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Temporal monotonicity of the solutions of some semilinear parabolic equations with fractional diffusion

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Abstract

Suppose that the functions g , φ and ψ are nonnegative and satisfy suitable regularity conditions. Then, we prove in this work that the parabolic semilinear problem

$$\begin{cases} \partial_t u(t, x) = \Delta_\alpha u(t, x) - g(x)f(u(t, x)) + \varphi(x), & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = \psi(x), & x \in \mathbb{R}^n, \end{cases}$$

has a unique positive and time-monotone solution. Here, Δ_α is the fractional Laplacian with $\alpha \in (0, 2]$, and the source term f is a convex function with $f(0) = 0$. Moreover, using the temporal monotonicity, we show that the elliptic equation

$$\Delta_\alpha v(x) = g(x)f(v(x)) - \varphi(x), \quad x \in \mathbb{R}^d,$$

with boundary condition $\lim_{\|x\| \rightarrow \infty} v(x) = 0$, has a positive solution. We provide also sufficient conditions for the integrability of both solutions.

Keywords: Temporal monotonicity, asymptotic behavior, positive solutions, semilinear parabolic equations, fractional diffusion.

2010 MSC: 35B40, 35B65, 35B09.

1. Introduction

In this paper, we study the temporal monotonicity of the positive solutions of the parabolic semilinear partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta_\alpha u(t, x) - g(x)f(u(t, x)) + \varphi(x), & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u(0, x) = \psi(x), & x \in \mathbb{R}^d, \end{cases} \quad (1)$$

where Δ_α is the fractional Laplacian with $\alpha \in (0, 2]$, the initial datum ψ is a nonnegative function in the domain of Δ_α , the source term f is a convex function with $f(0) = 0$, and both g and φ are nonnegative continuous functions.

There has been recently an increase in the interest on partial differential equations that involve the fractional Laplacian. This is due, in part, to the multiple applications of the fractional operator. Indeed, many interesting phenomena in molecular biology, mathematical finance, statistical physics and hydrodynamics can be modeled by a fractional Laplacian (see [1] and references therein). On the other hand, from

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11 a theoretical point of view and as opposed to the classical operators, the fractional Laplacian is non-local.
 12 As a consequence, different analytical techniques may be required to study fractional partial differential
 13 equations. Among the traditional references on the study of the fractional Laplacian, the books [2] and [3]
 14 clearly stand out, the first one in the context of fractional powers of closed operators and the second in the
 15 framework of Riesz potentials.

16 From a physical point of view, it is worthwhile to recall that certain discrete systems yield fractional
 17 derivatives in some continuous-limit processes [4]. In such way, fractional models in the form of ordinary
 18 or partial differential equations are obtained from discrete physical systems [5]. From this perspective, the
 19 use of fractional operators is physically justified, at least as the continuous limit of several discrete systems
 20 appearing in various branches of sciences [6]. Obviously, this fact has encouraged the mathematical modeling
 21 using fractional differential equations, as well as the analytical and the physical investigation of these models
 22 [7, 8]. Needless to mention that the specialized literature has benefited from the investigation of fractional
 23 equations [9]. Indeed various interesting reports have been published on the existence and the uniqueness of
 24 solutions of fractional forms of parabolic models, like the porous media equation [10], the nonlinear diffusion
 25 equation in multiple dimensions [11] and nonlinear degenerate diffusion equations in bounded domains [12].
 26 In particular, it is well known [13] that some constructions of superprocesses are based on the existence of
 27 solutions of equations like (1). For example, the (α, d, β) -superprocess is a Markov process whose Laplace
 28 functional involves a solution of (1), with $g = 1$, $\psi = 0$ and $f(x) = e^{-\lambda x + \beta}$, for $0 < \beta \leq 1$.

29 **Definition 1.** Any solution of (1) will be called a *classical solution*. By a *mild solution* of (1) we mean [14,
 30 Section 6.1] a continuous curve $u : [0, \infty) \rightarrow B(\mathbb{R}^d)$ satisfying

$$u(t) = S_t(\psi) + \int_0^t S_{t-s}(\varphi - f(u(s)))ds, \quad t \geq 0. \quad (2)$$

31 Here, $B(\mathbb{R}^d)$ denotes the space of bounded measurable real-valued functions defined on \mathbb{R}^d , and $\{S_t : t \geq 0\}$
 32 is the semigroup corresponding to the fractional Laplacian (see Section 2).

33 The purpose of the present work is to prove that (1) has a unique positive and bounded global (or
 34 entire) solution $u(t, x)$. As we mentioned before, the main difference between the classical and the fractional
 35 Laplacians is that the second is not local. In particular, this leads to a more elaborate proof on the positivity
 36 of the solution of (1). To this end, we need to ensure that the extremes are global (see Theorem 9) in order
 37 to follow the well-known methods of the literature. It is worth mentioning that if the initial datum of (1) is
 38 in $C_b^2(\mathbb{R}^d)$ then the solution also belongs in this space when $\alpha \in (1, 2)$ (see [15]). Moreover, we will see that
 39 u is monotone in time (see [16]) whence it makes sense to define $u_\infty(x) = \lim_{t \rightarrow \infty} u(t, x)$. As a result, we
 40 obtain that u_∞ is a solution of the similar elliptic equation

$$\Delta_\alpha v(x) = g(x)f(v(x)) - \varphi(x), \quad x \in \mathbb{R}^d, \quad (3)$$

41 with boundary condition $\lim_{\|x\| \rightarrow \infty} v(x) = 0$ (see Theorem 10 below).

42 To show that u_∞ solves the elliptic problem (3) with boundary data at infinity, the main difficulty is
 43 to verify that the solution is in the domain of the fractional Laplacian. This task is carried out using the
 44 closure of such operator. When $\alpha = 2$ and $d \in \{1, 2\}$, we obtain some bounds for $u(t, x)$ which are uniform
 45 in time. The main idea here is to reduce the estimation problem to the study of the asymptotic behavior of
 46 a second-order ordinary differential equation with boundary condition at infinity. In this case, the bounds
 47 depend on whether $f'(0+) = 0$ or $f'(0+) > 0$ (see Theorem 11).

48 **Example 2.** Let $t \geq 0$. Some typical examples of convex functions f that satisfy $f(0) = 0$ are given below.

- 49 (i) Let $f(t) = t^{1+\beta}$ where $\beta > 0$. Here, $f'(0+) = 0$, and u_∞ is integrable if $\beta < \frac{2}{d}$.
- 50 (ii) Let $f(t) = e^{\beta t} - 1$ with $\beta > 0$. In this case, $f'(0+) = \beta$, and we get that u_∞ is always integrable.

51 The existence of global solutions of (1) has been proved in some particular cases. For example, in the
 52 classical scenario ($\alpha = 2$), a stochastic method was used in [17] to estimate the solutions of (1) when
 53 $g = \varphi = 1$. Also, the existence of a unique bounded solution of an elliptic equation similar to (3) was proved

when $g = 0$. In that case, the author uses a Dirichlet problem to establish the existence. Similar results were obtained in [18, 19] for a more general source term f . It is worth mentioning that the number of reports for the classical case is very large. On the other hand, the number of works dealing with positive solutions of (1) when $\alpha \in (0, 2)$ has increased in recent years, though most of them study the existence of positive radial solutions (see [20, 21] as examples).

Other existing works on parabolic semilinear fractional partial differential equations are scarce. For example, in [22] the authors study a model similar to (1) for a fractional Laplacian with $\alpha \in (0, 2]$. The existence of solutions is established but no asymptotic behavior is studied. Some finite-element methods were introduced in [23] to approximate the solutions of an extended form of a model similar ours. Meanwhile, the convergence of solutions of a fractional heat equation is investigated in [24] considering $\alpha \in (0, 1)$, and a more general fractional heat equation is studied analytically in [25]. However, from our point of view, the study reported in this manuscript has not been performed previously in the literature.

This work is organized as follows. In Section 2, we recall the notions of α -stable densities, semigroups and fractional Laplacian. We present therein some preliminary results on such concepts. In that section, we study also asymptotic properties of a second-order ordinary differential equation which is used to obtain some bound for the solutions of (1) and (3). In Section 3, we provide the precise statement of the main results of this work and their proofs. In turn, Section 4 provides some discussions motivated by the comments of the reviewers. In the final section, we present some concluding remarks.

2. Preliminary results

In the following, the letter c will denote a generic positive constant whose value may change throughout this work. We use $p_\alpha(t, \cdot)$ to denote real-valued functions defined on \mathbb{R}^d , with Fourier transforms given by

$$\int_{\mathbb{R}^d} e^{z \cdot x i} p_\alpha(t, x) dx = e^{-t \|z\|^\alpha}, \quad \text{for all } t > 0, z \in \mathbb{R}^d. \quad (4)$$

Here, \cdot and $\|\cdot\|$ are the inner product and the Euclidean norm in \mathbb{R}^d , respectively.

Definition 3. The functions $p_\alpha(t, \cdot)$ are usually called α -stable densities.

An alternative definition of the functions $p_\alpha(t, \cdot)$ will be provided below. We establish now some useful properties of α -stable densities.

Proposition 4. Let $p_\alpha(t, \cdot)$ be any α -stable density.

(i) For each $t > 0$,

$$\int_{\mathbb{R}^d} p_\alpha(t, y) dy = 1, \quad (5)$$

and $p_\alpha(t, x) > 0$, for all $x \in \mathbb{R}^d$ (density property).

(ii) For each $t, s > 0$ and $x \in \mathbb{R}^d$, $p_\alpha(ts, x) = t^{-d/\alpha} p_\alpha(s, t^{-1/\alpha} x)$ (scale property). In particular, it follows that $p_\alpha(t, x) \leq t^{-d/\alpha} p_\alpha(1, 0)$ (unimodal property).

(iii) The function $(t, x) \mapsto p_\alpha(t, x)$ is in $C^\infty((0, \infty) \times \mathbb{R}^d)$.

(iv) The function $(0, \infty) \times \mathbb{R}^d \ni (t, x) \mapsto p_\alpha(t, x) \in L^1(\mathbb{R}^d)$ is continuous.

(v) For each $t > 0$,

$$\lim_{x \rightarrow 0} \int_0^t \|p_\alpha(s, x + \cdot) - p_\alpha(s, \cdot)\|_1 ds = 0. \quad (6)$$

Proof. Beforehand, note that properties (i) and (ii) were established in [26], and (iii) was proved in [15].

(iv) Following [2], we introduce the auxiliary function

$$f_\alpha(t, \lambda) = \begin{cases} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^{\alpha/2}} dz, & \sigma > 0, t > 0, \lambda \geq 0, \\ 0, & \lambda < 0. \end{cases} \quad (7)$$

From Proposition 11.1 in [2] and (4), we obtain the subordination formula

$$p_\alpha(t, x) = \int_0^\infty f_\alpha(t, \lambda) p_2(\lambda, x) d\lambda, \quad (8)$$

where p_2 is the Gaussian density [26]. Let $t, \tilde{t} > 0$, and note that the subordination formula yields

$$\int_{\mathbb{R}^d} |p_\alpha(t, x) - p_\alpha(\tilde{t}, x)| dx = \int_0^\infty |f_\alpha(t, \lambda) - f_\alpha(\tilde{t}, \lambda)| d\lambda = |t - \tilde{t}| \int_0^\infty \left| \frac{d}{d\lambda} f_\alpha(\zeta, \lambda) \right| d\lambda, \quad (9)$$

where ζ is a number between t and \tilde{t} , which is provided by the mean value theorem. In order to estimate $\partial_t f_\alpha$, we use formula (19) on page 263 of [2] to obtain

$$\left| \frac{\partial}{\partial t} f_\alpha(\zeta, \lambda) \right| \leq \frac{1}{\pi} \int_0^\infty \exp((\lambda r + \zeta r^{\alpha/2}) \cos \theta_\alpha) r^{\alpha/2} dr, \quad (10)$$

with $\theta_\alpha = 2\pi/(2 + \alpha)$. Fubini's theorem yields

$$\begin{aligned} \int_0^\infty \left| \frac{\partial}{\partial t} f_\alpha(\zeta, \lambda) \right| d\lambda &\leq \frac{-1}{\pi \cos \theta_\alpha} \int_0^\infty \exp(s r^{\alpha/2} \cos \theta_\alpha) r^{(\alpha/2)-1} dr \\ &\leq \frac{-1}{\pi \cos \theta_\alpha} \left(\frac{2}{\alpha} + \int_0^\infty \exp(s r^{\alpha/2} \cos \theta_\alpha) dr \right) < \infty, \end{aligned} \quad (11)$$

for $s < t, \tilde{t}$. Here, we have used the fact that $\cos \theta_\alpha < 0$. This implies the desired result.

(v) Let (x_n) be a sequence in \mathbb{R}^d such that $\lim_{n \rightarrow \infty} x_n = \infty$. Fatou's lemma implies that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} 2p_\alpha(s, y) dy ds + \liminf_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}^d} |p_\alpha(s, x_n + y) - p_\alpha(s, y)| dy ds \\ &= \liminf_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}^d} [p_\alpha(s, x_n + y) + p_\alpha(s, y) \pm |p_\alpha(s, x_n + y) - p_\alpha(s, y)|] dy ds \\ &\geq \int_0^t \int_{\mathbb{R}^d} \liminf_{n \rightarrow \infty} [p_\alpha(s, x_n + y) + p_\alpha(s, y) \pm |p_\alpha(s, x_n + y) - p_\alpha(s, y)|] dy ds \\ &= \int_0^t \int_{\mathbb{R}^d} 2p_\alpha(s, y) dy ds \end{aligned} \quad (12)$$

whence the result readily follows. \square

Recall that $B(\mathbb{R}^d)$ is Banach space with the supremum norm $\|\cdot\|_\infty$. We will consider the uniformly continuous semigroup $\{S_t : t \geq 0\}$ defined on $B(\mathbb{R}^d)$ as

$$S_t h(x) = (p_\alpha(t) * h)(x) = \int_{\mathbb{R}^d} p_\alpha(t, x - y) h(y) dy, \quad (13)$$

where $(S_0 h)(x) = h(x)$ (see [24]). Some properties of the semigroup $\{S_t : t \geq 0\}$ are presented below.

Proposition 5. Let $\psi \in L^\infty(\mathbb{R}^d)$ and $\phi \in L^1(\mathbb{R}^d)$.

- (i) For each $t > 0$, it follows that $\|S_t \psi\|_\infty \leq \|\psi\|_\infty$ and $\|S_t \phi\|_1 \leq \|\phi\|_1$.
- (ii) $\lim_{t \rightarrow \infty} t^{d/\alpha} (S_t \psi)(x) = p_1(0) \|\phi\|_1$ uniformly in $x \in \mathbb{R}^d$. In particular, $\lim_{t \rightarrow \infty} (S_t \phi)(x) = 0$ uniformly in $x \in \mathbb{R}^d$.
- (iii) $\limsup_{\|x\| \rightarrow \infty} |(S_t \psi)(x)| = 0$. In particular, $\limsup_{\|x\| \rightarrow \infty} |(S_t \phi)(x)| = 0$ uniformly in $t > 0$.
- (iv) If $\alpha > \alpha$ then

$$\limsup_{\|x\| \rightarrow \infty} \limsup_{t \rightarrow \infty} \int_0^t (S_s \phi)(x) ds = 0. \quad (14)$$

106 *Proof.* Property (i) readily follows from the definition of the norm (see [27]). Meanwhile, property (ii) is
 107 a consequence of the scale property of α -stable densities and the dominated convergence theorem (here we
 108 use the unimodal property of α -stable densities).

109 (iii) If $\limsup_{\|x\| \rightarrow \infty} |\phi(x)| = l > 0$, there is $M > 0$ sufficiently large so that in $\{\phi(x) : \|x\| \geq M\} > l/2$.
 110 As a consequence,

$$\|\phi\|_1 \geq \int_{\|x\| \geq M} |\phi(x)| dx \geq \int_{\|x\| \geq M} \frac{l}{2} dx = \infty. \quad (15)$$

111 The second assertion of (iii) readily follows now from property (i).

112 (iv) Let $\varepsilon > 0$, and take $M > 0$ such that $M^{1-d/\alpha}((d/\alpha) - 1)^{-1} < \varepsilon/2$. The scale and unimodal properties
 113 of α -stable densities imply

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_0^t (S_s \phi)(x) ds &\leq \int_0^M (S_s \phi)(x) ds + \limsup_{t \rightarrow \infty} \int_M^t e^{-s/\alpha} p_1(0) \|\phi\|_1 ds \\ &\leq \int_0^M (S_s \phi)(x) ds + p_1(0) \|\phi\|_1 \frac{\varepsilon}{2}, \quad \text{for all } x \in \mathbb{R}^d. \end{aligned} \quad (16)$$

114 The digression (iii) and the dominated convergence theorem yield

$$\limsup_{\|x\| \rightarrow \infty} \limsup_{t \rightarrow \infty} \int_0^t S_s \phi(x) ds \leq p_1(0) \|\phi\|_1 \frac{\varepsilon}{2}. \quad (17)$$

115 Finally, we reach the conclusion when we let $\varepsilon \rightarrow 0$. \square

116 Recall that the corresponding infinitesimal generator of the semigroup $\{S_t : t \geq 0\}$ is the fractional
 117 Laplacian Δ_α , whose domain is denoted by $D(\Delta_\alpha)$.

118 **Proposition 6.** *Let Δ_α represent the fractional Laplacian.*

- 119 (i) *If $x \in \mathbb{R}^d$ is a global minimum of $\phi \in D(\Delta_\alpha)$ then $\Delta_\alpha \phi(x) \geq 0$.*
 120 (ii) *Δ_α is a closed linear operator.*

121 *Proof.* The proof of (i) can be found in [14]. Meanwhile, proposition (ii) is Corollary 1.2.5 in [14]. \square

122 The following is an essential tool to investigate the integrability of u and v in (1) and (3), respectively.

123 **Proposition 7.** *Let $h : [0, \infty) \rightarrow \mathbb{R}$ be a continuous, non-decreasing function. The initial value problem*

$$\begin{cases} y''(r) = h(y(r)), & r > 0, \\ y(0) = 1, \end{cases} \quad (18)$$

124 *subject to*

$$y(r) > 0, \quad y'(r) < 0, \quad r \geq 0, \quad (19)$$

125 *has a unique solution χ that satisfies $\lim_{r \rightarrow \infty} \chi(r) = 0$ and $\lim_{r \rightarrow \infty} \chi'(r) = 0$.*

126 *Proof.* The existence property follows from Theorem 3.1 of [28]. We examine now the asymptotic behavior
 127 of the solutions. Since χ is nonincreasing, we can assume that $\liminf_{r \rightarrow \infty} \chi(r) = \eta \in [0, 1]$. If $\eta > 0$ then
 128 there exists $M > 0$ such that $\chi(r) > \eta/2$, for all $r > M$. The fundamental theorem of calculus implies that

$$\liminf_{r \rightarrow \infty} \chi'(r) = \chi'(M) + \liminf_{r \rightarrow \infty} \int_M^r h(\chi(s)) ds \geq \chi'(M) + \liminf_{r \rightarrow \infty} h\left(\frac{\eta}{2}\right)(r - M). \quad (20)$$

129 This means that $\liminf_{r \rightarrow \infty} \chi'(r) = \infty$, which is impossible. Using now the monotonicity of χ , we note that
 130 $\liminf_{r \rightarrow \infty} \chi'(r) = \limsup_{r \rightarrow \infty} \chi'(r) = 0$. To reach the second limit of the conclusion we argue as before to
 131 obtain

$$0 \leq \chi(0) + \limsup_{r \rightarrow \infty} (r\chi(r)) \leq \chi(0). \quad (21)$$

132 The monotonicity of the function χ' readily implies the existence of $\lim_{r \rightarrow \infty} \chi'(r) = l \leq 0$. Note that $l < 0$
 133 yields that $-\infty = \limsup_{r \rightarrow \infty} (r\chi(r)) \geq -\chi(0)$, so we must have $\lim_{r \rightarrow \infty} \chi'(r) = 0$. \square

3. Main results and proofs

Let $T > 0$. We will use E_T to denote the Banach space $C([0, T] : B(\mathbb{R}^d))$ with the norm

$$\|u\|_T = \sup\{\|u(t)\|_\infty : t \in [0, T]\}. \quad (22)$$

Meanwhile, $B_r(x)$ and $\bar{B}_r(x)$ will represent, respectively, the open and the closed ball with center at x and radius $r > 0$.

Lemma 8. *Suppose that*

- (i) $\psi \in D(\Delta_\alpha) \cap B(\mathbb{R}^d)$,
- (ii) $\varphi, g \in C(\mathbb{R}^d) \cap B(\mathbb{R}^d)$ and
- (iii) $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz continuous.

Then there exists a unique classical solution $u \in \bar{B}_{\|\psi\|_\infty + 1}(0) \subseteq E_T$ of (1), for some $T > 0$.

Proof. By taking $T > 0$ sufficiently small, we can apply the contraction principle to obtain a solution $u \in \bar{B}_{\|\psi\|_\infty + 1}(0)$ of the integral equation (2). To establish the temporal regularity of u , derive this function formally and consider the corresponding integral equation (which has a unique solution by means of another application of the contraction principle). Such solution is continuous and it is the time-derivative of u . Using classical arguments, we can see that u is a solution of the differential equation (1) [14, Theorem 6.1.5]. \square

The following theorem is one of our main results.

Theorem 9. *Suppose that*

- (i) $\psi, \varphi \in L^1(\mathbb{R}^d)$,
- (ii) $\psi \in D(\Delta_\alpha) \cap B(\mathbb{R}^d)$ and $\psi \geq 0$,
- (iii) $\varphi, g \in C(\mathbb{R}^d) \cap B(\mathbb{R}^d)$ and $\varphi, g \geq 0$, and
- (iv) $f : [0, \infty) \rightarrow \mathbb{R}$ is convex with $f(0) = 0$.

Then there exists a unique nonnegative global solution $u \in \bar{B}_{\|\psi\|_\infty + 1}(0)$ of (1).

Proof. In Lemma 8, consider the function $\tilde{f}(x) = f(\max\{0, x\})$ for $x \in \mathbb{R}$. Then we obtain a continuous solution $u : [0, T] \rightarrow B(\mathbb{R}^d)$ of (1) with $\|u\|_T \leq \|\psi\|_\infty + 1$, for some $T > 0$. To show that $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous, we show firstly that $u(\cdot, x)$ is continuous in t uniformly in x . The continuity in time uniformly in x of the first term in (2) follows from Proposition 4(iii). For the second term in (2), fix an arbitrary $\tilde{t} \in [0, T]$, and note that

$$\begin{aligned} & \left| \int_0^{\tilde{t}+h} p_\alpha(\tilde{t}+h-s) * (\varphi - g\tilde{f}(u(s, \cdot)))(x) ds - \int_0^{\tilde{t}} p_\alpha(\tilde{t}-s) * (\varphi - g\tilde{f}(u(s, \cdot)))(x) ds \right| \\ & \leq (\|\varphi\|_\infty + \|g\|_\infty \|f\|_{[0, \|\psi\|_\infty + 1]}) \left[\int_0^{\tilde{t}+h} \|p_\alpha(\tilde{t}+h-s) - p_\alpha(\tilde{t}-s)\|_1 ds + |h| \right], \end{aligned} \quad (23)$$

where $\|f\|_{[0, \|\psi\|_\infty + 1]}$ is the supremum norm of \tilde{f} on $[-\|\psi\|_\infty - 1, \|\psi\|_\infty + 1]$. The result follows from the dominated convergence theorem and Proposition 4(iii). Fix now $(\tilde{t}, \tilde{x}) \in [0, T] \times \mathbb{R}^d$, and observe that

$$\begin{aligned} |u(t, x) - u(\tilde{t}, \tilde{x})| & \leq |(p_\alpha(t) * \psi)(x) - (p_\alpha(\tilde{t}) * \psi)(x)| + |(p_\alpha(\tilde{t}) * \psi)(x) - (p_\alpha(\tilde{t}) * \psi)(\tilde{x})| \\ & + \left| \int_0^t (p_\alpha(t-s) * (\varphi - g\tilde{f}(u(s, \cdot))))(x) ds - \int_0^{\tilde{t}} (p_\alpha(\tilde{t}-s) * (\varphi - g\tilde{f}(u(s, \cdot))))(x) ds \right| \\ & + \left| \int_0^{\tilde{t}} (p_\alpha(\tilde{t}-s) * (\varphi - g\tilde{f}(u(s, \cdot))))(x) ds - \int_0^{\tilde{t}} (p_\alpha(\tilde{t}-s) * (\varphi - g\tilde{f}(u(s, \cdot))))(\tilde{x}) ds \right|. \end{aligned} \quad (24)$$

Proposition 4(ii) and the continuity in t uniformly in x of $u(\cdot, x)$ imply, respectively, that the first and the third terms on the right-hand side of the above inequality tend to zero when $(t, x) \rightarrow (\tilde{t}, \tilde{x})$. Moreover, using the dominated convergence theorem we can see that the second and the fourth term also tend to zero when $(t, x) \rightarrow (\tilde{t}, \tilde{x})$. As a consequence, $\lim_{(t, x) \rightarrow (\tilde{t}, \tilde{x})} u(t, x) = u(\tilde{t}, \tilde{x})$.

166 • **Positivity.** Define the function $h : [0, T] \rightarrow \mathbb{R}$ by $h(t) = \inf_{x \in \mathbb{R}^d} u(t, x)$. Note that h is well defined
167 because $u \in \overline{B}_{\|\psi\|_\infty + 1}(0)$. The continuity in time of u uniform in x implies that h is a continuous
168 function on the compact set $[0, T]$. Then, there exists $\tilde{t} \in [0, T]$ such that

$$\inf_{x \in \mathbb{R}^d} u(\tilde{t}, x) = h(\tilde{t}) = \inf_{(t, x) \in [0, T] \times \mathbb{R}^d} u(t, x). \quad (25)$$

169 On the other hand, the convexity of f implies that $f(z) \leq cz$, for each $z \in [\|\psi\|_\infty + 1]$. This and
170 the integral representation (2) of u yield

$$\int_{\mathbb{R}^d} |u(t, x)| dx \leq (\|\psi\|_1 + T\|\varphi\|_1) + c\|g\|_\infty \int_0^t \int_{\mathbb{R}^d} |u(s, x)| dx ds. \quad (26)$$

171 Gronwall's inequality implies that $\sup\{\|u(t, \cdot)\|_1 : t \in [0, T]\} < \infty$. As a result of (iii) in the previous
172 proposition and the dominated convergence theorem, we obtain that $\limsup_{\|x\| \rightarrow \infty} |u(t, x)| = 0$, for
173 each $t \in [0, T]$. Suppose that $\xi := h(\tilde{t}) < 0$. Using the fact that $\lim_{\|x\| \rightarrow \infty} |u(\tilde{t}, x)| = 0$ and the
174 continuity of u , there exists $\tilde{x} \in \mathbb{R}^d$ such that

$$\xi = u(\tilde{t}, \tilde{x}) = h(\tilde{t}) = \inf_{(t, x) \in [0, T] \times \mathbb{R}^d} u(t, x). \quad (27)$$

175 Since $\psi \geq 0$ then $\tilde{t} \in (0, T]$. Define now the auxiliary function $\tilde{h} : [0, \tilde{t}] \rightarrow \mathbb{R}$ as $\tilde{h}(t) = \inf_{x \in \mathbb{R}^d} v(t, x)$,
176 where $v(t, x) = u(t, x) - k(\tilde{t} - t)$ and $k = -\xi/(\tilde{t} - 0) > 0$. As before, there exists $\hat{t} \in [0, \tilde{t}]$ such that

$$\tilde{h}(\hat{t}) = \inf_{t \in [0, \tilde{t}]} \tilde{h}(t) = \inf_{(t, x) \in [0, \tilde{t}] \times \mathbb{R}^d} v(t, x) \quad (28)$$

177 and $\lim_{\|x\| \rightarrow \infty} v(\hat{t}, x) = -k(\tilde{t} - \hat{t})$. As a consequence, there exists $M > 0$ sufficiently large with the
178 property that $v(\hat{t}, x) > \frac{\xi}{6} - k(\tilde{t} - \hat{t})$, for $\|x\| > M$. On the other hand, $\lim_{t \rightarrow \tilde{t}} u(t, \tilde{x}) = \xi$ yields a
179 $\delta > 0$ such that $v(t, \tilde{x}) < \frac{\xi}{6} - k(\tilde{t} - \hat{t})$, for all $t \in (\tilde{t} - \delta, \tilde{t}]$. In this way, if $t \in (\tilde{t} - \delta, \tilde{t}]$ then

$$\inf_{x \in \mathbb{R}^d} v(t, x) \leq \tilde{h}(t) \leq v(t, \tilde{x}) < \frac{\xi}{6} - k(\tilde{t} - \hat{t}). \quad (29)$$

180 By the definition of infimum, there exists $z \in \mathbb{R}^d$ such that $v(\hat{t}, z) < \xi/6 - k(\tilde{t} - \hat{t})$ and, obviously,
181 $\|z\| \leq M$. Using the continuity of v , we can find $\hat{x} \in \mathbb{R}^d$ such that $\|\hat{x}\| \leq M$ and

$$\inf_{t \in [0, \hat{t}]} \inf_{x \in \mathbb{R}^d} v(t, x) = v(\hat{t}, \hat{x}) = \inf_{\|x\| \leq M} v(\hat{t}, x) < \frac{\xi}{6} - k(\tilde{t} - \hat{t}). \quad (30)$$

182 If $\hat{t} = 0$ then (30) implies that $\psi(\hat{x}) < \xi/6 < 0$. Therefore, \hat{t} must be in $(0, \tilde{t}]$, in which case

$$\frac{\partial}{\partial t} u(\hat{t}, \hat{x}) + k = \frac{\partial}{\partial t} v(\hat{t}, \hat{x}) = \lim_{h \downarrow 0} \frac{v(\hat{t}, \hat{x}) - v(\hat{t} - h, \hat{x})}{h} \leq 0. \quad (31)$$

183 On the other hand, since \hat{x} is a global minimum for $v(\hat{t}, \cdot)$ (see (30)), we have $\Delta_\alpha u(\hat{t}, \hat{x}) = \Delta_\alpha v(\hat{t}, \hat{x}) \geq 0$
184 by Proposition 6(i). Using this inequality along with (30), (31) and the identity $\tilde{f}(u(\hat{t}, \hat{x})) = 0$, we
185 obtain

$$-k \geq \frac{\partial}{\partial t} u(\hat{t}, \hat{x}) = \Delta_\alpha u(\hat{t}, \hat{x}) - g(\hat{x})\tilde{f}(u(\hat{t}, \hat{x})) + \varphi(\hat{x}) \geq 0. \quad (32)$$

186 This means that u is not a solution of (1) due to the fact that the point $(\hat{t}, \hat{x}) \in (0, T] \times \mathbb{R}^d$ does
187 not satisfy such equation. Hence $\xi \geq 0$. In this way, we have shown that the solution u of (1) is
188 nonnegative.

189 • **Global existence.** Let T_{\max} be the maximal time of existence of the mild solution u of (1), and
190 suppose that $T_{\max} < \infty$. Then $\lim_{t \uparrow T_{\max}} \|u(t)\|_\infty = \infty$ [14, Theorem 6.1.4], which is impossible
191 because the positivity of u implies that it is bounded on $[0, T_{\max}]$. \square

Next, we will use the time-monotonicity of the solution u of (1) in order to obtain a solution v of (3).

Theorem 10. *Let $d > \alpha$ and suppose that*

- (i) $\varphi \in C(\mathbb{R}^d) \cap B(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $\varphi \geq 0$, and
- (ii) $f : [0, \infty) \rightarrow \mathbb{R}$ is convex and satisfies $f(0) = 0$.

Then there is a solution $u_\infty \in \overline{B}_{\|\psi\|_\infty + 1}(0)$ of (3) satisfying the boundary condition $\lim_{\|x\| \rightarrow \infty} u_\infty(x) = 0$. If $\alpha = 2$ and $d \in \{1, 2\}$ we require that $\varphi \in C_c(\mathbb{R}^d)$, $\liminf_{\|x\| \rightarrow \infty} g(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}^d$.

Proof.

- **Case 1:** $d > \alpha$. Consider (1) with $\psi \equiv 0$, and let v be the solution of

$$\begin{cases} \frac{\partial}{\partial t} v(t, x) &= \Delta_\alpha v(t, x) - g(x)(\tilde{f}'(u(t, x))v(t, x)), \\ v(0, x) &= \varphi(x), \end{cases} \quad (33)$$

where $\tilde{f}(x) = f(\max\{0, x\})$, for each $x \in \mathbb{R}$. Proceeding as in the proof of Theorem 9, we readily see that $\partial_t u(t, x) = v(t, x) \geq 0$. Considering the corresponding mild equation of (33), we deduce that

$$\frac{\partial}{\partial t} u(t, x) \leq c\varphi(x). \quad (34)$$

From Proposition 5(ii), it follows that there exists $\delta > 0$ such that $S_t \varphi(x) \leq ct^{-d/\alpha}$, for all $x \in \mathbb{R}^d$ and $t \geq \delta$. If we take $t > s \geq \delta$ then (34) results in

$$u(t, x) - u(s, x) \leq \frac{\alpha c}{\alpha - 1} \left(t^{1-d/\alpha} - s^{1-d/\alpha} \right). \quad (35)$$

Using now (2) and $d > \alpha$, we reach that $u(t, x) \leq c(\delta + \delta^{1-d/\alpha})$ holds for all $x \in \mathbb{R}^d$ and $t \geq \delta$. This implies that $\lim_{t \rightarrow \infty} u(t, x) =: u_\infty(x)$ exists and, moreover, one can easily prove that the convergence is uniform in x . On the other hand, using (34) and Proposition 5(ii) we see that $\lim_{t \rightarrow \infty} \partial_t u(t, x) = 0$ holds uniformly in x . Thus, $\lim_{t \rightarrow \infty} u(t, x) = g(x)f(u_\infty(x)) - \varphi(x)$ is uniform in x . Since Δ_α is a closed operator by Proposition 6(ii), then $u_\infty \in D(\Delta_\alpha)$ is a solution of (3). Moreover $u_\infty \neq 0$ if $\varphi \neq 0$. The positivity of $u(t, x)$ implies then that

$$\limsup_{\|x\| \rightarrow \infty} u_\infty(x) \leq \limsup_{\|x\| \rightarrow \infty} \limsup_{t \rightarrow \infty} \int_0^t S_s \phi(x) ds. \quad (36)$$

Proposition 5(iv) implies that $\lim_{\|x\| \rightarrow \infty} u_\infty(x) = 0$.

- **Case 2:** $\alpha = 2$ and $d \in \{1, 2\}$. We show firstly that

$$\lim_{\|x\| \rightarrow \infty} u(t, x) = 0 \text{ uniformly in } t > 0. \quad (37)$$

By hypothesis, $\liminf_{\|x\| \rightarrow \infty} g(x) > 0$. Recall that $R > 0$ is such that $\text{supp}(\varphi) \subseteq \overline{B}(0, R)$ and $\inf\{g(x) : \|x\| > R\} \geq 1/2$. Set $v(x) = \chi(\|x\|)$, where $\chi : [R, \infty) \rightarrow \mathbb{R}$ is the corresponding solution of (18). Using (1), for all $\|x\| > R$ and $t > 0$ the following holds:

$$\begin{aligned} \Delta_\alpha(u(t, x) - v(x)) &\geq \frac{L}{2}f(u(t, x)) - \frac{L}{2}f(v(x)) - \frac{d-1}{\|x\|}\chi'(\|x\|) \\ &> \frac{L}{2}(f(u(t, x)) - f(v(x))). \end{aligned} \quad (38)$$

If there exists $(t_0, x_0) \in (0, \infty) \times (\mathbb{R}^d \setminus \overline{B}(0, R))$ such that $u(t_0, x_0) > v(x_0)$, then there is $\tilde{x}_0 \in \overline{B}(x_0, \tilde{R})$ with $\tilde{R} = (\|x_0\| - R)/2$, for which

$$u(t_0, \tilde{x}_0) - v(\tilde{x}_0) = \max\{u(t_0, x) - v(x) : x \in \overline{B}(x_0, \tilde{R})\}. \quad (39)$$

As a consequence,

$$0 = \Delta(u(t_0, \tilde{x}_0) - v(\tilde{x}_0)) > \frac{L}{2} (f(u(t_0, \tilde{x}_0)) - f(v(\tilde{x}_0))), \quad (40)$$

and the strict monotonicity of f yields $0 > u(t_0, \tilde{x}_0) - v(\tilde{x}_0) \geq u(t_0, x_0) - v(x_0) > 0$. It follows that

$$u(t, x) \leq v(x), \text{ for all } t > 0 \text{ and } \|x\| > R. \quad (41)$$

The desired uniform limits follow then from Proposition 7. Take now $M > R$ and $t > 0$, and let $x_t \in \overline{B}(0, M)$ such that $u(t, x_t) = \max\{u(t, x) : \|x\| \leq M\}$. If $\|x_t\| = R$, then (41) implies that

$$u(t, x) \leq \max\{v(x) : \|x\| = M\}, \text{ for } \|x\| \leq M. \quad (42)$$

In the case $\|x_t\| < M$, we readily note that $0 \leq \partial_t u(t, x_t) = \Delta u(t, x_t) - g(x_t)f(u(t, x_t)) + \varphi(x_t)$. Moreover, notice that

$$u(t, x) \leq f^{-1} \left(\frac{\|\varphi\|_\infty}{\inf\{g(x) : x \in \mathbb{R}^d\}} \right), \text{ for all } t > 0 \text{ and } \|x\| < M. \quad (43)$$

The strict positivity of the infimum follows now from the fact that $\inf\{g(x) : \|x\| > R\} \geq L/2$, and that $g(x) > 0$ for all $x \in \mathbb{R}^d$. On the other hand, the fact that $\lim_{\|x\| \rightarrow \infty} v(x) = 0$ and the expressions (41), (42) and (43) imply that $\sup\{u(t, x) : t \geq 0, x \in \mathbb{R}^d\} < \infty$. Hence, $\lim_{t \rightarrow \infty} u(t, x) := u_\infty(x)$ exists in this case. Moreover, we claim that the convergence is uniform on $\|x\| \leq M$, with $M > R$. To prove that, we check firstly that $\{u(m, \cdot)\}_{m=1}^\infty$ is equicontinuous on $\overline{B}(0, M)$. Indeed, let $m \in \mathbb{N}$ be arbitrary but fixed, and let $x, y \in \mathbb{R}^d$. Then

$$\begin{aligned} |u(m, x) - u(m, y)| &\leq \int_0^m \int_{\mathbb{R}^d} |\varphi(z) - g(z)f(u(s, z))| |p_\alpha(t-s, x-z) - p_\alpha(t-s, y-z)| dz ds \\ &\leq (\|\varphi\|_\infty + \|g\|_\infty \|f\|_\infty) \int_0^m \int_{\mathbb{R}^d} |p_\alpha(s, x-y+z) - p_\alpha(s, z)| dz ds; \end{aligned} \quad (44)$$

the equicontinuity of $\{u(m, \cdot)\}_{m=1}^\infty$ follows now from Proposition 4(v). Since $\lim_{m \rightarrow \infty} u(m, x) = u_\infty(x)$, Theorem 7.5.6 in [29] implies that the convergence is uniform on $\overline{B}(0, M)$, so u is continuous on \mathbb{R}^d . Dini's theorem implies next that the convergence $\lim_{t \rightarrow \infty} u(t, x) = u_\infty(x)$ is uniform on $\overline{B}(0, M)$, for each $M > R$. Now, we will check that the convergence is uniform in \mathbb{R}^d . To that end, let $\varepsilon > 0$. By (37), there exists $M_\varepsilon > R$ such that

$$|u(t, x)| < \frac{\varepsilon}{2}, \text{ for all } t > 0 \text{ and } \|x\| > M_\varepsilon. \quad (45)$$

Moreover, the uniform convergence on $\overline{B}(0, M)$ guarantees that there exists $t_\varepsilon > 0$ for which

$$|u(t, x) - u_\infty(x)| < \frac{\varepsilon}{2}, \text{ for all } t > t_\varepsilon \text{ and } \|x\| \leq M_\varepsilon. \quad (46)$$

Propositions (45) and (46) yield $|u(t, x) - u_\infty(x)| \leq \varepsilon$, for all $t > t_\varepsilon$ and $x \in \mathbb{R}^d$. To complete the proof, we proceed now as in Case 1. \square

Let $R > 0$ be such that $\text{supp}(\varphi) \subseteq \overline{B}(0, R)$ and $\inf\{g(x) : \|x\| > R\} \geq L/2$. Define the function

$$\mathbf{F}(t) = \int_t^{\chi(R)} \frac{ds}{(F(s))^{1/2}}, \quad 0 < t \leq \chi(R), \quad (47)$$

where χ is the solution of (18) and

$$F(t) = \int_0^t f(s) ds, \quad t \geq 0. \quad (48)$$

From (41), we readily check that $u_\infty(x) \leq \chi(\|x\|)$, for all $\|x\| \geq R$. The following result provides other explicit bounds

Theorem 11. Assume the hypotheses of Theorem 9, and let $\alpha = 2$ and $d \in \{1, 2\}$. If $f'(0+) = 0$ then

$$u(t, x) \leq \exp\left(-\frac{1}{2}\|x\|\sqrt{f'(0+)}\right), \text{ for all } \|x\| \geq R \text{ and } t > 0. \quad (49)$$

If $f'(0+) = 0$ then

$$u(t, x) \leq \mathbf{F}^{-1}(\sqrt{3}\|x\|), \text{ for all } \|x\| \geq R \text{ and } t > 0. \quad (50)$$

As a consequence, if $f'(0+) > 0$ then u_∞ is always integrable. However, if $f'(0+) = 0$ then u_∞ is integrable when $\int_R^\infty \mathbf{F}^{-1}(r)r^{d-1}dr < \infty$.

Proof. An application of L'Hôpital's rule yields

$$\lim_{r \rightarrow \infty} \left(\frac{\chi'(r)}{\chi(r)}\right)^2 = \lim_{r \rightarrow \infty} \frac{f(\chi(r))}{\chi(r)}. \quad (51)$$

On the other hand, the convexity of f implies that $\lim_{x \downarrow 0} f(x)/x = f'(0+) \geq 0$.

• **Case 1:** $f'(0+) > 0$. Since $\lim_{r \rightarrow \infty} \chi'(r)/\chi(r) = -\sqrt{f'(0+)}$, there is $M > 0$ large enough such that

$$\exp\left(-\frac{3}{2}r\sqrt{f'(0+)}\right) < \chi(r) < \exp\left(-\frac{1}{2}r\sqrt{f'(0+)}\right), \text{ for all } r > M. \quad (52)$$

The integrability of u readily follows now.

• **Case 2:** $f'(0+) = 0$. By L'Hôpital's rule, $\lim_{r \rightarrow \infty} (\chi'(r))^2 / F(\chi(r)) = 2$. So, for $M > 0$ large enough,

$$-\sqrt{3}(F(\chi(r)))^{1/2} < \chi'(r) < -\sqrt{F(\chi(r))}^{1/2}, \text{ for all } r \geq M. \quad (53)$$

The comparison lemma for ordinary differential equations yields $y(r) \leq \chi(r) \leq z(r)$, for all $r \geq M$. Here,

$$y'(r) = -\sqrt{F(y(r))}^{1/2}, \quad y(M) = \chi(M), \quad (54)$$

$$z'(r) = -\sqrt{F(z(r))}^{1/2}, \quad z(M) = \chi(M). \quad (55)$$

Moreover, the solutions are given by $y(r) = \mathbf{F}^{-1}(r)$ and $z(r) = \mathbf{F}^{-1}(\sqrt{3}r)$. We may use now L'Hôpital's rule to see that $\lim_{t \downarrow 0} \mathbf{F}(t) = \infty$, so that y and z are well defined for all $r \geq M$. To conclude the proof, it suffices to observe that (41) implies that $u(t, x) \leq v(x) = \chi(\|x\|)$, for all $t > 0$ and $\|x\| > R$. \square

4. Discussion

The present section is devoted to discuss the significance of the present study along with the mathematical methodology to prove the main results of this work.

4.1. Significance

Firstly, it is worth recalling that there are some Markov process in probability theory which are characterized through their Laplace functionals [13, 30]. Such processes are measure-valued and, intuitively, they represent the continuous-state of clouds of certain branching phenomenon. The study of their path properties is based mainly on the Laplace functionals and, consequently, on the properties of the solutions of equations like those studied in the present paper. For example, when the diffusion on \mathbb{R}^d is an α -stable process and the branching mechanism f is $x^{1+\beta}$, then we obtain the (α, d, β) -superprocess.

There is an extensive literature on path behavior properties of superprocesses. Using mild solutions of the equation $\partial u(t, x)/\partial t = \Delta_\alpha u(t, x) - (u(t, x))^{1+\beta}$, one of the authors studied the self-intersection local time of (α, d, β) -superprocess [30]. If $X = \{X_s : s \geq 0\}$ is a superprocess on \mathbb{R}^d then

$$Y_t(\varphi) = \int_0^t \int_{\mathbb{R}^d} \varphi(x) X_s(dx) ds \quad (56)$$

is called *occupation time process*. The Laplace functional for the occupation time of an (α, d, β) -superprocess has associated a partial differential equation of the form

$$\frac{\partial u(t, x)}{\partial t} = \Delta_\alpha u(t, x) - (u(t, x))^{1+\beta} + \varphi(x) \quad (57)$$

Clearly, the path properties of Y involve the study of (57). For example, the ergodic limit $\lim_{t \rightarrow \infty} Y_t/t$ is thoroughly investigated in [31]. Meanwhile, in the article [16] the author studied the time-monotonicity of $u(t, x)$ to prove that the Laplace transform of $Y_\infty = \lim_{t \rightarrow \infty} Y_t$ involves a solution of

$$\Delta_\alpha v(x) = (v(x))^{1+\beta} - \varphi(x), \quad (58)$$

with boundary condition $\lim_{\|x\| \rightarrow \infty} v(x) = 0$.

4.2. Novelty

Positivity. The novelty of the approach followed to establish positivity is arguably new, indeed. However, it is important to note that the newness lies in the particular way to tackle the technical difficulties to apply the classical approach. More precisely, Gronwall's inequality was used to show the uniform integrability of the solution. In turn, we employed this result to check that the local minimum is global. Afterwards, the proof proceeds as in the classic case.

Elliptic problem. Note that the case $d > \alpha$ follows from general methods described in semi-group theory (see [14] or [16]). However, to the best of our knowledge, the approach followed in the case when $\alpha = 2$ and $d \in \{1, 2\}$ is entirely new. In fact, we could not find any report in the literature which uses the asymptotic properties of the solutions of problem (8)–(19), in order to prove that the elliptic equation (3) has a solution. This step is crucial to prove that the local solution $u(t, x)$ is bounded in space and uniform in time. From that point, we established that u is in the domain of fractional Laplacian. In light of these facts, we are convinced that the results on the elliptic problem are novel, and may prove interesting for the researchers in applied mathematics.

Asymptotic behavior. To bound the solution $v = u_\infty$ of (3), we used the solution of (18) and showed that the integrability of v depends only on the source term f . On the other hand, it is well known [32] that the branching mechanism of a superprocess has the form

$$f(x) = ax + bx^2 + \int (e^{-rx} - 1 + rx)\pi(dr), \quad (59)$$

where $a \geq 0$, $b \geq 0$ and π is a σ -finite measure on $(0, \infty)$ such that $\int \min\{r, r^2\}\pi(dr) < \infty$. If $a = b = 0$ and $\pi(dr) = c r^{-\beta-1} dr$ with $\beta \in (0, 1)$, we have $f(x) = c' x^{1+\beta}$. In this case, we readily see that an $(2, d, \beta)$ -superprocess spends an infinite weighted occupation time in bounded Borel subset of \mathbb{R}^d if $\beta < \frac{2}{d}$ (see Example 2 and [16]). In this way, the integrability conditions deduced in the present work can be used to propose conditions on the convex function (59) (see [32]), in order to study integrability properties of the occupation time of a broader class of superprocesses.

5. Conclusions

In this work, we studied the temporal monotonicity of the positive solutions of a parabolic semilinear partial differential equation (1). Moreover, we used our results to establish the existence of solutions for the elliptic equation (3). In particular, we observed that the source term f is fundamental in order to guarantee the integrability of the solutions. In fact, the ordinary differential equation (18) only involves such term. That fact was noted in one of the examples of this manuscript, in which the integrability of u_∞ only requires to impose conditions on β . It is important to point out that the study of partial differential equations with fractional diffusion is an interesting topic of current research in mathematics. The present work represents a contribution in that field. As one of the problems which remain open after the conclusion of this work, it is still pending to check if the integrability conditions for u_∞ are necessary. Additionally, the case when $d = 1$ and $\alpha \in [1, 2)$ still needs to be studied, and new techniques must be developed to tackle that case. This is an interesting problem since the solution can be used to see how much time an $(\alpha, 1, \beta)$ -superprocess spends in bounded Borel subset of \mathbb{R}^d , see [16].

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