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Some remarks on Arslan's 2011 paper

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ABSTRACT

It is shown that the main theorem of Arslan's paper (Theorem 2, 2011), as stated, is incorrect. Under additional conditions, we present a short proof of the corrected version of the theorem. We also give a proof of a theorem of Rao and Shanbhag (1991) [2], employed by Arslan, without the use of the Kolmogorov Consistency Theorem.

1. Introduction

In [1], the author attempts to establish a variant of the Choquet–Deny Theorem that will be used to prove some characterization results. Here we state below his theorem (Theorem 2, p. 4533) which we will call "Theorem A".

Theorem A. *Let $H : (0, \beta) \to [0, \infty)$ be a continuous and bounded solution of*

$$H(x) = \int_{\beta-x}^{\beta} H(t)\, \nu_x(dt), \quad x \in (0, \beta), \tag{1.1}$$

where ν_x is a nondegenerate probability measure concentrated in $(\beta - x, \beta)$ for every $x \in (0, \beta)$. Then H is constant.

In proving Theorem A, Arslan tries to use the same method of solving an integral equation as employed by Rao and Shanbhag [2]. We show in Section 2 that application of this method is not possible under conditions of Theorem A. First, however, we present one of our counterexamples to show that Theorem A, as stated, is incorrect. Then in Section 2, we state Rao and Shanbhag's Theorem and give a short proof of this theorem without the use of Kolmogorov Consistency Theorem used in [2] and employed by Arslan [1] as well. In Section 3, we point out an error in the proof of Arslan's Theorem 3 [1, p. 4535] which we will call Theorem B, and make some remarks related to this theorem.

Example 1. Let $\beta = 1$ and set

$$H(x) = \begin{cases} 1 + 2x & \text{if } x \in \left(0, \dfrac{1}{2}\right), \\ 2 & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right], \\ 5 - 4x & \text{if } x \in \left(\dfrac{3}{4}, 1\right). \end{cases}$$

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Then H is positive, continuous and bounded on $(0, 1)$; it even has limits as $x \to 0^+$ and $x \to 1^-$. For every $x \in (0, 1)$ there exists a nondegenerate probability measure ν_x on $(1-x, 1)$ such that

$$H(x) = \int_{1-x}^{1} H(t)\ \nu_x(dt)\,. \tag{1.2}$$

In fact, we choose ν_x to be a uniform measure (normalized Lebesgue measure) on the following intervals (a, b):

$$(a,b) = \begin{cases} (1-x, 1) & \text{if } x \in \left(0, \dfrac{1}{4}\right], \\ \left(\dfrac{3}{4}, \dfrac{5}{4} - x\right) & \text{if } x \in \left(\dfrac{1}{4}, \dfrac{1}{2}\right), \\ \left(\dfrac{1}{2}, \dfrac{3}{4}\right) & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right], \\ \left(\dfrac{3}{4}, 2x - \dfrac{3}{4}\right) & \text{if } x \in \left(\dfrac{3}{4}, \dfrac{7}{8}\right), \\ (2x-1, 1) & \text{if } x \in \left[\dfrac{7}{8}, 1\right). \end{cases}$$

It is an easy calculation to verify that (1.2) holds in each case and that ν_x is concentrated in $(1-x, 1)$. Therefore Theorem A is false.

2. Main results

In [2], the authors consider the functional equation

$$h(x) = \int_0^\infty h(x+y)\ \mu(dy)\,. \tag{2.1}$$

(Actually, they consider a more general equation but the arguments are the same.) In (2.1), μ is a given probability measure on $[0, \infty)$. By a solution of (2.1) we understand a function $h : [0, \infty) \to [0, \infty)$ which is Borel-measurable and bounded (so that the integral on the right-hand side of (2.1) exists) and satisfies (2.1).

The proof of the following theorem is the same as the one given in [2] but the use of the Kolmogorov Consistency Theorem is avoided.

Theorem 2. *Let μ be a probability measure on $[0, \infty)$. Then h is a solution of (2.1) if and only if, for all $x \in [0, \infty)$,*

$$h(x+y) = h(x) \quad \text{for } \mu \text{ almost all } y \in [0, \infty)\,.$$

Proof. The "if" part is trivial. To prove the "only if" part, suppose that h is a solution of (2.1). It is enough to show that the existence of $x_0 \geq 0$ and a Borel set $B_0 \subset [0, \infty)$ such that

$$h(x_0)\ \mu(B_0) < \int_{B_0} h(x_0 + y)\ \mu(dy)\,, \tag{2.2}$$

leads to a contradiction. Note that (2.2) implies that $h(x_0) > 0$ and $\mu(B_0) > 0$.

Let $m \in \mathbb{N}$. By iterating the functional equation (2.1) m times we obtain

$$h(x) = \int_0^\infty \cdots \int_0^\infty h(x + y_1 + \cdots + y_m)\ \mu(dy_1) \cdots \mu(dy_m)\,. \tag{2.3}$$

Let ν_m be the probability measure on $[0, \infty)^m$ given by

$$\nu_m(B) = \frac{1}{h(x_0)} \int \cdots \int_B h(x_0 + y_1 + \cdots + y_m)\ \mu(dy_1) \cdots \mu(dy_m)\,.$$

We define random variables $X_1, \ldots, X_m$ on the corresponding probability space by

$$X_j(t_1, \ldots, t_m) = \begin{cases} 1 & \text{if } t_j \in B_0, \\ 0 & \text{if } t_j \notin B_0. \end{cases}$$

Define

$$a_k = \nu_k(B_0 \times \cdots \times B_0) = E\left(X_{j_1} \cdots X_{j_k}\right),$$

where $1 \le j_1 < \cdots < j_k \le m$. Then

$$\left(\frac{1}{m}\sum_{j=1}^{m} X_j\right)^2 = \frac{1}{m^2}\left(\sum_{j=1}^{m} X_j^2 + 2\sum_{i<j} X_i X_j\right),$$

so

$$E\left(\left(\frac{1}{m}\sum_{j=1}^{m} X_j\right)^2\right) = \frac{1}{m}\, a_1 + \left(1-\frac{1}{m}\right) a_2.$$

Using the Cauchy–Schwarz inequality,

$$a_1^2 = \left(E\left(\frac{1}{m}\sum_{j=1}^{m} X_j\right)\right)^2 \le E\left(\left(\frac{1}{m}\sum_{j=1}^{m} X_j\right)^2\right) = \frac{1}{m}\, a_1 + \left(1-\frac{1}{m}\right) a_2.$$

As $m \to \infty$, we obtain $a_1^2 \le a_2$. In a similar way we show that

$$a_k^2 \le a_{2k} \quad \textit{for } k \in \mathbb{N}$$

and so

$$\frac{a_{2^n}}{\left(\mu\left(B_0\right)\right)^{2^n}} \ge \left(\frac{a_1}{\mu\left(B_0\right)}\right)^{2^n}. \tag{2.4}$$

This leads to a contradiction since the left-hand side of (2.4) is bounded but the right-hand side converges to infinity as $n \to \infty$ by (2.2). □

Remark 3. In [2] the Kolmogorov Consistency Theorem is used to show the existence of the random variables X_j, which explains the idea of considering all $X_j, j \in \mathbb{N}$, on the same probability space. However, when working only with $X_1, \dots, X_m$ with fixed m there is no need for the Kolmogorov Theorem. When we let $m \to \infty$ we work with real sequences.

As we mentioned earlier, one can correct Theorem A by adding assumptions as follows.

Theorem 4. *In addition to the assumptions of Theorem A, suppose that* $\lim_{x\to\beta^-} H(x)$ *exists in* $[0, \infty)$ *and that, for every* $x \in (0, \beta)$*, the support of the probability measure* ν_x *contains* β*, that is,* $\nu_x((\beta-\delta, \beta)) > 0$ *for every* $\delta > 0$*. Then H is constant.*

Proof. Eq. (1.1) shows that $\lim_{x\to 0^+} H(x)$ also exists and is equal to $c = \lim_{x\to\beta^-} H(x)$ for some nonnegative constant c. Suppose that H is not constant. Then H attains its absolute extremum at $x_0 \in (0, \beta)$ and $H(x_0) \neq c$. Suppose that H attains its absolute maximum at x_0 and $H(x_0) > c$ (the proof is similar for an absolute minimum). Then $H(t) \le H(x_0)$ for $t \in (0, \beta)$ and there is $\delta \in (0, x_0)$ such that $H(t) \le b = \frac{1}{2}(c + H(x_0)) < H(x_0)$ for $t \in (\beta-\delta, \beta)$. Then

$$\begin{aligned} H(x_0) &= \int_{(\beta-x_0,\beta-\delta]} H(t)\ \nu_{x_0}(dt) + \int_{(\beta-\delta,\beta)} H(t)\ \nu_{x_0}(dt) \\ &\le H(x_0)\,\nu_{x_0}((\beta-x_0, \beta-\delta]) + b\nu_{x_0}((\beta-\delta, \beta)) < H(x_0) \end{aligned}$$

which is a contradiction. Therefore, H is a constant. □

Remark 5. The author of [1] employs the same method of arguments as used in the proof of Theorem 2 by Rao and Shanbhag [2] to treat (1.1). This seems impossible because one cannot iterate Arslan's functional equation as was done in the proof of Theorem 2 above. A reviewer suggested that the iteration of the functional equation may be one way of proving the theorem.

3. Further remarks

In [1], Arslan states the following characterization result based on the distribution of the spacing of generalized order statistics. For the definition of generalized order statistics $X(r, n, m, k)'s$ we refer the reader to [1, p. 4533].

Theorem B. *Let* $X_1, X_2, \dots, X_n$ *be a sequence of i.i.d. nonnegative random variables with an absolutely continuous distribution function F and symmetric about* $\beta/2$*. Given the following statements:*

(1) $X(r, n, m, k) - X(r-1, n, m, k) \stackrel{d}{=} X_l(r, n, m, k)$ *is true for some* $\gamma_r = k + (n-r)(m+1) \ge 1$ *with* $r > 1$.
(2) $\gamma_r = 1$ *and* $F \sim U(0, \beta)$,
it follows that $(1) \Longrightarrow (2)$.

Remarks 6. (i) In proving Theorem B based on Theorem A, Arslan defines

$$H(x) = \overline{F}^{\gamma_r - 1}(x)\, f(x).$$

Since $H(x)$ must be continuous, then the density $f(x)$ must be assumed continuous. This, however, is missing from the assumptions of Theorem B.

(ii) Arslan uses (2) (ordinary order statistics case) of Theorem B above to give a characterization of one of the present authors' special cases, namely $s = r + 1$ for $r = n - 1$ (please see [3] for details), as mentioned in [1, p. 4535, last line]. We have the following characterization of the uniform distribution without the assumption of symmetry and continuity of the density function: Let $X_1, X_2, \ldots, X_n$ be *i.i.d.* nonnegative random variables with an absolutely continuous distribution function F with support $[0, \beta]$. Then $X_{2:n} - X_{1:n} \overset{d}{=} X_{1:n}$ implies $F \sim U(0, \beta)$, where $X_{2:n}$ and $X_{1:n}$ are 2nd and 1st order statistics corresponding to X_j's. For the proof see [4, Theorem 2.2] for the special case $r = 1$; we state this theorem here for the sake of completeness. Theorem 2.2 of [4]. Let $X_1, X_2, \ldots, X_n$ be absolutely continuous *i.i.d.* random variables satisfying $X_{r+1:n} - X_{r:n} \overset{d}{=} X_{r:n}$ for some $r \in \{1, 2, \ldots, n-1\}$. Then there is a $c > 0$ such that their common *cdf* $F(x)$ is given by

$$F(x) = \begin{cases} \left(\dfrac{x}{c}\right)^{1/r} & \text{if } 0 \le x < c \\ 1 & \text{if } x \ge c. \end{cases}$$

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