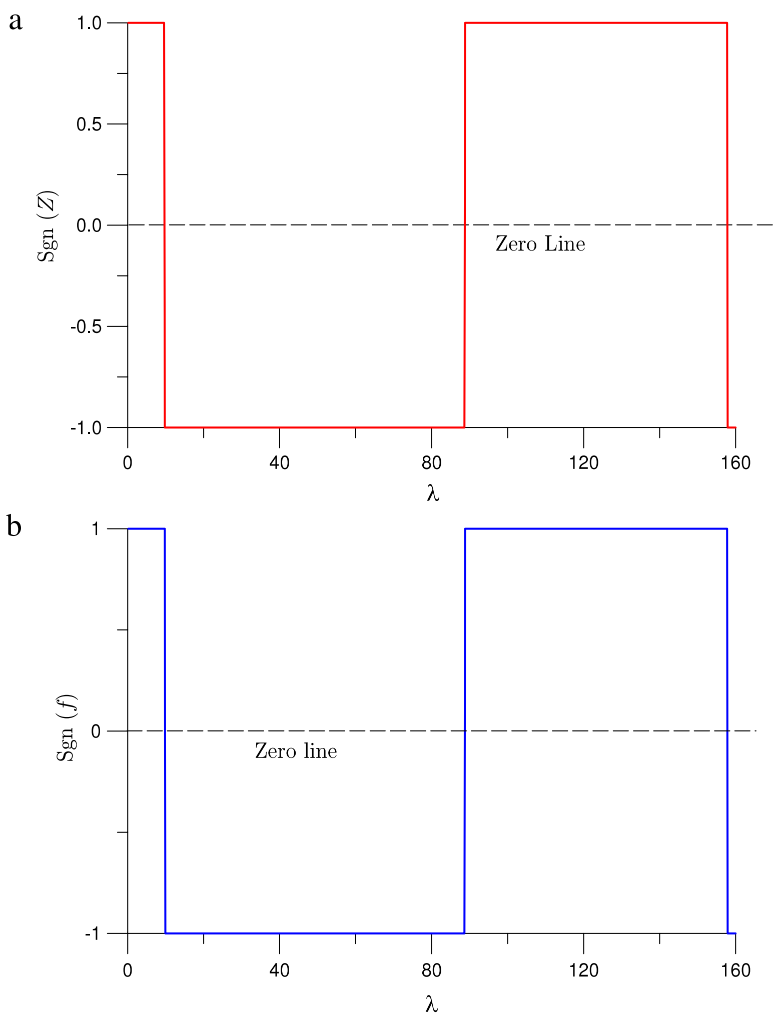

Fig. 5. For Example 5: (a) plotting the eigenvalue curve, and (b) the two curves determine the eigenvalues.

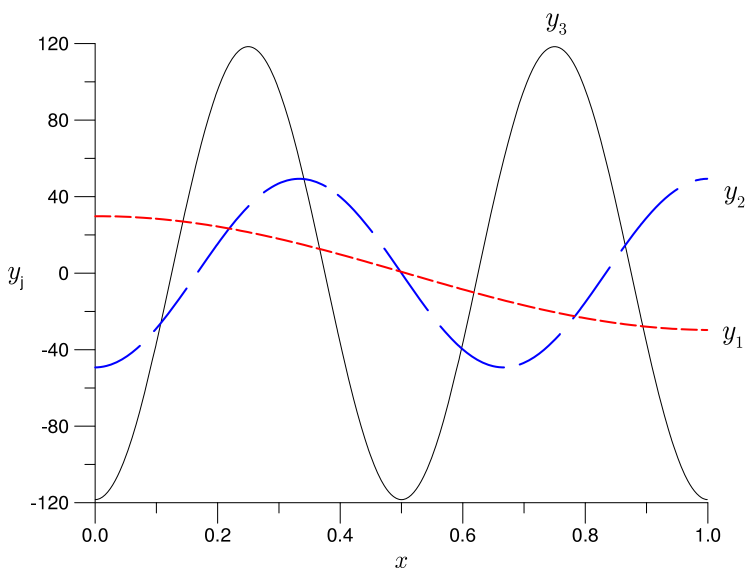

Fig. 6. For Example 5 plotting the first three eigenfunctions.

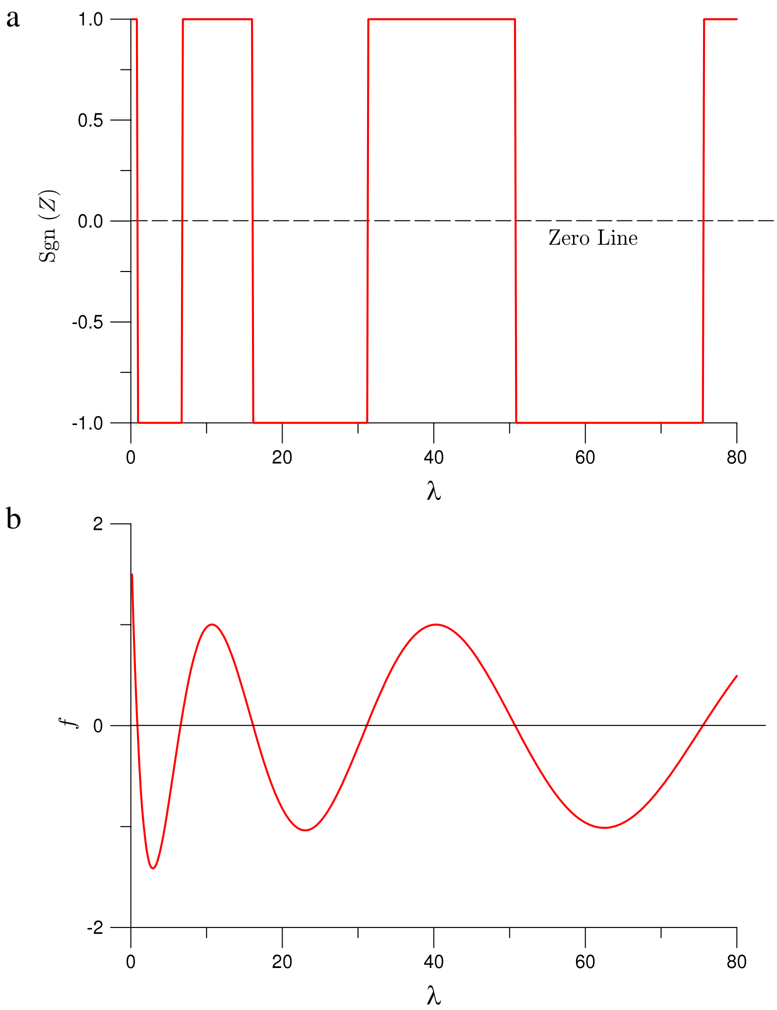

Fig. 7. For Example 6: (a) plotting the eigenvalue curve, and (b) the curve determines the eigenvalues.

$$y(0) = y(1) = 0. \tag{56}$$

Here $q = 1/(\lambda + x^2)^2$ is a nonlinear function of the eigen-parameter λ.

When we apply the $SL(2, \mathbb{R})$ Lie-group shooting method to calculate the eigenvalues in a range of $0 < \lambda < 0.2$, we can see that the curve of $y(1)$ is intersected with the zero line at ten points as shown in Fig. 8, of which the number of eigenvalues is coincident with that obtained in [24,25].

5.6. Example 8

Finally, we consider an example of the mixed-type eigen-parameter dependent boundary conditions:

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < 1, \tag{57}$$

$$y(0) = 0, \qquad y'(0) + \sqrt{\lambda} y(1) + e^{\sqrt{\lambda}} y'(1) = 0. \tag{58}$$

When we apply the $SL(2, \mathbb{R})$ Lie-group shooting method to calculate the eigenvalues in a range of $0 < \lambda < 210$, we can see that the curve of $\operatorname{sgn}(Z)$ is intersected with the zero line at five points as shown in Fig. 9, of which the number of eigenvalues is coincident with that obtained from the following characteristic equation:

$$f(\lambda) = 1 + \sin\sqrt{\lambda} + e^{\sqrt{\lambda}} \cos\sqrt{\lambda} = 0, \tag{59}$$

which is derived from Eq. (42). This example shows again the power of the present method to compute the exact eigenvalues for a problem with general non-separated eigen-parameter dependent boundary conditions.

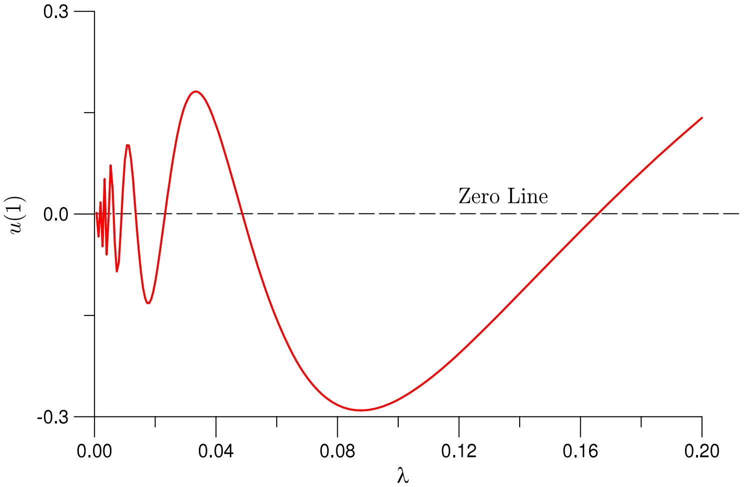

Fig. 8. For Example 7 displaying the eigenvalue curve.

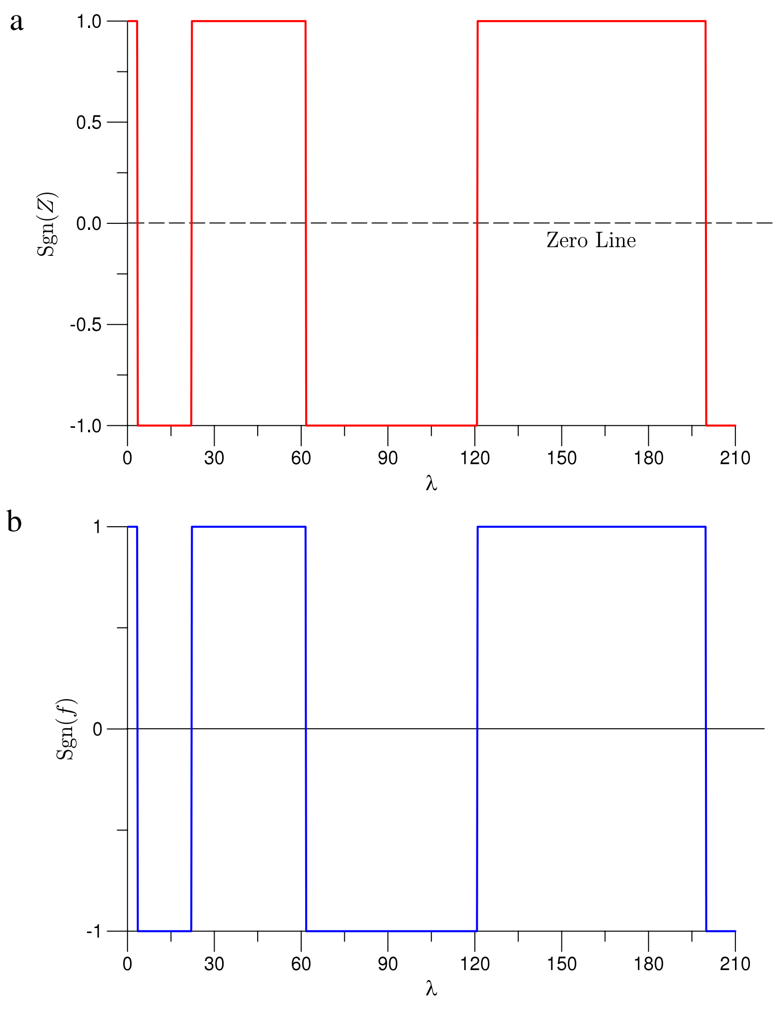

Fig. 9. For Example 8 displaying the eigenvalue curve.

6. Conclusions

A new Lie-group shooting method based on the Lie-group $SL(2,\mathbb{R})$ was developed in this paper, which can be used to compute the eigenvalues of the generalized Sturm–Liouville problems. The theoretical foundation based on the Lie-group $SL(2,\mathbb{R})$ and the closure property are proven. When the eigenvalues were available, we can also apply the Lie-group shooting method to compute the eigenfunctions. The key point is relied on the derived closed-form formulas for expressing the unknown boundary values in terms of λ. Several numerical examples were given to confirm the efficiency and accuracy of the present $SL(2,\mathbb{R})$ Lie-group shooting method, which is much easy to be numerically implemented with a very low cost. Moreover, the accuracy of the derived closed-form formulas to compute eigenvalues is confirmed.

Acknowledgments

The author sincerely appreciates the constructive comments from anonymous referees, which improved the quality of this paper. The project NSC-100-2221-E-002-165-MY3 and the 2011 Outstanding Research Award from Taiwan's National Science Council, and the 2011 Taiwan Research Front Award from Thomson Reuters, granted to the author, are highly appreciated.

Appendix

In this appendix we derive Eqs. (21) and (22). Finding **G**, which is corresponding to the constant matrix **A** in Eq. (20), is equivalent to find the transition matrix for the following two first-order ordinary differential equations:

$$w'(x) = c_1 z(x), \qquad z'(x) = c_2 w(x). \tag{A.1}$$

It follows that

$$w''(x) = c_1 z'(x) = c_1 c_2 w(x) = \kappa w(x). \tag{A.2}$$

If $\kappa = c_1 c_2 < 0$, through some elementary operations we can find

$$\mathbf{G}(x) = \begin{bmatrix} \cos\left[\sqrt{-\kappa}(x-x_0)\right] & \dfrac{c_1}{\sqrt{-\kappa}}\sin\left[\sqrt{-\kappa}(x-x_0)\right] \\ -\dfrac{\sqrt{-\kappa}}{c_1}\sin\left[\sqrt{-\kappa}(x-x_0)\right] & \cos\left[\sqrt{-\kappa}(x-x_0)\right] \end{bmatrix}. \tag{A.3}$$

It can be seen that $\mathbf{G}(x)$ satisfies the governing equation $\mathbf{G}'(x) = \mathbf{A}\mathbf{G}(x)$ and $\mathbf{G}(x_0) = \mathbf{I}_2$. By taking $c_1 = 1/p(\hat{x})$, $c_2 = -p(\hat{x})b(\hat{x})$, $\kappa = -b(\hat{x}) < 0$ and $x = x_f$, from Eq. (A.3) we can derive Eq. (21).

Similarly, if $\kappa = c_1 c_2 > 0$ we can find

$$\mathbf{G}(x) = \begin{bmatrix} \cosh\left[\sqrt{\kappa}(x-x_0)\right] & \dfrac{c_1}{\sqrt{\kappa}}\sinh\left[\sqrt{\kappa}(x-x_0)\right] \\ \dfrac{\sqrt{\kappa}}{c_1}\sinh\left[\sqrt{\kappa}(x-x_0)\right] & \cosh\left[\sqrt{\kappa}(x-x_0)\right] \end{bmatrix}. \tag{A.4}$$

By using $c_1 = 1/p(\hat{x})$, $c_2 = -p(\hat{x})b(\hat{x})$, $\kappa = -b(\hat{x}) > 0$ and $x = x_f$, from Eq. (A.4) we can derive Eq. (22).

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