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A conservative fourth-order stable finite difference scheme for the generalized Rosenau–KdV equation in both 1D and 2D

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HIGHLIGHTS

- A new conservative fourth-order stable finite difference scheme is proposed.
- The scheme is convergent with $O(\tau^2 + h^4)$ and unconditionally stable.
- The scheme is mass and energy conserved.
- The scheme can be applied to study the solitary wave traveling in a long time.

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ABSTRACT

In the present work, a conservative fourth-order stable finite difference scheme is proposed to solve the generalized Rosenau–KdV equation in both 1D and 2D. The existence, uniqueness, and mass and energy conservations of the numerical solution are proved using the discrete energy method. The new scheme is convergent with $O(\tau^2+h^4)$ and unconditionally stable. Numerical experiments are carried out to show that the scheme is efficient and reliable.

1. Introduction

The well-known Korteweg–de Vries (KdV) equation is [1–4]

$$u_t + uu_x + u_{xxx} = 0,$$

which has been used to study nonlinear wave phenomena, such as magnetic fluid wave, ion sound wave, and longitudinal astigmatic wave [5–12]. However, in the study of the dynamics of dense discrete systems, the case of wave–wave and wave–wall interactions cannot be properly described using the well-known KdV equation. To overcome this drawback, Rosenau proposed a modified equation, so-called the Rosenau equation [13–15]

$$u_t + u_x + uu_x + u_{xxxxt} = 0.$$

The existence and uniqueness of the solution for the Rosenau equation were proved by Park [16]. For the further consideration of nonlinear waves, it was suggested that the viscous term u_{xxx} should be included in the Rosenau equation. This equation is usually called the Rosenau–KdV equation [17–20]

$$u_t + u_x + u_{xxx} + u_{xxxxt} + uu_x = 0.$$

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Zuo [21] discussed the solitary wave solutions and periodic solutions for the Rosenau–KdV equation.

The 1D initial–boundary value problem of generalized Rosenau–KdV equation has the following form [22]

$$u_t + u_x + u_{xxx} + u_{xxxxt} + (u^p)_x = 0, \quad x \in \Omega, \quad t \in [0, T], \tag{1}$$

with an initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{2}$$

and boundary conditions

$$u(\alpha, t) = u(\beta, t) = 0, \quad u_x(\alpha, t) = u_x(\beta, t) = 0, \quad t \in [0, T], \tag{3}$$

where $p \geq 2$ is a positive integer, $\Omega = [\alpha, \beta]$, $u_0(x)$ is a given smooth function. Here, $u(x, t)$ is the nonlinear wave profile, and x and t are the spatial and temporal variables, respectively. Since $u_0(x)$ tends to zero when $\alpha \ll 0$ and $\beta \gg 0$ in most cases, it is reasonable to assume that $u(\alpha, t) = u(\beta, t) = 0$ and $u_x(\alpha, t) = u_x(\beta, t) = 0$, so that the initial condition in Eq. (2) and the boundary conditions in Eq. (3) are consistent with each other.

We define the following Sobolev space [23]

$$H^k(\Omega) = \left\{u(x) : \int_\Omega [u^{(j)}]^2 dx < +\infty, \;\; j = 0, 1, 2, \ldots, k-1, k\right\},$$

$$H_0^k(\Omega) = \left\{u(x) \in H^k(\Omega) : \frac{\partial^i u}{\partial x^i}\Big|_{\partial\Omega} = 0, \;\; i = 0, 1, 2, \ldots, k-1\right\},$$

where $u^{(j)}$ is the jth order derivative. The initial–boundary value problem in Eqs. (1)–(3) has the following conservative properties.

Theorem 1.1 (*See [24]*). *Suppose $u_0(x) \in H_0^2([\alpha, \beta])$, then the solution of the problem in Eqs. (1)–(3) satisfies*

$$Q(t) = \int_\alpha^\beta u(x, t) dx = \int_\alpha^\beta u(x, 0) dx = \int_\alpha^\beta u_0(x) dx = Q(0).$$

Theorem 1.2. *Suppose $u_0(x) \in H_0^2([\alpha, \beta])$, then the solution of the problem in Eqs. (1)–(3) satisfies*

$$E(t) = \int_\alpha^\beta (u^2 + u_{xx}^2) dx = \|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0),$$

where $\|\cdot\|_{L_2}$ denotes L_2-norm.

Proof. Rewriting Eq. (1) as

$$u_t + u_{xxxxt} = -u_x - u_{xxx} - (u^p)_x,$$

and considering the boundary conditions in Eq. (3), we have

$$\begin{aligned}
\frac{dE(t)}{dt} &= 2\int_\alpha^\beta (uu_t + u_{xx}u_{xxt}) dx = 2\int_\alpha^\beta (uu_t - u_x u_{xxxt}) dx + 2u_x u_{xxt}\Big|_\alpha^\beta \\
&= 2\int_\alpha^\beta u(u_t + u_{xxxxt}) dx = -2\int_\alpha^\beta u\Big[u_x + u_{xxx} + (u^p)_x\Big] dx \\
&= -\Big[u^2 + \frac{2p}{p+1}u^{p+1} + 2uu_{xx} - (u_x)^2\Big]\Big|_\alpha^\beta = 0.
\end{aligned}$$

Therefore, $E(t)$ is a constant function only depending on the initial data. This completes the proof.

Theorem 1.3. *Suppose $u_0(x) \in H_0^2([\alpha, \beta])$, then the solution of the problem in Eqs. (1)–(3) satisfies $\|u\|_{L_2} \leq C$, $\|u_x\|_{L_2} \leq C$, $\|u_{xx}\|_{L_2} \leq C$, and hence $\|u\|_{L_\infty} \leq C$, $\|u_x\|_{L_\infty} \leq C$, where $\|\cdot\|_{L_\infty}$ denotes L_∞-norm.*

Proof. From Theorem 1.2, we know that $\|u\|_{L_2} \leq C$, $\|u_{xx}\|_{L_2} \leq C$. Using the Hölder inequality and the Schwartz inequality [24], we get

$$\|u_x\|_{L_2}^2 = \int_\alpha^\beta u_x u_x dx = -\int_\alpha^\beta uu_{xx} dx \leq \|u\|_{L_2}\|u_{xx}\|_{L_2} \leq \frac{1}{2}\Big[\|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2\Big] \leq C.$$

Furthermore, using the Sobolev inequality [25], we obtain

$$\|u\|_{L_\infty} \leq C_1\|u\|_{L_2} + C_2\|u_x\|_{L_2} \leq C, \quad \|u_x\|_{L_\infty} \leq C_1\|u_x\|_{L_2} + C_2\|u_{xx}\|_{L_2} \leq C.$$

This completes the proof.

Because of the nonlinearity in Eq. (1), the above problem has to be solved numerically. Up to date, Razborova et al. [26] discussed the solitary solutions for the generalized Rosenau–KdV equation using the common solitary ansatz method. Hu et al. [27] proposed a conservative three-level in time linear finite difference scheme with second-order convergence for the initial–boundary value problem of the Rosenau–KdV equation. Zheng et al. [24] developed an average linear finite difference scheme for the numerical solution of the generalized Rosenau–KdV equation, in which the scheme is second-order convergent in both time and space variables and unconditionally stable.

On the other hand, as pointed out in [28], the overall accuracy of a specific numerical method is affected by not only the order of accuracy of the numerical method but also the conservative approximation property. Recently, numerical schemes that inherit conservation properties have attracted much attention [29,30]. Conservation laws play a very crucial role in the solution and reduction of partial differential equations [31]. Pan et al. [32] pointed out that the non-conservative difference schemes may easily show non-linear blow-up and the conservative finite difference schemes are better than the non-conservative ones. And these conservative properties are advantageous because the conservation of mass should contribute to stability, and the conservation of energy indicates that the scheme inherits the physical background and should yield better solutions in a physical point of view [33]. Thus, higher-order accurate and conservative numerical schemes have developed [15,20,34,35]. In particular, Ghiloufi and Omrani [35] developed fourth-order accurate and conservative finite difference schemes for the generalized Rosenau–KdV equations. However, these schemes were developed only for 1D cases. The motivation of this research is to develop a fourth-order accurate, conservative, and stable finite difference scheme for the initial–boundary value problem of the generalized Rosenau–KdV equation in both 1D and 2D, in order to provide more accurate solutions.

The rest of this article are organized as follows: Section 2 gives the detailed description of the three-level linearly implicit finite difference method and its numerical analysis for 1D generalized Rosenau–KdV equation. Section 3 extends the numerical method and theoretical analysis to the 2D generalized Rosenau–KdV equation case. In Section 4, we give some numerical simulations to test the accuracy of the obtained schemes and verify our theoretical analysis. Finally, we wrap up our paper by concluding remarks in Section 5.

2. Numerical method and analysis in 1D

2.1. Fourth-order finite difference scheme and its conservation

In this section, we propose a fourth-order accurate and conservative stable finite difference scheme for the initial–boundary value problem in Eqs. (1)–(3). To this end, we first describe our solution domain and its grids. The solution domain is defined to be $\{(x,t)|\alpha \le x \le \beta, 0 \le t \le T\}$. Let $h = (\beta-\alpha)/J$ and $\tau = T/N$ be the uniform step sizes in the spatial and temporal directions, respectively. Denote $x_j = \alpha + jh$, $t_n = n\tau$, $0 \le j \le J$, $0 \le n \le N$, $u_j^n \approx u(x_j, t_n)$, and $Z_h^0 = \{u = (u_j)|u_{-1} = u_0 = u_J = u_{J+1} = 0, -1 \le j \le J+1\}$. We use the following notations, the inner product, l_2-norm and l_∞-norm as

$$(u_j^n)_{\tilde{x}} = \frac{1}{h}(u_{j+1}^n - u_j^n),\ \ (u_j^n)_{\bar{x}} = \frac{1}{h}(u_j^n - u_{j-1}^n),\ \ (u_j^n)_{\hat{x}} = \frac{1}{2h}(u_{j+1}^n - u_{j-1}^n),$$
$$u_j^{n+\frac{1}{2}} = \frac{1}{2}(u_j^{n+1} + u_j^n),\ \ (u_j^n)_{\hat{t}} = \frac{1}{2\tau}(u_j^{n+1} - u_j^{n-1}),\ \ \bar{u}_j^n = \frac{1}{2}(u_j^{n+1} + u_j^{n-1}),$$
$$\langle u^n, v^n\rangle = h\sum_{j=1}^{J-1} u_j^n v_j^n,\ \ \|u^n\|^2 = \langle u^n, u^n\rangle,\ \ \|u^n\|_\infty = \max_{1\le j\le J-1}|u_j^n|.$$

For the third-order derivative u_{xxx} ($u^{(3)}$ for simplicity) in Eq. (1), we design a finite difference operator as follows

$$\begin{aligned}(u_j)_{\widehat{\hat{x}\hat{x}\hat{x}}} &= a_1(u_{j+3} - u_{j-3}) + a_2(u_{j+2} - u_{j-2}) + a_3(u_{j+1} - u_{j-1})\\ &= u_j^{(3)} + \frac{h^2}{6}u_j^{(5)} + O(h^4),\end{aligned} \tag{4}$$

where a_1, a_2, a_3 are constants to be determined. Expanding each term in Eq. (4) using a Taylor series at x_j and then matching both sides, we obtain a system of equations for solving a_1, a_2, a_3 as follows:

$$3a_1 + 2a_2 + a_3 = 0,\ \ \frac{h^3}{3!}\left(3^3a_1 + 2^3a_2 + a_3\right) = \frac{1}{2},\ \ \frac{h^3}{5!}\left(3^5a_1 + 2^5a_2 + a_3\right) = \frac{1}{12},$$

which gives

$$a_1 = -\frac{1}{24h^3},\ \ a_2 = \frac{2}{3h^3},\ \ a_5 = -\frac{29}{24h^3}. \tag{5}$$

Substituting Eq. (5) into Eq. (4) gives the following fourth-order approximation for $u_j^{(3)} + \frac{h^2}{6}u_j^{(5)}$ as

$$(u_j^n)_{\widehat{\hat{x}\hat{x}\hat{x}}} = \frac{1}{24h^3}[u_{j-3}^n - u_{j+3}^n - 16(u_{j-2}^n - u_{j+2}^n) + 29(u_{j-1}^n - u_{j+1}^n)]. \tag{6}$$

By setting

$$w = -u_x - u_{xxx} - u_{xxxxt} - (u^p)_x, \tag{7}$$

Eq. (1) can be written as $w = u_t$. Using the Taylor series expansion at (x_j, t_n), we obtain

$$\begin{aligned} w_j^n = &-\left[(u_j^n)_{\hat{x}} - \frac{h^2}{6}(\partial_x^3 u)_j^n\right] - \left[(u_j^n)_{\widehat{\bar{x}x\hat{x}}} - \frac{h^2}{6}(\partial_x^5 u)_j^n\right] - \left[[(u_j^n)^p]_{\hat{x}} - \frac{h^2}{6}(\partial_x^3 u^p)_j^n\right] \\ &- \left[(u_j^n)_{\bar{x}x\bar{x}x\hat{t}} - \frac{h^2}{6}(\partial_x^6 \partial_t u)_j^n\right] + O(h^4). \end{aligned} \tag{8}$$

By taking the second-order derivative with respect to x to Eq. (7), we obtain

$$(\partial_x^6 \partial_t u)_j^n = -(\partial_x^3 u)_j^n - (\partial_x^5 u)_j^n - (\partial_x^3 u^p)_j^n - (\partial_x^2 w)_j^n. \tag{9}$$

Substituting Eq. (9) into Eq. (8) gives

$$w_j^n = -(u_j^n)_{\hat{x}} - (u_j^n)_{\widehat{\bar{x}x\hat{x}}} - [(u_j^n)^p]_{\hat{x}} - (u_j^n)_{\bar{x}x\bar{x}x\hat{t}} - \frac{h^2}{6}(\partial_x^2 w)_j^n + O(h^4).$$

Using the second-order accurate approximations as follows:

$$w_j^n = (\partial_t u)_j^n = (u_j^n)_{\hat{t}} + O(\tau^2), \;\; (\partial_x^2 w)_j^n = (w_j^n)_{\bar{x}x} + O(h^2), \;\; \bar{u}_j^n = u_j^n + O(\tau^2),$$

we then obtain a fourth-order accurate finite difference scheme for solving the problem in Eqs. (1)–(3) as follows

$$(u_j^n)_{\hat{t}} + \frac{h^2}{6}(u_j^n)_{\bar{x}x\hat{t}} + (u_j^n)_{\hat{x}} + (\bar{u}_j^n)_{\widehat{\bar{x}x\hat{x}}} + (u_j^n)_{\bar{x}x\bar{x}x\hat{t}} + [(u_j^n)^p]_{\hat{x}} = 0, \tag{10}$$

where $2 \le j \le J-2,\; 2 \le n \le N$, and the discrete initial–boundary value conditions are

$$u_j^0 = u_0(x_j), \quad 0 \le j \le J, \tag{11}$$

$$u_0^n = u_J^n = 0,\; u_{-1}^n = u_1^n = 0,\; u_{J-1}^n = u_{J+1}^n = 0,\; 1 \le n \le N. \tag{12}$$

It should be pointed out that since $u_0^n = 0$ and $(u_x)_0^n = 0$ based on Eq. (3), we may assume $u_{-1}^n = u_1^n = 0$ for simplicity. Similarly, we assume $u_{J-1}^n = u_J^n = u_{J+1}^n = 0$, where $j = -1$ and $J+1$ are ghost points, $1 \le n \le N$. Since a three-level in time method is used for the time discretization in the above obtained scheme, we employ a two-level in time method to estimate the solution u^1 by

$$(u_j^0)_{\tilde{t}} + \frac{h^2}{6}(u_j^0)_{\bar{x}x\tilde{t}} + (u_j^{0.5})_{\hat{x}} + (u_j^{0.5})_{\widehat{\bar{x}x\hat{x}}} + (u_j^0)_{\bar{x}x\bar{x}x\tilde{t}} + (u_j^0)_{\hat{x}}^p = 0, \tag{13}$$

where $(u_j^0)_{\tilde{t}} = (u_j^1 - u_j^0)/\tau$, $u_j^{0.5} = (u_j^0 + u_j^1)/2$, $j = 2, \dots, J-2$. The matrix system of the scheme in Eqs. (10)–(13) is symmetric and diagonal dominated, which can be solved effectively by the Thomas algorithm. The nonlinear term of Eq. (1) has been handled by using the linear implicit approximation. Therefore, the algebraic system of equations is solved easily by using the presented method since it does not require extra effort to deal with the nonlinear term in the Rosenau–KdV equation.

Furthermore, it should be pointed out that in [35], a three-level linearized and fourth-order finite difference scheme was presented. Although there are some similarities, the derivatives are different. In particular, their scheme was obtained based on taking a weighted average on each derivative term and then discretizing them. Our method discretizes the differential equation as a whole without any weighted average. As such, our scheme looks much concise as compared with Scheme D in [35].

The following lemmas and theorems give some properties of the above finite difference scheme in Eqs. (10)–(13).

Lemma 2.1 (*See [15,32,34,36]*). *For any two mesh functions, $u, v \in Z_h^0$, we have*

$$\langle u_{\bar{x}}, v\rangle = -\langle u, v_{\bar{x}}\rangle, \;\; \langle u_{\hat{x}}, v\rangle = -\langle u, v_{\hat{x}}\rangle,$$

$$\langle u_{\bar{x}x}, v\rangle = -\langle u_{\bar{x}}, v_{\bar{x}}\rangle, \;\; \langle u_{\bar{x}x}, u\rangle = -\|u_{\bar{x}}\|^2.$$

Furthermore, if $(u_0^n)_{\bar{x}x} = (u_J^n)_{\bar{x}x} = 0$, then $\langle u, u_{\bar{x}x\bar{x}x}\rangle = \|u_{\bar{x}x}\|^2$.

Theorem 2.2. *Suppose $u_0(x) \in H_0^2([\alpha, \beta])$, then the scheme in Eqs. (10)–(13) is conservative for discrete mass under the assumption $u \in Z_h^0$; that is, $Q^n = Q^{n-1} = \cdots = Q^0$, where*

$$Q^n = \frac{h}{2}\sum_{j=1}^{J-1}(u_j^{n+1} + u_j^n).$$

Proof. By multiplying Eq. (10) by h, summing up for j from 1 to $J-1$, we obtain

$$\sum_{j=1}^{J-1}\Big[(u_j^n)_{\hat t}+\frac{h^2}{6}(u_j^n)_{\tilde x\bar x\hat t}+(u_j^n)_{\hat x}+(\bar u_j^n)_{\widehat{\dot x\dot x\dot x}}+(u_j^n)_{\tilde x\tilde x\bar x\bar x\hat t}+(u_j^n)^p_{\hat x}\Big]h=0.$$

Based on the discrete boundary conditions in Eq. (12) together with Lemma 2.1, we obtain

$$\frac{h}{2}\sum_{j=1}^{J-1}(u_j^{n+1}-u_j^{n-1})=0,$$

and hence

$$\sum_{j=1}^{J-1}u_j^{n+1}=\sum_{j=1}^{J-1}u_j^{n-1}.$$

Thus, this gives

$$Q^n=\frac{h}{2}\sum_{j=1}^{J-1}(u_j^{n+1}+u_j^n)=\frac{h}{2}\sum_{j=1}^{J-1}(u_j^{n-1}+u_j^n)=Q^{n-1}=\cdots=Q^0,$$

hence, we complete the proof.

Lemma 2.3 (*Discrete Sobolev's Inequality [37]*)**.** *For any discrete function u_j^n on the finite interval $[\alpha,\beta]$, $j=0,1,2,\ldots,J$, $n=0,1,2,\ldots,N$, there exist two positive constants C_1 and C_2 such that*

$$\|u^n\|_\infty\le C_1\|u^n\|+C_2\|u^n_{\tilde x}\|.$$

Lemma 2.4 (*See [28]*)**.** *For any mesh function $u\in Z_h^0$, we have $\langle u_{\hat x},u\rangle=0$.*

Lemma 2.5. *For any mesh function $u\in Z_h^0$, we have $\langle u_{\widehat{\dot x\dot x\dot x}},u\rangle=0$.*

Proof. For any mesh function $u\in Z_h^0$, we obtain

$$\sum_{j=1}^{J-1}(u_{j-1}^n-u_{j+1}^n)u_j^n=\sum_{j=1}^{J-1}u_j^nu_{j-1}^n-\sum_{j=2}^{J}u_j^nu_{j-1}^n=0, \tag{14}$$

$$\sum_{j=1}^{J-1}(u_{j-2}^n-u_{j+2}^n)u_j^n=\sum_{j=1}^{J-1}u_j^nu_{j-2}^n-\sum_{j=3}^{J+1}u_j^nu_{j-2}^n=0, \tag{15}$$

$$\sum_{j=1}^{J-1}(u_{j-3}^n-u_{j+3}^n)u_j^n=\sum_{j=1}^{J-1}u_j^nu_{j-3}^n-\sum_{j=4}^{J+2}u_j^nu_{j-3}^n=0. \tag{16}$$

Thus, from Eqs. (14)–(16) and the definition of $u_{\widehat{\dot x\dot x\dot x}}$, we obtain $\langle u_{\widehat{\dot x\dot x\dot x}},u\rangle=0$. This completes the proof.

Lemma 2.6 (*See [15]*)**.** *For any discrete function $u_j^n\in Z_h^0$, we have*

$$\|u^n_{\hat x}\|\le\|u^n_{\tilde x}\|,\ \ \|u^n_{\tilde x}\|^2\le\frac{4}{h^2}\|u^n\|^2,\ \ j=0,1,2,\ldots,J,\ \ n=0,1,2,\ldots,N.$$

Theorem 2.7. *Suppose $u_0(x)\in H_0^2([\alpha,\beta])$, then the solution u^n of the scheme in Eqs. (10)–(13) satisfies $\|u^n\|\le C$, $\|u^n_{\tilde x}\|\le C$, $\|u^n_{\tilde x\bar x}\|\le C$, which yield $\|u^n\|_\infty\le C$, $\|u^n_{\tilde x}\|_\infty\le C$, $\|u^n_{\bar x}\|_\infty\le C$, $\|u^n_{\hat x}\|_\infty\le C$, $n=1,2,3,\ldots,N$.*

Proof. We use the mathematical induction method to prove it. It follows from the discrete initial condition in Eq. (11) that $\|u^0\|\le C$ and $\|u^0\|_\infty\le C$. The first level $\{u_j^1\}$ $(j=0,1,2,\ldots,J)$ is computed by the scheme in Eq. (13). Taking the inner product of Eq. (13) with $2u^{0.5}$ (i.e., u^1+u^0), we obtain

$$\langle (u^0)_{\tilde t}+\frac{h^2}{6}(u^0)_{\tilde x\bar x\tilde t}+(u^{0.5})_{\hat x}+(u^{0.5})_{\widehat{\dot x\dot x\dot x}}+(u^0)_{\tilde x\tilde x\bar x\bar x\tilde t}+(u^0)^p_{\hat x},2u^{0.5}\rangle=0. \tag{17}$$

By Lemma 2.1, we obtain

$$\langle (u^0)_{\tilde t},2u^{0.5}\rangle=\|u^0\|^2_{\tilde t},\ \ \langle (u^0)_{\tilde x\bar x\tilde t},2u^{0.5}\rangle=-\|u^0_{\tilde x}\|^2_{\tilde t},\ \ \langle (u^0)_{\tilde x\tilde x\bar x\bar x\tilde t},2u^{0.5}\rangle=\|u^0_{\tilde x\bar x}\|^2_{\tilde t}. \tag{18}$$

By Lemmas 2.4 and 2.5, we obtain

$$\langle (u^{0.5})_{\hat{x}}, 2u^{0.5}\rangle = 0,\quad \langle (u^{0.5})_{\widehat{\bar{x}\bar{x}\hat{x}}}, 2u^{0.5}\rangle = 0. \tag{19}$$

Substituting Eqs. (18)–(19) into Eq. (17) gives

$$\|u^0\|_{\hat{t}}^2 - \frac{h^2}{6}\|u_{\bar{x}}^0\|_{\hat{t}}^2 + \|u_{\bar{x}\bar{x}}^0\|_{\hat{t}}^2 + \langle (u^0)_{\hat{x}}^p, 2u^{0.5}\rangle = 0. \tag{20}$$

According to the Cauchy–Schwarz inequality [15] and Lemma 2.1, we obtain

$$\langle (u^0)_{\hat{x}}^p, 2u^{0.5}\rangle = -h\sum_{j=1}^{J-1}\Big[(u_j^0)^p(u_j^1+u_j^0)_{\hat{x}}\Big] \le C(\|u^0\|^2 + \frac{1}{2}\|u_{\bar{x}}^1\|^2 + \frac{1}{2}\|u_{\bar{x}}^0\|^2). \tag{21}$$

Denoting

$$E^0 \equiv \|u^0\|^2 - \frac{h^2}{6}\|u_{\bar{x}}^0\|^2 + \|u_{\bar{x}\bar{x}}^0\|^2,\quad E^1 \equiv \|u^1\|^2 - \frac{h^2}{6}\|u_{\bar{x}}^1\|^2 + \|u_{\bar{x}\bar{x}}^1\|^2, \tag{22}$$

we obtain from Eqs. (20)–(22) that

$$E^1 - E^0 \le C\tau(E^1 + E^0). \tag{23}$$

Thus, if τ is sufficiently small such that $\tau \le \frac{k-2}{Ck}$ when $k > 2$, then we have

$$E^1 \le \frac{1+C\tau}{1-C\tau}E^0 \le (1+Ck\tau)E^0 \le \exp(kCT)E^0. \tag{24}$$

Furthermore, from Lemma 2.6, we can obtain

$$\|u^1\|^2 - \frac{h^2}{6}\|u_{\bar{x}}^1\|^2 \ge \|u^1\|^2 - \frac{h^2}{6}\frac{4}{h^2}\|u^1\|^2 \ge \frac{1}{3}\|u^1\|^2 \ge 0, \tag{25}$$

implying that

$$E^1 \ge \frac{1}{3}\|u^1\|^2 + \|u_{\bar{x}\bar{x}}^1\|^2 \ge 0. \tag{26}$$

From Eqs. (25)–(26), we obtain $\|u^1\| \le C$, $\|u_{\bar{x}\bar{x}}^1\| \le C$. By Lemma 2.1 and the Schwartz inequality [15], we obtain

$$\|u_{\bar{x}}^1\|^2 \le \|u^1\|\|u_{\bar{x}\bar{x}}^1\| \le \frac{1}{2}(\|u^1\|^2 + \|u_{\bar{x}\bar{x}}^1\|^2) \le C,$$

and hence $\|u^1\|_\infty \le C$ according to Lemma 2.3.

We now assume that $\|u^k\| \le C$, $\|u^k\|_\infty \le C$, $k = 0, 1, 2, \ldots, n$. Taking the inner product of Eq. (10) with $2\bar{u}^n$ (i.e. $u^{n+1} + u^{n-1}$), we obtain

$$\langle (u^n)_{\hat{t}} + \frac{h^2}{6}(u^n)_{\bar{x}x\hat{t}} + (u^n)_{\hat{x}} + (\bar{u}^n)_{\widehat{\bar{x}\bar{x}\hat{x}}} + (u^n)_{\bar{x}\bar{x}xx\hat{t}} + (u^n)_{\hat{x}}^p, 2\bar{u}^n\rangle = 0. \tag{27}$$

By Lemma 2.1, we obtain

$$\langle (u^n)_{\hat{t}}, 2\bar{u}^n\rangle = \|u^n\|_{\hat{t}}^2,\quad \langle (u^n)_{\bar{x}x\hat{t}}, 2\bar{u}^n\rangle = -\|u_{\bar{x}}^n\|_{\hat{t}}^2,\quad \langle (u^n)_{\bar{x}\bar{x}xx\hat{t}}, 2\bar{u}^n\rangle = \|u_{\bar{x}\bar{x}}^n\|_{\hat{t}}^2. \tag{28}$$

Substituting Eq. (28) into Eq. (27) gives

$$\begin{aligned}&\|u^{n+1}\|^2 - \|u^{n-1}\|^2 - \frac{h^2}{6}(\|u_{\bar{x}}^{n+1}\|^2 - \|u_{\bar{x}}^{n-1}\|^2) + \|u_{\bar{x}\bar{x}}^{n+1}\|^2 - \|u_{\bar{x}\bar{x}}^{n-1}\|^2\\ &= -2\tau[\langle (u^n)_{\hat{x}}, 2\bar{u}^n\rangle + \langle (\bar{u}^n)_{\widehat{\bar{x}\bar{x}\hat{x}}}, 2\bar{u}^n\rangle + \langle (u^n)_{\hat{x}}^p, 2\bar{u}^n\rangle].\end{aligned} \tag{29}$$

According to the Cauchy–Schwarz inequality [15], Lemmas 2.1, 2.5, 2.6 and using some direct calculations, we obtain

$$\langle (\bar{u}^n)_{\widehat{\bar{x}\bar{x}\hat{x}}}, 2\bar{u}^n\rangle = 0, \tag{30}$$

$$\langle (u^n)_{\hat{x}}, 2\bar{u}^n\rangle \le \|u_{\bar{x}}^n\|^2 + \frac{1}{2}(\|u^{n+1}\|^2 + \|u^{n-1}\|^2), \tag{31}$$

$$\begin{aligned}\langle [(u^n)^p]_{\hat{x}}, 2\bar{u}^n\rangle &= -h\sum_{j=1}^{J-1}\Big[(u_j^n)^p\cdot(u_j^{n+1}+u_j^{n-1})_{\hat{x}}\Big]\\ &\le C(\|u^n\|^2 + \frac{1}{2}\|u_{\bar{x}}^{n+1}\|^2 + \frac{1}{2}\|u_{\bar{x}}^{n-1}\|^2).\end{aligned} \tag{32}$$

Denoting

$$E^n \equiv \|u^{n+1}\|^2 + \|u^n\|^2 - \frac{h^2}{6}(\|u_{\tilde{x}}^{n+1}\|^2 + \|u_{\tilde{x}}^n\|^2) + \|u_{\bar{x}x}^{n+1}\|^2 + \|u_{\bar{x}x}^n\|^2,\ n > 1, \tag{33}$$

we obtain from Eqs. (29)–(33) that

$$E^n - E^{n-1} \leq C\tau(E^n + E^{n-1}),\ n > 1.$$

Thus, if τ is sufficiently small such that $\tau \leq \frac{k-2}{Ck}$ when $k > 2$, then we have

$$E^n \leq \frac{1 + C\tau}{1 - C\tau}E^{n-1} \leq (1 + Ck\tau)E^{n-1} \leq (1 + Ck\tau)^n E^0 \leq \exp(kCT)E^0. \tag{34}$$

Similarly, from Lemma 2.6, we can obtain

$$\|u^n\|^2 - \frac{h^2}{6}\|u_{\tilde{x}}^n\|^2 \geq \|u^n\|^2 - \frac{h^2}{6}\frac{4}{h^2}\|u^n\|^2 \geq \frac{1}{3}\|u^n\|^2 \geq 0,\ n > 1,$$

implying that

$$E^n \geq \frac{1}{3}\|u^{n+1}\|^2 + \frac{1}{3}\|u^n\|^2 + \|u_{\bar{x}x}^{n+1}\|^2 + \|u_{\bar{x}x}^n\|^2 \geq 0,\ n > 1. \tag{35}$$

From Eqs. (34)–(35), we obtain $\|u^{n+1}\| \leq C$, $\|u_{\bar{x}x}^{n+1}\| \leq C, n > 1$. By Lemma 2.1 and the Schwartz inequality, we obtain

$$\|u_{\tilde{x}}^{n+1}\|^2 \leq \|u^{n+1}\|\|u_{\bar{x}x}^{n+1}\| \leq \frac{1}{2}(\|u^{n+1}\|^2 + \|u_{\bar{x}x}^{n+1}\|^2) \leq C,\ n > 1,$$

and hence $\|u^{n+1}\|_\infty \leq C$ according to Lemma 2.3. This completes the proof.

2.2. Solvability, convergence and stability

Theorem 2.8. *The finite difference scheme in Eqs. (10)–(13) is uniquely solvable.*

Proof. We use the mathematical induction to prove it. First, we can determine u^0 uniquely by the discrete initial condition in Eq. (11) and then choose a fourth-order method, such as Eq. (13), to compute u^1. Suppose that $u^0, u^1, u^2, \ldots, u^n$ can be solved uniquely. We consider the homogeneous form of Eq. (10) for u^{n+1} as

$$\frac{1}{2\tau}u_j^{n+1} + \frac{h^2}{6}\frac{1}{2\tau}(u_j^{n+1})_{\bar{x}x} + \frac{1}{2\tau}(u_j^{n+1})_{\bar{x}x\bar{x}x} + \frac{1}{2}(u_j^{n+1})_{\widehat{\bar{x}x}} = 0. \tag{36}$$

Taking the inner product of Eq. (36) with u^{n+1}, we obtain from the discrete boundary condition in Eq. (12) and Lemmas 2.1, 2.5 that

$$\|u^{n+1}\|^2 - \frac{h^2}{6}\|u_{\tilde{x}}^{n+1}\|^2 + \|u_{\bar{x}x}^{n+1}\|^2 = 0. \tag{37}$$

Based on the Cauchy–Schwarz inequality [15] and Eq. (37), we obtain

$$\|u_{\tilde{x}}^{n+1}\|^2 \leq \frac{1}{2}(\|u^{n+1}\|^2 + \|u_{\bar{x}x}^{n+1}\|^2) = \frac{h^2}{12}\|u_{\tilde{x}}^{n+1}\|^2, \tag{38}$$

implying that $\|u_{\tilde{x}}^{n+1}\|^2 = 0$ when h is small and $\|u^{n+1}\|^2 + \|u_{\bar{x}x}^{n+1}\|^2 = 0$. By Lemma 2.3, we obtain $\|u^{n+1}\|_\infty = 0$. Hence, there uniquely admits a zero solution satisfying the scheme in Eqs. (10)–(13). Therefore, u^{n+1} is uniquely solvable, and the proof completes.

Lemma 2.9 (*Discrete Gronwall's Inequality [15,35]*)**.** *Suppose that $w(k)$ and $\rho(k)$ are nonnegative functions and $\rho(k)$ is nondecreasing. If*

$$w(k) \leq \rho(k) + C\tau\sum_{l=0}^{k-1} w(l), C > 0,$$

then $w(k) \leq \rho(k)e^{C\tau k}$.

Theorem 2.10. *Supposing $u_0(x) \in H_0^2([\alpha, \beta])$, then the solution u^n of the scheme in Eqs. (10)–(13) converges to the solution of the problem in Eqs. (1)–(3) with the convergence rate of $O(\tau^2 + h^4)$ in both l_2-norm and l_∞-norm.*

Proof. Let $e_j^n = v_j^n - u_j^n$, where v_j^n and u_j^n are the solutions of the problem in Eqs. (1)–(3) and the scheme in Eqs. (10)–(13), respectively. Then, we obtain the following error equation

$$R_j^n = (e_j^n)_{\hat{t}} + \frac{h^2}{6}(e_j^n)_{\bar{x}x\hat{t}} + (e_j^n)_{\hat{x}} + (\bar{e}_j^n)_{\widehat{\bar{x}x\hat{x}}} + (e_j^n)_{\bar{x}x\bar{x}x\hat{t}} + [(v_j^n)^p - (u_j^n)^p]_{\hat{x}}, \tag{39}$$

and $e_{-1}^n = e_0^n = e_1^n = 0$, $e_{J-1}^n = e_J^n = e_{J+1}^n = 0$, where $\bar{e}_j^n = (e_j^{n+1} + e_j^{n-1})/2$. Taking the inner product of Eq. (39) with $2\bar{e}^n$ (i.e., $e^{n+1} + e^{n-1}$), we have

$$\begin{aligned}
&\frac{1}{2\tau}\left\{\|e^{n+1}\|^2 - \|e^{n-1}\|^2 - \frac{h^2}{6}[\|e_{\bar{x}}^{n+1}\|^2 - \|e_{\bar{x}}^{n-1}\|^2] + \|e_{\bar{x}x}^{n+1}\|^2 - \|e_{\bar{x}x}^{n-1}\|^2\right\} \\
&= \langle R^n, 2\bar{e}^n\rangle - \langle (e^n)_{\hat{x}}, 2\bar{e}^n\rangle - \langle (\bar{e}^n)_{\widehat{\bar{x}x\hat{x}}}, 2\bar{e}^n\rangle + \langle [(u^n)^p - (v^n)^p]_{\hat{x}}, 2\bar{e}^n\rangle.
\end{aligned} \tag{40}$$

By Lemma 2.1 and Theorem 2.7, we have

$$\begin{aligned}
&\langle [(u^n)^p - (v^n)^p]_{\hat{x}}, 2\bar{e}^n\rangle \\
&= -h\sum_{j=1}^{J-1}\left\{\left[(u_j^n)^p - (v_j^n)^p\right]\cdot 2(\bar{e}_j^n)_{\hat{x}}\right\} \\
&= -h\sum_{j=1}^{J-1}\left\{\sum_{k=1}^{p-1}\left[(u_j^n)^{p-k}(v_j^n)^k(\bar{e}_j^n)\right]\cdot 2(\bar{e}_j^n)_{\hat{x}}\right\} \\
&\le C(\|e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e_{\bar{x}}^{n-1}\|^2 + \|e_{\bar{x}}^{n+1}\|^2).
\end{aligned} \tag{41}$$

By the Cauchy–Schwarz inequality [15], Lemmas 2.1 and 2.5, we obtain

$$\|e_{\hat{x}}^n\|^2 \le \|e_{\bar{x}}^n\|^2 = -\langle e^n, e_{\bar{x}x}^n\rangle \le \frac{1}{2}(\|e^n\|^2 + \|e_{\bar{x}x}^n\|^2), \tag{42}$$

$$\langle e_{\hat{x}}^n, 2\bar{e}^n\rangle \le \|e_{\hat{x}}^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2), \tag{43}$$

$$\langle \bar{e}_{\widehat{\bar{x}x\hat{x}}}^n, 2\bar{e}^n\rangle = 0, \tag{44}$$

$$\langle R^n, 2\bar{e}^n\rangle = \langle R^n, e^{n+1} + e^{n-1}\rangle \le \|R^n\|^2 + \frac{1}{2}\Big(\|e^{n+1}\|^2 + \|e^{n-1}\|^2\Big). \tag{45}$$

Substituting Eqs. (41)–(45) into Eq. (40) gives

$$\begin{aligned}
&\|e^{n+1}\|^2 - \|e^{n-1}\|^2 - \frac{h^2}{6}(\|e_{\bar{x}}^{n+1}\|^2 - \|e_{\bar{x}}^{n-1}\|^2) + \|e_{\bar{x}x}^{n+1}\|^2 - \|e_{\bar{x}x}^{n-1}\|^2 \\
&\le 2\tau\|R^n\|^2 + C\tau(\|e_{\bar{x}x}^{n+1}\|^2 + \|e_{\bar{x}x}^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2).
\end{aligned} \tag{46}$$

Denoting

$$\Gamma^n \equiv \|e^{n+1}\|^2 + \|e^n\|^2 - \frac{h^2}{6}(\|e_{\bar{x}}^{n+1}\|^2 + \|e_{\bar{x}}^n\|^2) + \|e_{\bar{x}x}^{n+1}\|^2 + \|e_{\bar{x}x}^n\|^2,$$

then we have from Lemma 2.6 that

$$\Gamma^n \ge \frac{1}{3}\|e^{n+1}\|^2 + \frac{1}{3}\|e^n\|^2 + \|e_{\bar{x}x}^{n+1}\|^2 + \|e_{\bar{x}x}^n\|^2 \ge 0,$$

and Eq. (46) can be simplified to

$$\Gamma^n - \Gamma^{n-1} \le 2\tau\|R^n\|^2 + C\tau(\Gamma^n + \Gamma^{n-1}).$$

Hence, we obtain

$$(1 - C\tau)(\Gamma^n - \Gamma^{n-1}) \le 2\tau\|R^n\|^2 + 2C\tau\Gamma^{n-1}.$$

If τ is sufficiently small such that $1 - C\tau > 1/2$, then

$$\Gamma^n - \Gamma^{n-1} \le C\tau\|R^n\|^2 + C\tau\Gamma^{n-1}. \tag{47}$$

Summarizing Eq. (47) from 1 to n, we obtain

$$\Gamma^n \le \Gamma^0 + C\tau\sum_{l=1}^{n}\|R^l\|^2 + C\tau\sum_{l=1}^{n}\Gamma^{l-1}.$$

Note that

$$\tau\sum_{l=1}^{n}\|R^l\|^2 \le n\tau\max_{1\le l\le n}\|R^l\|^2 \le T\cdot O(\tau^2+h^4)^2,$$

if we use a fourth-order method such as Eq. (13) to compute u^1 such that

$$e^0 = 0,\ \Gamma^0 \le O(\tau^2+h^4),$$

then we have

$$\Gamma^n \le O(\tau^2+h^4)^2 + C\tau\sum_{k=1}^{n}\Gamma^{k-1}.$$

By Lemma 2.9, we obtain that $\Gamma^n \le O(\tau^2+h^4)^2$, implying

$$\|e^{n+1}\| \le O(\tau^2+h^4),\ \ \|e^{n+1}_{\bar{x}\bar{x}}\| \le O(\tau^2+h^4). \tag{48}$$

Furthermore, by Lemma 2.1 and the Cauchy–Schwarz inequality [15], we have

$$\|e^{n+1}_{\bar{x}}\| \le \|e^{n+1}\|^{\frac12}\|e^{n+1}_{\bar{x}\bar{x}}\|^{\frac12} \le \frac{1}{2}\left[\|e^{n+1}\| + \|e^{n+1}_{\bar{x}\bar{x}}\|\right] \le O(\tau^2+h^4),$$

and hence $\|e^{n+1}\|_\infty \le O(\tau^2+h^4)$ according to Lemma 2.3. Therefore, the solution u^n of the scheme in Eqs. (10)–(13) converges to the solution of the problem in Eqs. (1)–(3) in both l_2-norm and l_∞-norm with the convergence rate of $O(\tau^2+h^4)$.

Using a similar argument as the proof for Theorem 2.10, one may obtain Theorem 2.11.

Theorem 2.11. *Supposing $u_0(x) \in H_0^2([\alpha,\beta])$, then the scheme in Eqs. (10)–(13) is unconditionally stable in the sense of l_2-norm and l_∞-norm.*

3. Numerical method and analysis in 2D

In this section, we would like to extend our scheme to a 2D case and consider the 2D generalized Rosenau–KdV equation as

$$u_t + \nabla\cdot u + u_{xxx} + u_{yyy} + u_{xxxxt} + u_{yyyyt} + \nabla\cdot u^p = 0,\ \ (x,y,t)\in\Omega\times[0,T], \tag{49}$$

subject to the initial condition

$$u(x,y,0) = u_0(x,y),\ \ (x,y)\in\Omega, \tag{50}$$

and boundary conditions

$$u(\alpha,y,t) = u(\beta,y,t) = 0,\ \ u(x,\alpha,t) = u(x,\beta,t) = 0, \tag{51}$$

$$u_x(\alpha,y,t) = u_x(\beta,y,t) = 0,\ \ u_y(x,\alpha,t) = u_y(x,\beta,t) = 0,\ \ t\in[0,T], \tag{52}$$

where $\nabla\cdot u = \partial u/\partial x + \partial u/\partial y$, $\Omega = [\alpha,\beta]\times[\alpha,\beta]$, $p\ge 2$ is a positive integer, $u_0(x,y)$ is a given smooth function.

Here, the solution domain is defined as $\{(x,y,t)|\alpha\le x\le\beta, \alpha\le y\le\beta, 0\le t\le T\}$, which is covered by a uniform mesh

$$\{(x_i,y_j,t_n)|x_i = ih,\ y_j = jh,\ t_n = n\tau,\ 0\le i\le J,\ 0\le j\le J,\ 0\le n\le N\},$$

with spacing $h = (\beta-\alpha)/J,\ \tau = T/N$. We denote $u^n_{i,j}$ to be the numerical approximation of $u(x_i,y_j,t_n)$ and

$$Z^0_{h,h} = \{u = (u_{i,j})|u_{-1,j} = u_{0,j} = u_{J,j} = u_{J+1,j} = 0,\ u_{i,-1} = u_{i,0} = u_{i,J} = u_{i,J+1} = 0\},$$

where $-1\le i,j\le J+1$. Similar to the 1D case, the difference operators are defined as follows:

$$(u^n_{i,j})_{\tilde{x}} = \frac{u^n_{i+1,j}-u^n_{i,j}}{h},\ \ (u^n_{i,j})_{\bar{x}} = \frac{u^n_{i,j}-u^n_{i-1,j}}{h},\ \ (u^n_{i,j})_{\hat{x}} = \frac{u^n_{i+1,j}-u^n_{i-1,j}}{2h},$$

$$(u^n_{i,j})_{\tilde{y}} = \frac{u^n_{i,j+1}-u^n_{i,j}}{h},\ \ (u^n_{i,j})_{\bar{y}} = \frac{u^n_{i,j}-u^n_{i,j-1}}{h},\ \ (u^n_{i,j})_{\hat{y}} = \frac{u^n_{i,j+1}-u^n_{i,j-1}}{2h},$$

$$(u^n_{i,j})_{\hat{t}} = \frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2\tau},\ \ (u^n_{i,j})_{\tilde{t}} = \frac{u^{n+1}_{i,j}-u^n_{i,j}}{\tau},\ \ \bar{u}^n_{i,j} = \frac{1}{2}(u^{n+1}_{i,j}+u^{n-1}_{i,j}),$$

$$\nabla_h u^n_{i,j} = (u^n_{i,j})_{\hat{x}} + (u^n_{i,j})_{\hat{y}}, \quad \Delta_h u^n_{i,j} = (u^n_{i,j})_{\tilde{x}\bar{x}} + (u^n_{i,j})_{\tilde{y}\bar{y}}.$$

Furthermore, for $u, v \in Z^0_{h,h}$, the inner product and Sobolev norms (or seminorms) are defined as

$$\langle u, v\rangle = h^2 \sum_{i,j=1}^{J} u_{i,j} v_{i,j}, \quad \|u\| = \sqrt{\langle u, u\rangle},$$

$$|u|_1 = \sqrt{\|u_{\tilde{x}}\|^2 + \|u_{\tilde{y}}\|^2}, \quad \|u\|_\infty = \max_{1\le i,j\le J} |u_{i,j}|,$$

$$\|\nabla_h u\| = \sqrt{h^2 \sum_{i,j=1}^{J} \Big(|(u_{i,j})_{\hat{x}}|^2 + |(u_{i,j})_{\hat{y}}|^2\Big)}, \quad \|\Delta_h u\| = \sqrt{h^2 \sum_{i,j=1}^{J} |\Delta_h u_{i,j}|^2}.$$

Using the result in the 1D case, the third-order derivatives u_{xxx} and u_{yyy} on the left-hand side of Eq. (49) are approximated by

$$(u^n_{i,j})_{\widehat{\dot{x}\dot{x}\dot{x}}} = \frac{1}{24h^3}[u^n_{i-3,j} - u^n_{i+3,j} - 16(u^n_{i-2,j} - u^n_{i+2,j}) + 29(u^n_{i-1,j} - u^n_{i+1,j})],$$

$$(u^n_{i,j})_{\widehat{\dot{y}\dot{y}\dot{y}}} = \frac{1}{24h^3}[u^n_{i,j-3} - u^n_{i,j+3} - 16(u^n_{i,j-2} - u^n_{i,j+2}) + 29(u^n_{i,j-1} - u^n_{i,j+1})],$$

with fourth-order truncation errors at $(x_i = ih, y_j = jh, t_n = n\tau)$.

Thus, the extended fourth-order accurate finite difference scheme for 2D generalized Rosenau–KdV equation can be written as

$$\begin{aligned}
&(u^n_{i,j})_{\hat{t}} + \frac{h^2}{6}\Delta_h(u^n_{i,j})_{\hat{t}} + \nabla_h u^n_{i,j} + [(\bar{u}^n_{i,j})_{\widehat{\dot{x}\dot{x}\dot{x}}} + (\bar{u}^n_{i,j})_{\widehat{\dot{y}\dot{y}\dot{y}}}] + \nabla_h(u^n_{i,j})^p \\
&+\frac{h^2}{6}[(u^n_{i,j})_{\tilde{x}\bar{x}\hat{y}} + (u^n_{i,j})_{\hat{x}\tilde{y}\bar{y}} + (u^n_{i,j})^p_{\tilde{x}\bar{x}\hat{y}} + (u^n_{i,j})^p_{\hat{x}\tilde{y}\bar{y}}] \\
&+[(u^n_{i,j})_{\tilde{x}\tilde{x}\bar{x}\bar{x}\hat{t}} + (u^n_{i,j})_{\tilde{y}\tilde{y}\bar{y}\bar{y}\hat{t}}] + \frac{h^2}{6}[(u^n_{i,j})_{\tilde{x}\bar{x}\tilde{y}\tilde{y}\bar{y}\bar{y}\hat{t}} + (u^n_{i,j})_{\tilde{x}\tilde{x}\bar{x}\bar{x}\tilde{y}\bar{y}\hat{t}}] = 0,
\end{aligned} \tag{53}$$

where $p \ge 2$, $3 \le i, j \le J-3$, $2 \le n \le N$, and the discrete initial–boundary value conditions are given as

$$u^0_{i,j} = u_0(x_i, y_j), \quad 0 \le i, j \le J, \tag{54}$$

$$u^n_{0,j} = u^n_{J,j} = u^n_{-1,j} = u^n_{1,j} = u^n_{J-1,j} = u^n_{J+1,j} = 0, \tag{55}$$

$$u^n_{i,0} = u^n_{i,J} = u^n_{i,-1} = u^n_{i,1} = u^n_{i,J-1} = u^n_{i,J+1} = 0, \quad 1 \le n \le N. \tag{56}$$

Since the scheme is a three-level method, to start the computation, a two-level in time method to estimate the solution u^1 is given

$$\begin{aligned}
&(u^0_{i,j})_{\tilde{t}} + \frac{h^2}{6}\Delta_h(u^0_{i,j})_{\tilde{t}} + \nabla_h u^{0.5}_{i,j} + [(u^{0.5}_{i,j})_{\widehat{\dot{x}\dot{x}\dot{x}}} + (u^{0.5}_{i,j})_{\widehat{\dot{y}\dot{y}\dot{y}}}] + \nabla_h(u^0_{i,j})^p \\
&+\frac{h^2}{6}[(u^0_{i,j})_{\tilde{x}\bar{x}\hat{y}} + (u^0_{i,j})_{\hat{x}\tilde{y}\bar{y}} + (u^0_{i,j})^p_{\tilde{x}\bar{x}\hat{y}} + (u^0_{i,j})^p_{\hat{x}\tilde{y}\bar{y}}] + [(u^0_{i,j})_{\tilde{x}\tilde{x}\bar{x}\bar{x}\tilde{t}} + (u^0_{i,j})_{\tilde{y}\tilde{y}\bar{y}\bar{y}\tilde{t}}] \\
&+\frac{h^2}{6}[(u^0_{i,j})_{\tilde{x}\bar{x}\tilde{y}\tilde{y}\bar{y}\bar{y}\tilde{t}} + (u^0_{i,j})_{\tilde{x}\tilde{x}\bar{x}\bar{x}\tilde{y}\bar{y}\tilde{t}}] = 0,
\end{aligned} \tag{57}$$

where $(u^0_{i,j})_{\tilde{t}} = (u^1_{i,j} - u^0_{i,j})/\tau$, $u^{0.5}_{i,j} = (u^0_{i,j} + u^1_{i,j})/2$, $i, j = 2, \ldots, J-2$.

We will analyse the conservation, stability, uniqueness and convergence of the above scheme using a similar method in the previous section.

Lemma 3.1 (*See [38–40]*)**.** *For any two mesh functions $u, v \in Z^0_{h,h}$, we have*

$$\langle \nabla_h u, v\rangle = -\langle u, \nabla_h v\rangle, \quad \langle \Delta_h u, u\rangle = -|u|_1^2.$$

Lemma 3.2. *For any two mesh functions $u, v \in Z^0_{h,h}$, we have $\langle \Delta_h u, v\rangle = \langle u, \Delta_h v\rangle$.*

Theorem 3.3. *For any mesh function $u \in Z^0_{h,h}$, we have*

$$\langle u_{\tilde{x}\bar{x}\tilde{y}\tilde{y}\bar{y}\bar{y}}, u\rangle = -\|u_{\tilde{x}\tilde{y}\bar{y}}\|^2, \quad \langle u_{\tilde{y}\bar{y}\tilde{x}\tilde{x}\bar{x}\bar{x}}, u\rangle = -\|u_{\tilde{y}\tilde{x}\bar{x}}\|^2.$$

Proof. By Lemma 2.1, we obtain

$$\langle u_{\bar{x}\bar{x}\bar{y}\bar{y}\bar{y}\bar{y}}, u\rangle = \langle (u_{\bar{y}\bar{y}\bar{y}\bar{y}})_{\bar{x}\bar{x}}, u\rangle = -\langle (u_{\bar{y}\bar{y}\bar{y}\bar{y}})_{\bar{x}}, u_{\bar{x}}\rangle = -\langle (u_{\bar{x}})_{\bar{y}\bar{y}\bar{y}\bar{y}}, u_{\bar{x}}\rangle = -\|u_{\bar{x}\bar{y}\bar{y}}\|^2,$$

and similarly $\langle u_{\bar{y}\bar{y}\bar{x}\bar{x}\bar{x}\bar{x}}, u\rangle = -\|u_{\bar{y}\bar{x}\bar{x}}\|^2$.

Lemma 3.4 (*See [40,41]*). *For any mesh function $u \in Z^0_{h,h}$, we have*

$$|u|_1 \le \frac{1}{2\sqrt{3}}\|\Delta_h u\|.$$

Lemma 3.5 (*See [41]*). *For any mesh function $u \in Z^0_{h,h}$, we have*

$$\|\nabla_h u\|^2 \le \|u\|\cdot\|\Delta_h u\|,\quad \|u\|^2_\infty \le C\|u\|\cdot(\|\Delta_h u\| + \|u\|).$$

Theorem 3.6. *Suppose that $u_0(x, y) \in H^2_0(\Omega)$. Then the finite difference scheme in Eqs. (53)–(57) is conservative for discrete mass in the sense of $Q^n = Q^{n-1} = \cdots = Q^0$, where*

$$Q^n = \frac{h^2}{2}\sum_{i,j=1}^{J}\Big[(u^{n+1}_{i,j} + u^n_{i,j})\Big]. \tag{58}$$

Theorem 3.7. *Suppose that $u_0 \in H^2_0(\Omega)$. If τ is sufficiently small, then the finite difference scheme in Eqs. (53)–(57) satisfies*

$$|u|_1 \le C,\quad \|u^n\| \le C,\quad \|\nabla_h u^n\| \le C,\quad \|\Delta_h u^n\| \le C,\quad \|u^n\|_\infty \le C,\quad n = 1, 2, 3, \dots, N.$$

Proof. We use the mathematical induction method to prove it similar to the proof for Theorem 2.7. In particular, taking the inner product of Eq. (53) with $2\bar{u}^n$ (i.e., $u^{n+1} + u^{n-1}$), we obtain

$$\begin{aligned}
&\langle (u^n)_{\hat{t}}, 2\bar{u}^n\rangle + \frac{h^2}{6}\langle \Delta_h(u^n)_{\hat{t}}, 2\bar{u}^n\rangle + \langle \nabla_h u^n, 2\bar{u}^n\rangle + \langle (\bar{u}^n)_{\widehat{\bar{x}\bar{x}\bar{x}}} + (\bar{u}^n)_{\widehat{\bar{y}\bar{y}\bar{y}}}, 2\bar{u}^n\rangle\\
&+\langle \nabla_h (u^n)^p, 2\bar{u}^n\rangle + \frac{h^2}{6}\langle (u^n)_{\bar{x}\bar{x}\hat{y}} + (u^n)_{\hat{x}\bar{y}\bar{y}} + (u^n)^p_{\bar{x}\bar{x}\hat{y}} + (u^n)^p_{\hat{x}\bar{y}\bar{y}}, 2\bar{u}^n\rangle\\
&+\langle (u^n)_{\bar{x}\bar{x}\bar{x}\bar{x}\hat{t}} + (u^n)_{\bar{y}\bar{y}\bar{y}\bar{y}\hat{t}}, 2\bar{u}^n\rangle + \frac{h^2}{6}\langle (u^n)_{\bar{x}\bar{x}\bar{y}\bar{y}\bar{y}\bar{y}\hat{t}} + (u^n)_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}\hat{t}}, 2\bar{u}^n\rangle = 0.
\end{aligned} \tag{59}$$

According to Lemmas 2.1 and 3.1, we obtain

$$\langle (u^n)_{\hat{t}}, 2\bar{u}^n\rangle = \|u^n\|^2_{\hat{t}},\quad \langle \Delta_h(u^n)_{\hat{t}}, 2\bar{u}^n\rangle = -(|u^n|_1)^2_{\hat{t}}, \tag{60}$$

$$\langle (u^n)_{\bar{x}\bar{x}\bar{x}\bar{x}\hat{t}}, 2\bar{u}^n\rangle = \|u^n_{\bar{x}\bar{x}}\|^2_{\hat{t}},\quad \langle (u^n)_{\bar{y}\bar{y}\bar{y}\bar{y}\hat{t}}, 2\bar{u}^n\rangle = \|u^n_{\bar{y}\bar{y}}\|^2_{\hat{t}}, \tag{61}$$

$$\langle (u^n)_{\bar{x}\bar{x}\bar{y}\bar{y}\bar{y}\bar{y}\hat{t}}, 2\bar{u}^n\rangle = \|u^n_{\bar{x}\bar{y}\bar{y}}\|^2_{\hat{t}},\quad \langle (u^n)_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}\hat{t}}, 2\bar{u}^n\rangle = \|u^n_{\bar{x}\bar{x}\bar{y}}\|^2_{\hat{t}}. \tag{62}$$

Substituting Eqs. (60)–(62) into Eq. (59) gives

$$\begin{aligned}
&\|u^n\|^2_{\hat{t}} - \frac{h^2}{6}(|u^n|_1)^2_{\hat{t}} + \|u^n_{\bar{x}\bar{x}}\|^2_{\hat{t}} + \|u^n_{\bar{y}\bar{y}}\|^2_{\hat{t}} + \frac{h^2}{6}(\|u^n_{\bar{x}\bar{y}\bar{y}}\|^2_{\hat{t}} + \|u^n_{\bar{x}\bar{x}\bar{y}}\|^2_{\hat{t}})\\
&= -\langle \nabla_h u^n, 2\bar{u}^n\rangle - \langle (\bar{u}^n)_{\widehat{\bar{x}\bar{x}\bar{x}}} + (\bar{u}^n)_{\widehat{\bar{y}\bar{y}\bar{y}}}, 2\bar{u}^n\rangle - \langle \nabla_h(u^n)^p, 2\bar{u}^n\rangle\\
&\quad - \frac{h^2}{6}\langle (u^n)_{\bar{x}\bar{x}\hat{y}} + (u^n)_{\hat{x}\bar{y}\bar{y}} + (u^n)^p_{\bar{x}\bar{x}\hat{y}} + (u^n)^p_{\hat{x}\bar{y}\bar{y}}, 2\bar{u}^n\rangle.
\end{aligned} \tag{63}$$

Based on the Cauchy–Schwarz inequality [15], Lemmas 2.5, 3.1, 3.4 and the assumption of mathematical induction, we obtain

$$\langle (\bar{u}^n)_{\widehat{\bar{x}\bar{x}\bar{x}}} + (\bar{u}^n)_{\widehat{\bar{y}\bar{y}\bar{y}}}, 2\bar{u}^n\rangle = 0, \tag{64}$$

$$\langle \nabla_h u^n, 2\bar{u}^n\rangle \le |u^n|^2_1 + \|u^{n+1}\|^2 + \|u^{n-1}\|^2, \tag{65}$$

$$\langle (u^n)_{\bar{x}\bar{x}\hat{y}}, 2\bar{u}^n\rangle \le \|u^n_{\bar{x}\bar{x}\bar{y}}\|^2 + \frac{1}{2}(\|u^{n+1}\|^2 + \|u^{n-1}\|^2), \tag{66}$$

$$\langle (u^n)_{\hat{x}\bar{y}\bar{y}}, 2\bar{u}^n\rangle \le \|u^n_{\bar{x}\bar{y}\bar{y}}\|^2 + \frac{1}{2}(\|u^{n+1}\|^2 + \|u^{n-1}\|^2), \tag{67}$$

$$\langle \nabla_h (u^n)^p, 2\bar{u}^n\rangle \le C\Big[\|u^n\|^2 + \frac{1}{2}(|u^{n+1}|_1^2 + |u^{n-1}|_1^2)\Big], \tag{68}$$

$$\langle (u^n)^p_{\bar{x}\hat{x}\hat{y}}, 2\bar{u}^n\rangle \le C\Big[\|u^n\|^2 + \frac{1}{2}\|u^{n+1}_{\bar{x}\bar{x}\bar{y}}\|^2 + \frac{1}{2}\|u^{n-1}_{\bar{x}\bar{x}\bar{y}}\|^2\Big], \tag{69}$$

$$\langle (u^n)^p_{\hat{x}\bar{y}\bar{y}}, 2\bar{u}^n\rangle \le C\Big[\|u^n\|^2 + \frac{1}{2}\|u^{n+1}_{\bar{x}\bar{y}\bar{y}}\|^2 + \frac{1}{2}\|u^{n-1}_{\bar{x}\bar{y}\bar{y}}\|^2\Big]. \tag{70}$$

Denoting

$$\begin{aligned} E^n \equiv\ & \|u^{n+1}\|^2 + \|u^n\|^2 - \frac{h^2}{6}(|u^{n+1}|_1^2 + |u^n|_1^2) + \|u^{n+1}_{\bar{x}\bar{x}}\|^2 + \|u^n_{\bar{x}\bar{x}}\|^2 \\ & + \|u^{n+1}_{\bar{y}\bar{y}}\|^2 + \|u^n_{\bar{y}\bar{y}}\|^2 + \frac{h^2}{6}(\|u^{n+1}_{\bar{x}\bar{y}\bar{y}}\|^2 + \|u^n_{\bar{x}\bar{y}\bar{y}}\|^2 + \|u^{n+1}_{\bar{y}\bar{x}\bar{x}}\|^2 + \|u^n_{\bar{y}\bar{x}\bar{x}}\|^2), \end{aligned} \tag{71}$$

we obtain from Eqs. (63)–(71) that

$$E^n - E^{n-1} \le C\tau(E^n + E^{n-1}),\ \ n > 1.$$

Thus, if τ is sufficiently small such that $\tau \le \frac{k-2}{Ck}$ when $k > 2$, then we have

$$E^n \le \frac{1+C\tau}{1-C\tau}E^{n-1} \le (1+Ck\tau)E^{n-1} \le (1+Ck\tau)^n E^0 \le \exp(kCT)E^0. \tag{72}$$

From Lemma 2.6, we can obtain

$$\|u^n\|^2 - \frac{h^2}{6}(|u^n|_1)^2 \ge \|u^n\|^2 - \frac{h^2}{6}\frac{4}{h^2}\|u^n\|^2 \ge \frac{1}{3}\|u^n\|^2 \ge 0,\ \ n > 1,$$

implying that

$$\begin{aligned} E^n \ge\ & \frac{1}{3}(\|u^{n+1}\|^2 + \|u^n\|^2) + \|u^{n+1}_{\bar{x}\bar{x}}\|^2 + \|u^n_{\bar{x}\bar{x}}\|^2 + \|u^{n+1}_{\bar{y}\bar{y}}\|^2 + \|u^n_{\bar{y}\bar{y}}\|^2 \\ & + \frac{h^2}{6}(\|u^{n+1}_{\bar{x}\bar{y}\bar{y}}\|^2 + \|u^n_{\bar{x}\bar{y}\bar{y}}\|^2 + \|u^{n+1}_{\bar{y}\bar{x}\bar{x}}\|^2 + \|u^n_{\bar{y}\bar{x}\bar{x}}\|^2) \ge 0,\ \ n > 1. \end{aligned} \tag{73}$$

From Eqs. (72)–(73), we obtain $\|u^n\| \le C$, $\|u^n_{\bar{x}\bar{x}}\| \le C$, $\|u^n_{\bar{y}\bar{y}}\| \le C$, $n > 1$. By the definition of $\|\Delta_h u\|$, we have $\|\Delta_h u^n\| \le C$. From Lemmas 3.4 and 3.5, we obtain $|u^n|_1 \le C$, $\|\nabla_h u^n\| \le C$, $\|u^n\|_\infty \le C$, $n = 1, 2, 3, \ldots, N$. This completes the proof.

Theorem 3.8. *The finite difference scheme in Eqs.* (53)–(57) *has a unique solution.*

Proof. We use the mathematical induction method to prove it similar to that for Theorem 2.8. Suppose that $u^1, u^2, \ldots, u^n$ $(1 \le n \le N-1)$ can be solved uniquely. We consider the homogeneous form of Eq. (53) for u^{n+1} as

$$\begin{aligned} & \frac{1}{2\tau}u^{n+1}_{i,j} + \frac{1}{2\tau}[(u^{n+1}_{i,j})_{\bar{x}\bar{x}\bar{x}\bar{x}} + (u^{n+1}_{i,j})_{\bar{y}\bar{y}\bar{y}\bar{y}}] + \frac{1}{2}[(u^{n+1}_{i,j})_{\widehat{\bar{x}\bar{x}}} + (u^{n+1}_{i,j})_{\widehat{\bar{y}\bar{y}}}] \\ & + \frac{h^2}{6}\frac{1}{2\tau}[\Delta_h(u^{n+1}_{i,j}) + (u^{n+1}_{i,j})_{\bar{x}\bar{x}\bar{y}\bar{y}\bar{y}\bar{y}} + (u^{n+1}_{i,j})_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}}] = 0. \end{aligned} \tag{74}$$

Taking the inner product of Eq. (74) with u^{n+1}, we obtain from the discrete boundary conditions in Eqs. (55)–(56) and by Lemmas 3.1, Lemma 3.3 that

$$\|u^{n+1}\|^2 - \frac{h^2}{6}|u^{n+1}|_1^2 + \|u^{n+1}_{\bar{x}\bar{x}}\|^2 + \|u^{n+1}_{\bar{y}\bar{y}}\|^2 - \frac{h^2}{6}(\|u^{n+1}_{\bar{x}\bar{y}\bar{y}}\|^2 + \|u^{n+1}_{\bar{x}\bar{x}}\bar{y}\|^2) = 0. \tag{75}$$

By Lemma 2.6, we obtain

$$\|u^{n+1}\|^2 - \frac{h^2}{6}|u^{n+1}|_1^2 \ge \|u^{n+1}\|^2 - \frac{h^2}{6}\frac{4}{h^2}\|u^{n+1}\|^2 = \frac{1}{3}\|u^{n+1}\|^2, \tag{76}$$

$$\|u^{n+1}_{\bar{x}\bar{x}}\|^2 - \frac{h^2}{6}\|u^{n+1}_{\bar{x}\bar{x}\bar{y}}\|^2 \ge \|u^{n+1}_{\bar{x}\bar{x}}\|^2 - \frac{h^2}{6}\frac{4}{h^2}\|u^{n+1}_{\bar{x}\bar{x}}\|^2 = \frac{1}{3}\|u^{n+1}_{\bar{x}\bar{x}}\|^2, \tag{77}$$

$$\|u^{n+1}_{\bar{y}\bar{y}}\|^2 - \frac{h^2}{6}\|u^{n+1}_{\bar{x}\bar{y}\bar{y}}\|^2 \ge \|u^{n+1}_{\bar{y}\bar{y}}\|^2 - \frac{h^2}{6}\frac{4}{h^2}\|u^{n+1}_{\bar{y}\bar{y}}\|^2 = \frac{1}{3}\|u^{n+1}_{\bar{y}\bar{y}}\|^2. \tag{78}$$

From Eqs. (75)–(78), we obtain

$$\|u^{n+1}\|^2 + \|u^{n+1}_{\bar{x}\bar{x}}\|^2 + \|u^{n+1}_{\bar{y}\bar{y}}\|^2 \le 0,$$

implying that

$$\|u^{n+1}\|^2=0,\quad \|u^{n+1}_{\bar{x}\bar{x}}\|^2=0,\quad \|u^{n+1}_{\bar{y}\bar{y}}\|^2=0.$$

By Lemma 3.5, we obtain $\|u^{n+1}\|_\infty=0$. Hence, there uniquely admits a zero solution satisfying the scheme in Eqs. (53)–(57). Therefore, u^{n+1} is uniquely solvable, and this completes the proof.

Theorem 3.9. *Supposing $u_0(x,y)\in H_0^2(\Omega)$, then the solution u^n of the scheme in Eqs. (53)–(57) converges to the solution of the problem in Eqs. (49)–(52) with the convergence rate of $O(\tau^2+h^4)$ in both l_2-norm and l_∞-norm when τ and h are small.*

Proof. The proof is similar to that for Theorem 2.10. Let $e^n_{i,j}=v^n_{i,j}-u^n_{i,j}$, where $v^n_{i,j}$ and $u^n_{i,j}$ are the solutions of the problem in Eqs. (49)–(52) and the scheme in Eqs. (53)–(57), respectively. Then, we obtain the following error equation

$$\begin{aligned}R^n_{i,j}&=(e^n_{i,j})_{\hat{t}}+\frac{h^2}{6}\Delta_h(e^n_{i,j})_{\hat{t}}+\nabla_h e^n_{i,j}+[(\bar{e}^n_{i,j})_{\widehat{\bar{x}\bar{x}x}}+(\bar{e}^n_{i,j})_{\widehat{\bar{y}\bar{y}y}}]+\nabla_h[(v^n_{i,j})^p-(u^n_{i,j})^p]\\&+\frac{h^2}{6}[(e^n_{i,j})_{\bar{x}\bar{x}\hat{y}}+(e^n_{i,j})_{\hat{x}\bar{y}\bar{y}}]+\frac{h^2}{6}[(v^n_{i,j})^p_{\bar{x}\bar{x}\hat{y}}-(u^n_{i,j})^p_{\bar{x}\bar{x}\hat{y}}+(v^n_{i,j})^p_{\hat{x}\bar{y}\bar{y}}-(u^n_{i,j})^p_{\hat{x}\bar{y}\bar{y}}]\\&+[(e^n_{i,j})_{\bar{x}\bar{x}\bar{x}\bar{x}\hat{t}}+(e^n_{i,j})_{\bar{y}\bar{y}\bar{y}\bar{y}\hat{t}}]+\frac{h^2}{6}[(e^n_{i,j})_{\bar{x}\bar{x}\bar{y}\bar{y}\bar{y}\bar{y}\hat{t}}+(e^n_{i,j})_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{y}\bar{y}\hat{t}}]=0,\end{aligned}\tag{79}$$

where $\bar{e}^n_{i,j}=(e^{n+1}_{i,j}+e^{n-1}_{i,j})/2$. Taking the inner product of Eq. (79) with $2\bar{e}^n$ (i.e. $e^{n+1}+e^{n-1}$), and according to Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned}&\|e^n\|^2_{\hat{t}}-\frac{h^2}{6}(|e|_1)^2_{\hat{t}}+\|e^n_{\bar{x}\bar{x}}\|^2_{\hat{t}}+\|e^n_{\bar{y}\bar{y}}\|^2_{\hat{t}}+\frac{h^2}{6}(\|e^n_{\bar{x}\bar{y}\bar{y}}\|^2_{\hat{t}}+\|e^n_{\bar{y}\bar{x}\bar{x}}\|^2_{\hat{t}})\\&=\langle R^n,2\bar{e}^n\rangle-\langle\nabla_h e^n,2\bar{e}^n\rangle-\langle(\bar{e}^n_{i,j})_{\widehat{\bar{x}\bar{x}x}}+(\bar{e}^n_{i,j})_{\widehat{\bar{y}\bar{y}y}},2\bar{e}^n\rangle\\&\quad+\langle\nabla_h[(u^n_{i,j})^p-(v^n_{i,j})^p],2\bar{e}^n\rangle-\frac{h^2}{6}\langle(e^n_{i,j})_{\bar{x}\bar{x}\hat{y}}+(e^n_{i,j})_{\hat{x}\bar{y}\bar{y}},2\bar{e}^n\rangle\\&\quad+\frac{h^2}{6}\langle(u^n_{i,j})^p_{\bar{x}\bar{x}\hat{y}}-(v^n_{i,j})^p_{\bar{x}\bar{x}\hat{y}},2\bar{e}^n\rangle+\frac{h^2}{6}\langle(u^n_{i,j})^p_{\hat{x}\bar{y}\bar{y}}-(v^n_{i,j})^p_{\hat{x}\bar{y}\bar{y}},2\bar{e}^n\rangle.\end{aligned}\tag{80}$$

Based on the Cauchy–Schwarz inequality [15], Lemmas 3.1 and 3.4, we have

$$\langle\nabla_h e^n,2\bar{e}^n\rangle\le|e^n|_1^2+\|e^{n+1}\|^2+\|e^{n-1}\|^2,\tag{81}$$

$$\langle(\bar{e}^n)_{\widehat{\bar{x}\bar{x}x}}+(\bar{e}^n)_{\widehat{\bar{y}\bar{y}y}},2\bar{e}^n\rangle=0,\tag{82}$$

$$\langle(u^n)_{\bar{x}\bar{x}\hat{y}},2\bar{e}^n\rangle\le\|e^n_{\bar{x}\bar{x}\bar{y}}\|^2+\frac{1}{2}(\|e^{n+1}\|^2+\|e^{n-1}\|^2),\tag{83}$$

$$\langle(e^n)_{\hat{x}\bar{y}\bar{y}},2\bar{e}^n\rangle\le\|e^n_{\bar{x}\bar{y}\bar{y}}\|^2+\frac{1}{2}(\|e^{n+1}\|^2+\|e^{n-1}\|^2).\tag{84}$$

Using a similar argument for the 1D case, one can obtain

$$\begin{aligned}&\langle\nabla_h[(u^n_{i,j})^p-(v^n_{i,j})^p],2\bar{e}^n\rangle\\&\le C(\|e^{n-1}\|^2+\|e^{n+1}\|^2+\|e^{n-1}_{\bar{x}}\|^2+\|e^{n+1}_{\bar{x}}\|^2+\|e^{n-1}_{\bar{y}}\|^2+\|e^{n+1}_{\bar{y}}\|^2),\end{aligned}\tag{85}$$

$$\langle(u^n_{i,j})^p_{\bar{x}\bar{x}\hat{y}}-(v^n_{i,j})^p_{\bar{x}\bar{x}\hat{y}},2\bar{e}^n\rangle\le C(\|e^{n-1}\|^2+\|e^{n+1}\|^2+\|e^{n-1}_{\bar{x}\bar{x}\bar{y}}\|^2+\|e^{n+1}_{\bar{x}\bar{x}\bar{y}}\|^2),\tag{86}$$

$$\langle(u^n_{i,j})^p_{\hat{x}\bar{y}\bar{y}}-(v^n_{i,j})^p_{\hat{x}\bar{y}\bar{y}},2\bar{e}^n\rangle\le C(\|e^{n-1}\|^2+\|e^{n+1}\|^2+\|e^{n-1}_{\bar{x}\bar{y}\bar{y}}\|^2+\|e^{n+1}_{\bar{x}\bar{y}\bar{y}}\|^2),\tag{87}$$

$$\|e^n_{\bar{x}}\|^2=-\langle e^n,e^n_{\bar{x}\bar{x}}\rangle\le\frac{1}{2}(\|e^n\|^2+\|e^n_{\bar{x}\bar{x}}\|^2),\tag{88}$$

$$\|e^n_{\bar{y}}\|^2=-\langle e^n,e^n_{\bar{y}\bar{y}}\rangle\le\frac{1}{2}(\|e^n\|^2+\|e^n_{\bar{y}\bar{y}}\|^2).\tag{89}$$

Substituting an equation similar to Eqs. (45) and (81)–(89) into Eq. (80) gives

$$\begin{aligned}&\|e^{n+1}\|^2-\|e^{n-1}\|^2-\frac{h^2}{6}(|e^{n+1}|_1^2-|e^{n-1}|_1^2)+\|e^{n+1}_{\bar{x}\bar{x}}\|^2-\|e^{n-1}_{\bar{x}\bar{x}}\|^2\\&+\|e^{n+1}_{\bar{y}\bar{y}}\|^2-\|e^{n-1}_{\bar{y}\bar{y}}\|^2+\frac{h^2}{6}(\|e^{n+1}_{\bar{x}\bar{y}\bar{y}}\|^2-\|e^{n-1}_{\bar{x}\bar{y}\bar{y}}\|^2+\|e^{n+1}_{\bar{x}\bar{x}\bar{y}}\|^2-\|e^{n-1}_{\bar{x}\bar{x}\bar{y}}\|^2),\\&\le 2\tau\|R^n\|^2+C\tau(\|e^{n+1}_{\bar{x}\bar{x}}\|^2+\|e^{n-1}_{\bar{x}\bar{x}}\|^2+\|e^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2)\end{aligned}$$

$$+ C\tau(\|e^{n+1}_{\bar{y}y}\|^2 + \|e^{n-1}_{\bar{y}y}\|^2 + |e^{n+1}|_1^2 + |e^{n-1}|_1^2)$$
$$+ C(\|e^{n+1}_{\bar{x}\bar{y}y}\|^2 - \|e^{n-1}_{\bar{x}\bar{y}y}\|^2 + \|e^{n+1}_{\bar{x}x\bar{y}}\|^2 - \|e^{n-1}_{\bar{x}x\bar{y}}\|^2). \quad (90)$$

Denoting

$$\Pi^n \equiv \|e^{n+1}\|^2 + \|e^n\|^2 - \frac{h^2}{6}(|e^{n+1}|_1^2 + |e^n|_1^2) + \|e^{n+1}_{\bar{x}x}\|^2 + \|e^n_{\bar{x}x}\|^2$$
$$+ \|e^{n+1}_{\bar{y}y}\|^2 + \|e^n_{\bar{y}y}\|^2 + \frac{h^2}{6}(\|e^{n+1}_{\bar{x}\bar{y}y}\|^2 + \|e^n_{\bar{x}\bar{y}y}\|^2 + \|e^{n+1}_{\bar{x}x\bar{y}}\|^2 + \|e^n_{\bar{x}x\bar{y}}\|^2), \quad (91)$$

then Eq. (90) can be simplified to

$$\Pi^n - \Pi^{n-1} \le 2\tau\|R^n\|^2 + C\tau(\Pi^n + \Pi^{n-1}).$$

Hence, we obtain

$$(1 - C\tau)(\Pi^n - \Pi^{n-1}) \le 2\tau\|R^n\|^2 + 2C\tau\Pi^{n-1}.$$

If τ is sufficiently small such that $1 - C\tau > 1/2$, then

$$\Pi^n - \Pi^{n-1} \le C\tau\|R^n\|^2 + C\tau\Pi^{n-1}. \quad (92)$$

Summarizing Eq. (92) with respect to n from 1 to n, we obtain

$$\Pi^n \le \Pi^0 + C\tau\sum_{l=1}^{n}\|R^l\|^2 + C\tau\sum_{l=1}^{n}\Pi^{l-1}.$$

Note that

$$\tau\sum_{l=1}^{n}\|R^l\|^2 \le n\tau \max_{1\le l\le n}\|R^l\|^2 \le T\cdot O(\tau^2 + h^4)^2,$$

if we use a fourth-order method such as Eq. (57) to compute u^1 such that

$$e^0 = 0,\ \ \Pi^0 \le O(\tau^2 + h^4),$$

we then have

$$\Pi^n \le O(\tau^2 + h^4)^2 + C\tau\sum_{k=1}^{n}\Pi^{k-1}.$$

By Lemma 2.9, we obtain that $\Pi^n \le O(\tau^2 + h^4)^2$, implying

$$\|e^{n+1}\| \le O(\tau^2 + h^4),\ \ \|e^{n+1}_{\bar{x}x}\| \le O(\tau^2 + h^4),\ \ \|e^{n+1}_{\bar{y}y}\| \le O(\tau^2 + h^4), \quad (93)$$

and hence $\|\Delta_h e^{n+1}\| \le O(\tau^2 + h^4)$ and $\|e^{n+1}\|_\infty \le O(\tau^2 + h^4)$ according to Lemma 3.5. Therefore, the solution u^n of the scheme in Eqs. (53)–(57) converges to the solution of the problem in Eqs. (49)–(52) in both l_2-norm and l_∞-norm with the convergence rate of $O(\tau^2 + h^4)$.

Using a similar argument as the proof for Theorem 3.9, one may obtain Theorem 3.10.

Theorem 3.10. *Supposing $u_0(x, y) \in H_0^2(\Omega)$, then the scheme in Eqs. (53)–(57) is unconditionally stable in the sense of l_2-norm and l_∞-norm.*

4. Numerical experiments

4.1. Example 1: 1D problems

For the case of $p = 3$, we considered the generalized Rosenau–KdV equation in Eq. (1) as

$$u_t + u_x + u_{xxx} + u_{xxxxt} + (u^3)_x = 0,\ \ \alpha \le x \le \beta,\ \ t \in [0, T], \quad (94)$$

and chose the initial condition to be $u_0(x) = k_{11}\,\mathrm{sech}^2(k_{12}x)$ so that the analytical solitary wave solution is $u(x, t) = k_{11}\,\mathrm{sech}^2[k_{12}(x - k_{13}t)]$, where [24,26]

$$k_{11} = \frac{1}{4}\sqrt{-15 + 3\sqrt{41}},\ k_{12} = \frac{1}{4}\sqrt{\frac{-5 + \sqrt{41}}{2}},\ k_{13} = \frac{1}{10}(5 + \sqrt{41}).$$

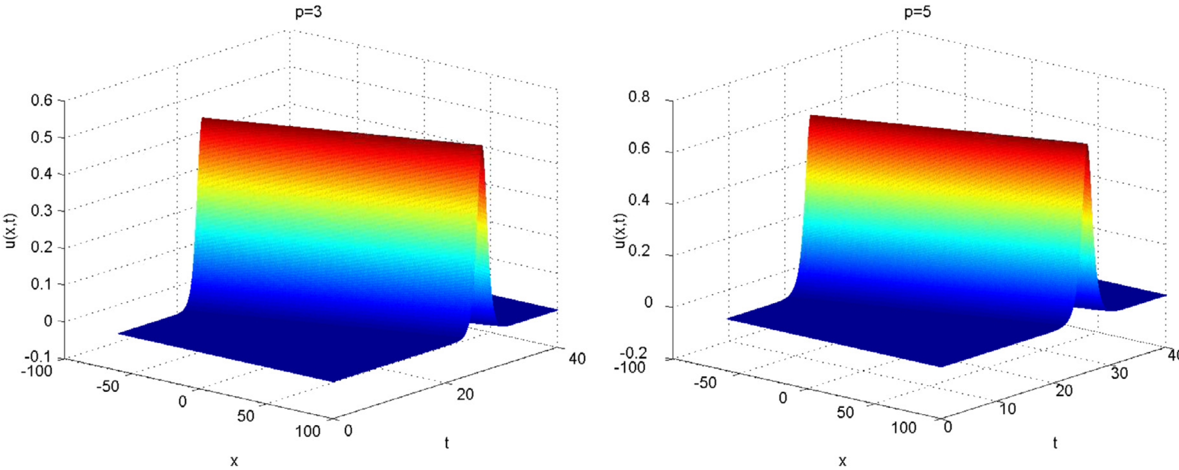

Fig. 1. Numerical solutions of $u(x,t)$ when $\alpha=-60$, $\beta=100$, $h=0.25$, $\tau=h^2$ at $T=40$.

Table 1
Errors and convergence rates of the present scheme in Eqs. (10)–(13) when $h=0.5$ and $\tau=h^2$ at $T=40$.

p		h,τ	$h/2,\tau/4$	$h/4,\tau/16$
$p=3$	$\|e\|_2$	2.6393848E−02	1.6159943E−03	1.0089225E−04
	Rate	–	4.029708	4.0015348
	$\|e\|_\infty$	9.9701002E−03	6.1045529E−04	3.8105323E−05
	Rate	–	4.0296505	4.0018212
$p=5$	$\|e\|_2$	4.4059552E−02	2.6675119E−03	1.6644060E−04
	Rate	–	4.0458882	4.0024154
	$\|e\|_\infty$	1.6616612E−02	1.0039841E−03	6.2629441E−05
	Rate	–	4.0488179	4.0027516

For the case of $p=5$, we considered the generalized Rosenau–KdV equation in Eq. (1) as

$$u_t+u_x+u_{xxx}+u_{xxxxt}+(u^5)_x=0,\quad \alpha\le x\le\beta,\quad t\in[0,T], \tag{95}$$

and chose the initial condition to be $u_0(x)=k_{21}\,\mathrm{sech}(k_{22}x)$ so that the analytical solitary wave solution is $u(x,t)=k_{21}\,\mathrm{sech}[k_{22}(x-k_{23}t)]$, where [42]

$$k_{21}=\sqrt[4]{\frac{4}{15}(-5+\sqrt{34})},\ k_{22}=\frac{1}{3}\sqrt{-5+\sqrt{34}},\ k_{23}=\frac{1}{10}(5+\sqrt{34}).$$

In our experiments, we chose $\alpha=-60$ and $\beta=90$. Numerical results in term of errors and rates of convergence at time $T=40$ were listed in Table 1, where $h=0.5$ and $\tau=h^2$. From Table 1, one may see that the convergence rates obtained based on the present fourth-order difference scheme in Eqs. (10)–(13) for both $p=3$ and $p=5$ are close to 4.0, which coincides with the theoretical prediction of convergence rates. We then compared errors and convergence rates between the present scheme in Eqs. (10)–(13) and the schemes proposed in [22,24]. Results at time $T=40$ were given in Table 2. It can be seen from Table 2 that the errors in l_∞-norm obtained based on the present scheme are much smaller than those obtained based on the schemes in [22,24], and the fourth-order convergence rate of the present scheme is verified as compared with the second-order convergence rate of the schemes in [22,24]. We computed the conservative invariants Q^n and E^n at various times as listed in Table 3. Results justify that the present scheme is conservative for both mass and energy.

Fig. 1 presents the numerical solitary wave travelling obtained based on the present scheme within $0\le t\le 40$ for cases $p=3$ and $p=5$, respectively, where $\alpha=-60$, $\beta=100$, $h=0.25$, and $\tau=h^2$. Figs. 2 and 3 show the error distributions in absolute value along the x-direction at various times $T=15,30,45,60$, which were calculated using the present scheme with $h=0.25$ and $\tau=h^2$ for cases $p=3$ and $p=5$, respectively. It is observed that the maximum error is taking place around the peak of the solitary wave.

To see the advantage of the present scheme when simulating the solitary wave travelling for a much longer time as compared with the scheme in [24], we plotted the solitary wave travelling within $0\le t\le 250$ simulated based on the present scheme and the scheme in [24], respectively, as shown in Figs. 4 and 5. It can be seen from Fig. 4 that based on the scheme in [24] with $h=0.25$, $\tau=h^2$ for cases $p=3$ and $p=5$, the solitary wave was travelling well within $0\le t\le 50$, and however, was then destroyed after $T>50$. This implies that the scheme in [24] does not keep the conservations for mass and energy after $T>50$. On the other hand, Fig. 5 shows that the present scheme simulated well the solitary wave travelling within $0\le t\le 250$, and the numerical solitary wave is in excellent agreement with the analytical solitary wave all the times,

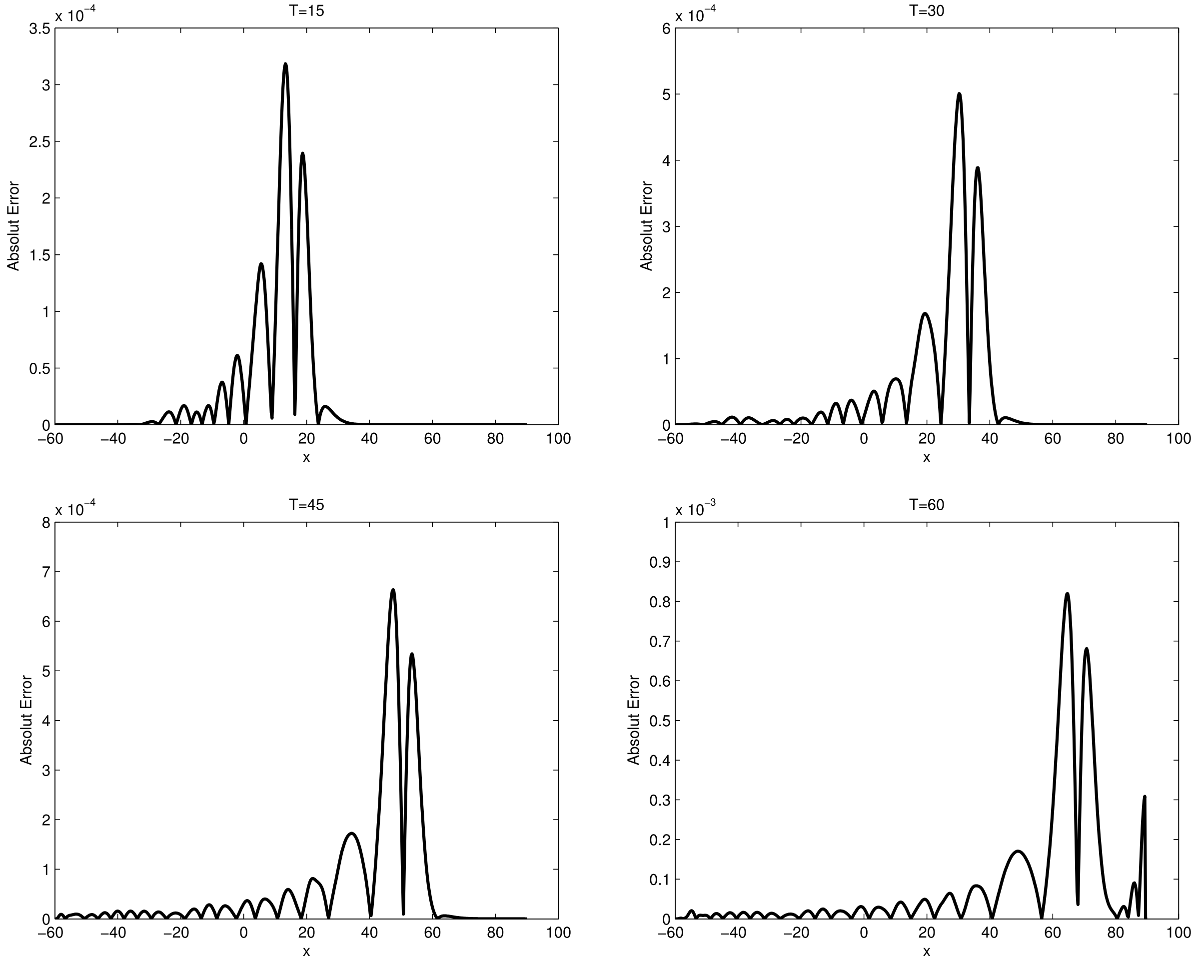

Fig. 2. Absolute error distribution when $p = 3$, $h = 0.25$, $\tau = h^2$ at $T = 15, 30, 45, 60$.

Table 2
Comparison of errors in the sense of l_∞ at $T = 40$.

p	Scheme	h	$\|e_h\|_\infty$	$\frac{\|e_h\|_\infty}{\|e_{h/2}\|_\infty}$	Rate
	Zhou [22]	0.25	7.70544E−03	–	–
		0.125	1.94252E−03	3.96672	1.98794
		0.0625	4.86553E−04	3.99242	1.99726
$p = 3$	Zheng [24]	0.25	1.34986E−02	–	–
		0.125	3.42489E−03	3.94134	1.97869
		0.0625	8.59570E−04	3.98441	1.99436
	Present	0.25	2.45605E−03	–	–
		0.125	1.52654E−04	16.08893	4.00799
		0.0625	9.48832E−06	16.08868	4.00797
	Zhou [22]	0.25	7.70544E−03	–	–
		0.125	1.94252E−03	3.96036	1.98563
		0.0625	4.86553E−04	3.98994	1.99636
$p = 5$	Zheng [24]	0.25	1.79985E−02	–	–
		0.125	4.56804E−03	3.94009	1.97823
		0.0625	1.14689E−03	3.98299	1.99385
	Present	0.25	4.04781E−03	–	–
		0.125	2.50920E−04	16.13187	4.01184
		0.0625	1.55345E−05	16.15236	4.01367

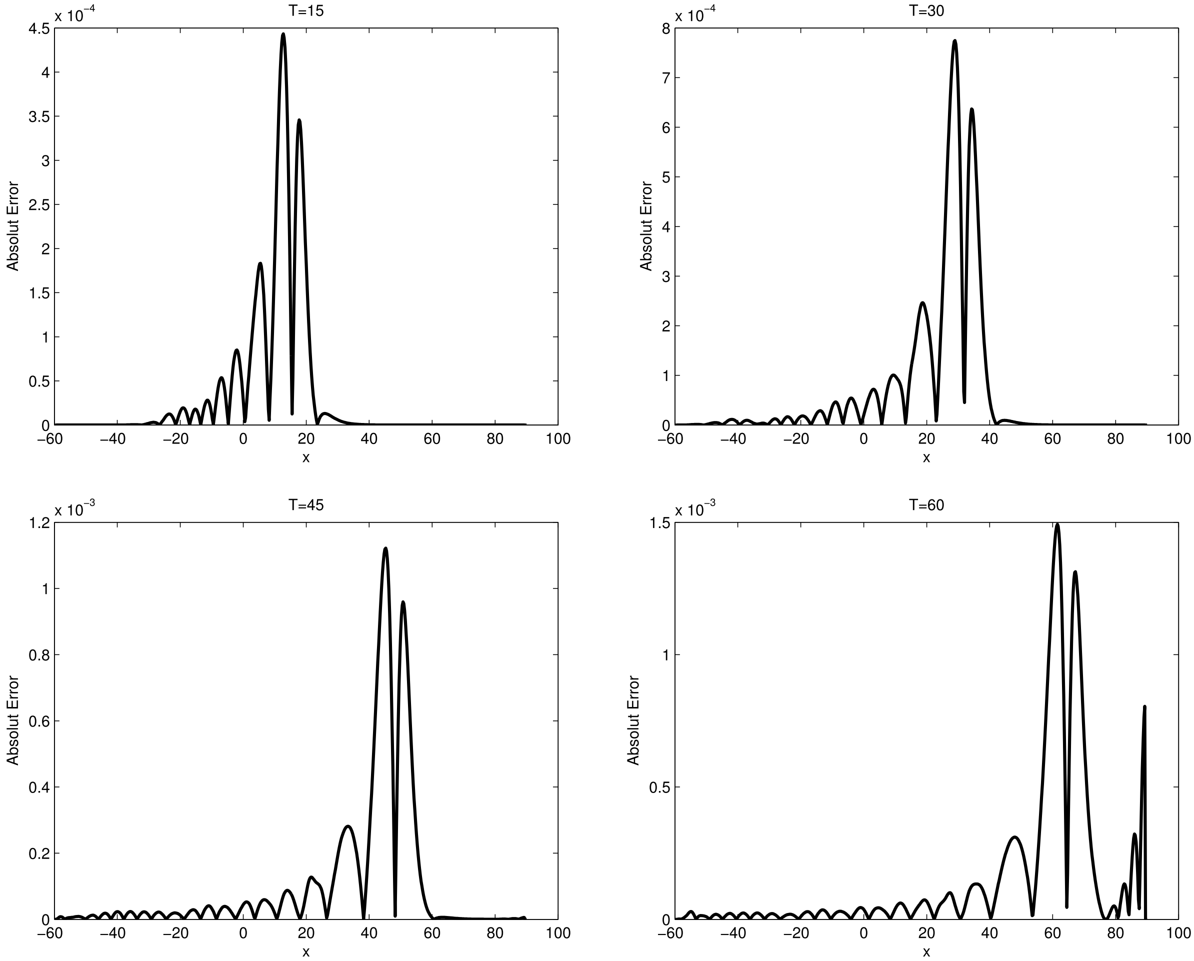

Fig. 3. Absolute error distribution when $p = 5$, $h = 0.25$, $\tau = h^2$ at $T = 15, 30, 45, 60$.

Table 3
Discrete mass Q^n and discrete energy E^n in various times for 1D Rosenau–KdV equation when $h = 0.25$ and $\tau = h^2$.

	$p = 3$		$p = 5$	
T	Q^n	E^n	Q^n	E^n
0	4.898979485518449	1.681957861861872	7.093643137390966	3.109735165395684
10	4.898979485576961	1.681956437155376	7.093643135900470	3.109732974034659
20	4.898979485736063	1.681956292932830	7.093643130766687	3.109732758960448
30	4.898979473518048	1.681956184705329	7.093642968577600	3.109732533953539
40	4.898980423386804	1.681956107381031	7.093640511749809	3.109732332758717
50	4.898974912587071	1.681956051420233	7.093529220712367	3.109732153146640
100	4.898968968207399	1.681956046692215	7.093481395237128	3.109732136106961
150	4.898959456288376	1.681956042203882	7.093414869512039	3.109732119414538
200	4.898944499111343	1.681956037849454	7.093322677988481	3.109732102990418
250	4.898920998338203	1.681956033564348	7.093195103057346	3.109732086419298

where conservative invariants Q^n and E^n can be seen in Table 3. The comparison indicates that there is a significant difference between these two schemes and the present scheme is more efficient for longer time solution simulations. Furthermore, Fig. 5 shows that the solitary wave travels at the same speed under two different grid sizes, implying that our scheme is grid-independent.

To compare with the recently developed linearized fourth-order scheme [35], we considered the following Rosenau–KdV equation

$$u_t + u_x + u_{xxx} + u_{xxxxt} + uu_x = 0, \quad \alpha \le x \le \beta, \quad t \in [0, T], \tag{96}$$

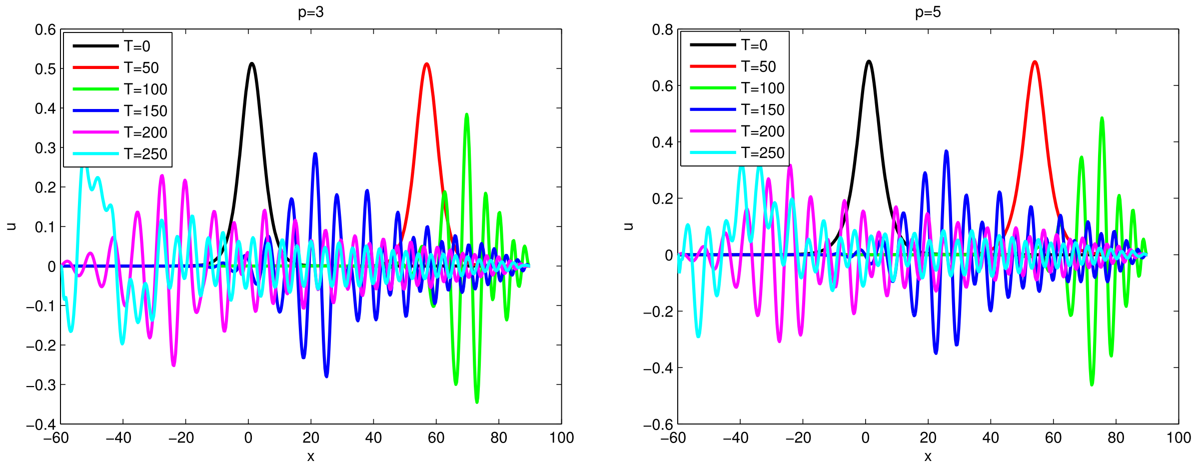

Fig. 4. Numerical solutions in [24] when $p = 3, p = 5, h = 0.25, \tau = h^2$ at $T = 0, 50, 100, 150, 200, 250$.

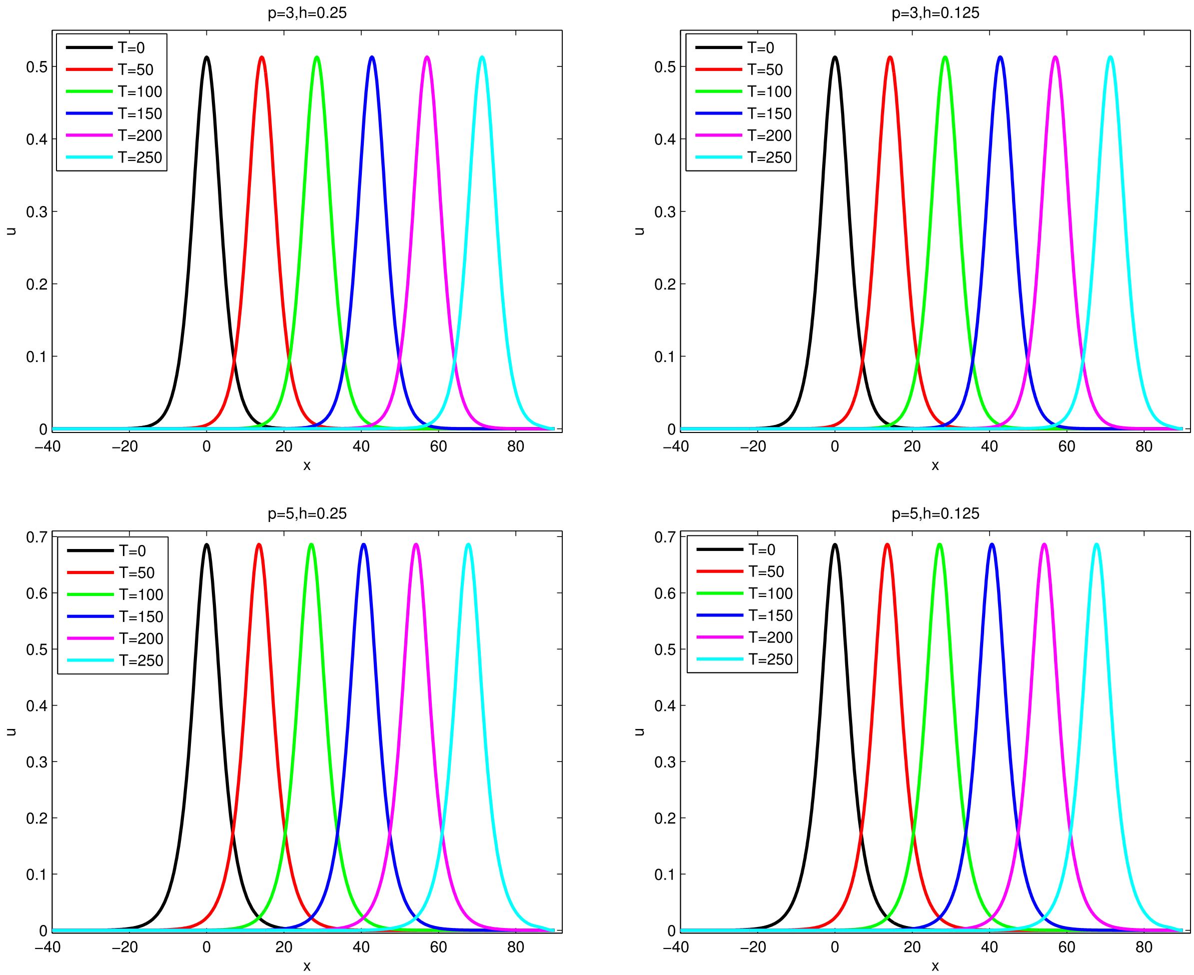

Fig. 5. Exact solutions of $u(x, t)$ at $T = 0$ and numerical solutions when $p = 3, p = 5, h = 0.25, h = 0.125, \tau = 0.0625$ at $T = 50, 100, 150, 200, 250$, respectively.

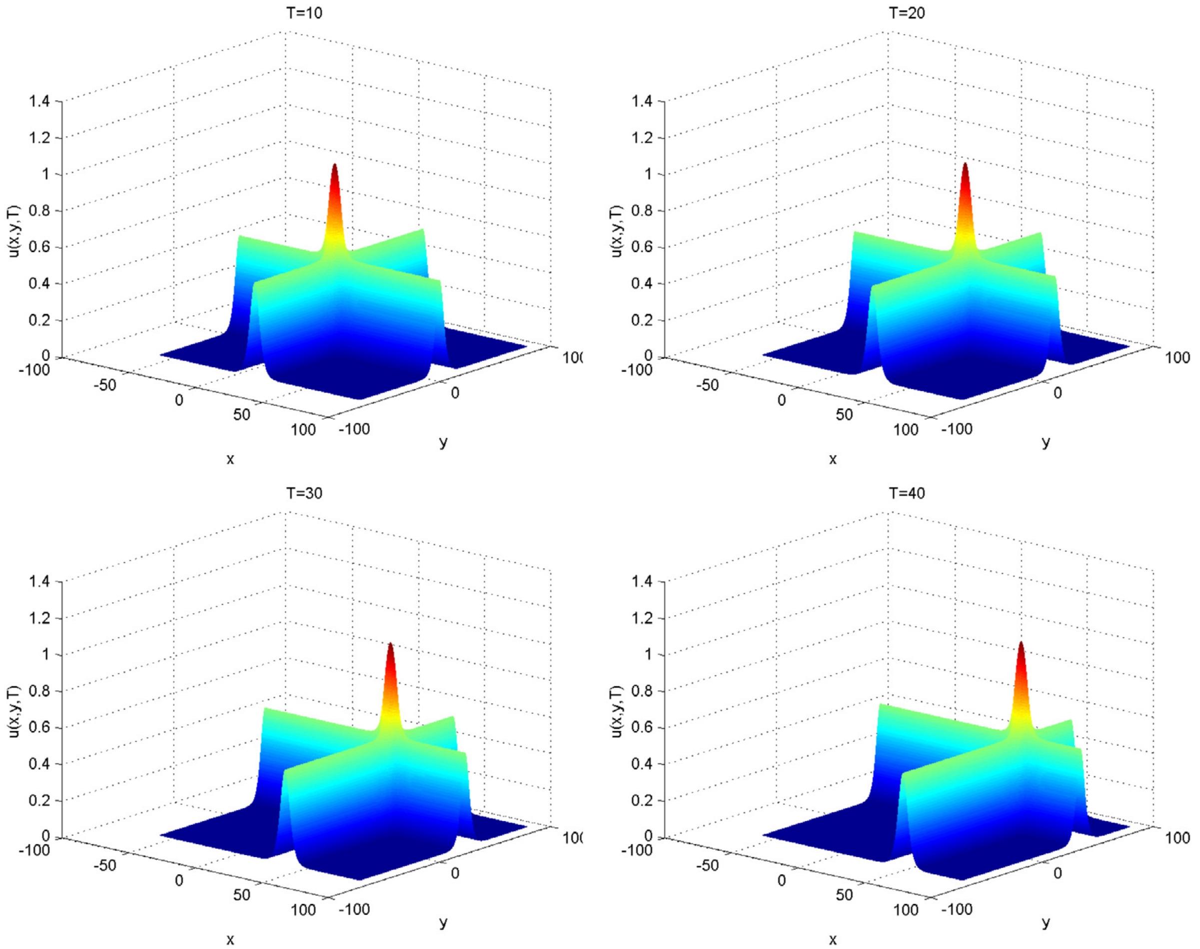

Fig. 6. Numerical solution for $p = 3$ at $T = 10$, 20, 30 and 40 when $h = 0.1$ and $\tau = 0.25$.

Table 4
Comparison of errors in maximum norm $\|u^n\|_\infty$ and CPU times when $h = 0.1$ and $\tau = h^2$.

T	Scheme D [35]	CPU (s)	Present	CPU (s)
5	$6.000056996952052 \times 10^{-5}$	23.484	$2.662060476388284 \times 10^{-6}$	13.828
10	$8.508635753025351 \times 10^{-5}$	67.469	$4.687559301774780 \times 10^{-6}$	47.047
15	$9.567596016540891 \times 10^{-5}$	89.282	$6.229245215261248 \times 10^{-6}$	58.953
20	$1.000789659701996 \times 10^{-4}$	135.688	$7.507679379514798 \times 10^{-6}$	94.672
25	$1.007744439283902 \times 10^{-4}$	162.078	$8.616310676501371 \times 10^{-6}$	112.500
30	$1.013011979855601 \times 10^{-4}$	202.984	$9.606567987152648 \times 10^{-6}$	141.438

and chose the initial condition to be $u_0(x) = k_{31}\,\mathrm{sech}^4(k_{32}x)$ so that the analytical solitary wave solution is $u(x,t) = k_{31}\,\mathrm{sech}^4[k_{32}(x - k_{33}t)]$, where

$$k_{31} = -\frac{35}{24} + \frac{35}{312}\sqrt{313},\ k_{32} = \frac{1}{24}\sqrt{-26 + 2\sqrt{313}},\ k_{33} = \frac{1}{2} + \frac{1}{26}\sqrt{313}.$$

The comparison of errors $\|u^n\|_\infty$ and CPU times between the present scheme and Scheme D in [35] (Scheme D) was presented in Table 4, where $\alpha = -70$, $\beta = 100$ and $h = 0.1$, $\tau = h^2$. From Table 4, one may see that the present scheme provides more accurate solutions and has less CPU times than Scheme D in [35] does.

4.2. Example 2: 2D problems

For the case of $p = 3$, we considered a 2D generalized Rosenau–KdV equation in Eq. (49) as

$$u_t + \nabla \cdot u + u_{xxx} + u_{yyy} + u_{xxxxt} + u_{yyyyt} + \nabla \cdot u^3 = 0, \tag{97}$$

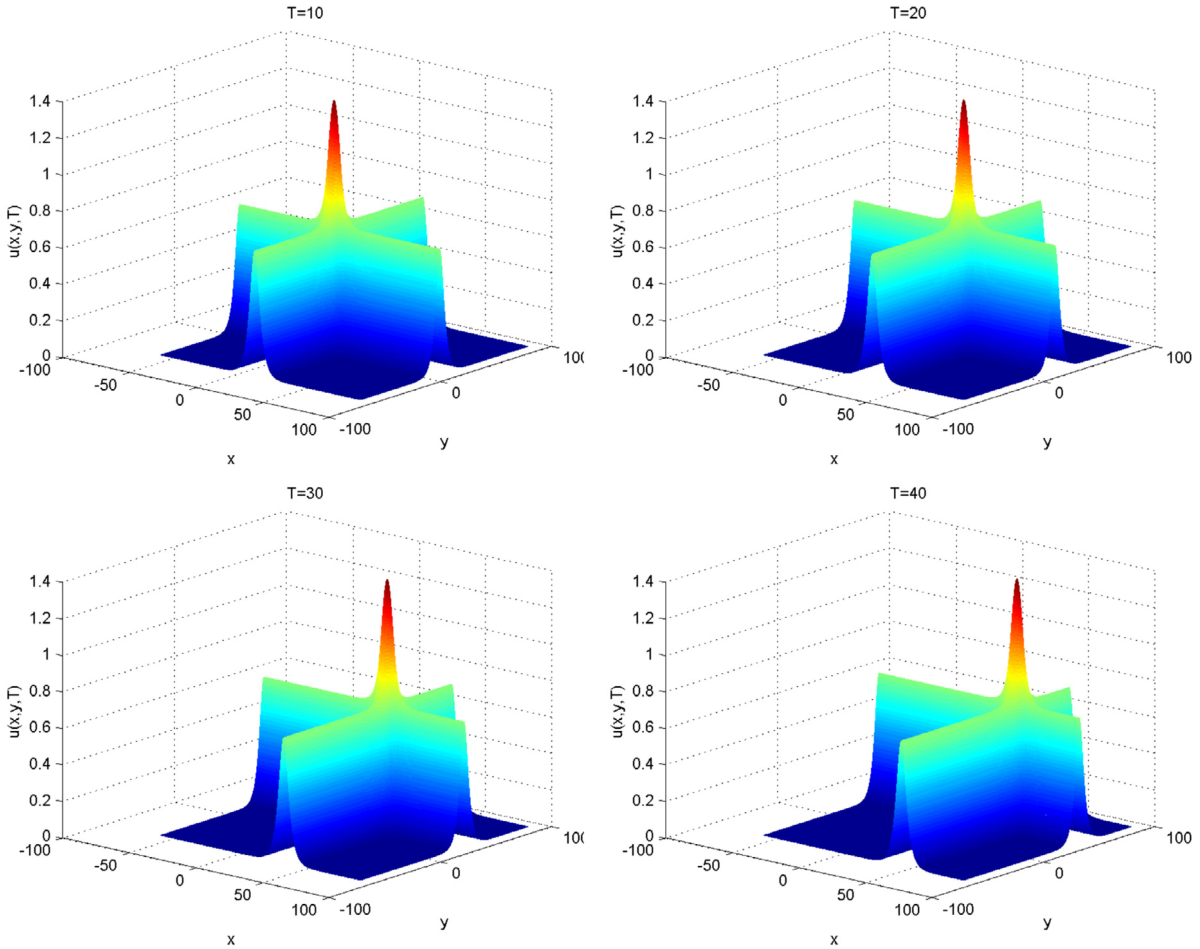

Fig. 7. Numerical solution for $p = 5$ at $T = 10, 20, 30$ and 40 when $h = 0.1$ and $\tau = 0.25$.

Table 5
Discrete mass Q^n and discrete energy E^n in various times for 2D Rosenau–KdV equation when $h = 0.1$ and $\tau = 0.25$.

	$p = 3$		$p = 5$	
T	Q^n	E^n	Q^n	E^n
0	1470.673641549157	1102.103195662962	2129.511670328377	2061.386333378561
5	1470.673641565020	1102.102957642978	2129.511683343761	2061.385550383789
10	1470.673641565538	1102.102931750660	2129.511685817504	2061.385443099122
15	1470.673641566190	1102.102925918361	2129.511686102712	2061.385368158929
20	1470.673641565720	1102.102918540323	2129.511685162606	2061.385255963096
25	1470.673641557803	1102.102908947192	2129.511679822136	2061.385244164556
30	1470.673641463808	1102.102903141113	2129.511652049340	2061.385215356977
35	1470.673640450367	1102.102895903703	2129.511508064930	2061.385150433415
40	1470.673629400097	1102.102889284785	2129.510761686757	2061.384945931876

where $(x, y, t) \in [\alpha, \beta] \times [\alpha, \beta] \times [0, T]$, and chose the initial condition to be

$$u_0(x, y) = \frac{1}{4}\sqrt{-15 + 3\sqrt{41}}\,\mathrm{sech}^2\Big[\frac{1}{4}\sqrt{\frac{-5 + \sqrt{41}}{2}}(x + y)\Big].$$

For the case of $p = 5$, we considered a 2D generalized Rosenau–KdV equation in Eq. (49) as

$$u_t + \nabla \cdot u + u_{xxx} + u_{yyy} + u_{xxxxt} + u_{yyyyt} + \nabla \cdot u^5 = 0, \tag{98}$$

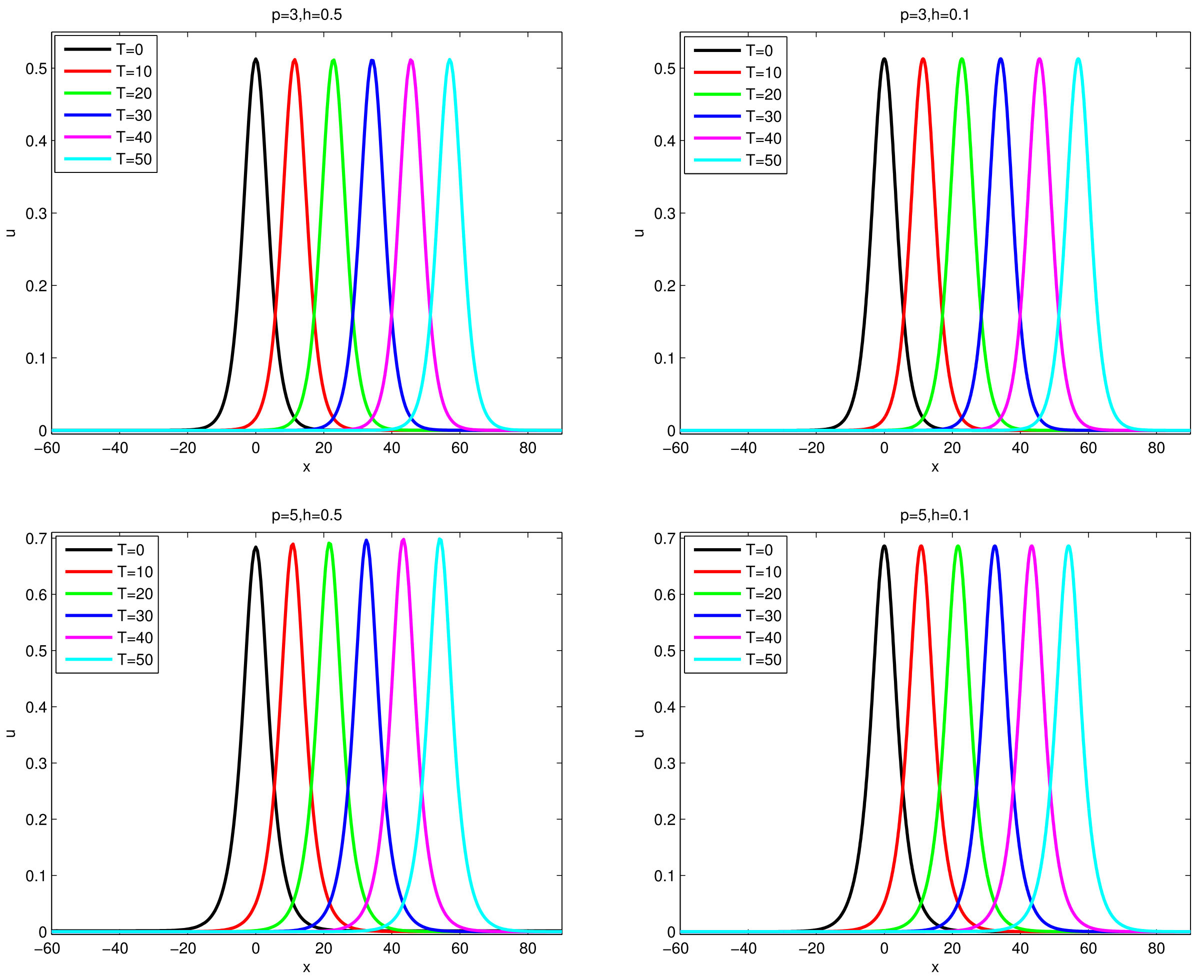

Fig. 8. Numerical solutions of $u(x, y, t)$ along the middle plane $y = (\beta - \alpha)/2$ at $T = 0, 10, 20, 30, 40, 50$ when $p = 3$, $p = 5$, $h = 0.5$, $h = 0.1$, $\tau = 0.25$, respectively.

where $(x, y, t) \in [\alpha, \beta] \times [\alpha, \beta] \times [0, T]$, and chose the initial condition to be

$$u_0(x, y) = \sqrt[4]{\frac{4}{15}(-5+\sqrt{34})}\,\mathrm{sech}\Big[\frac{1}{3}\sqrt{-5+\sqrt{34}}(x+y)\Big].$$

In our experiments, we chose the computational domain to be $(x, y, t) \in [-60, 90] \times [-60, 90] \times [0, T]$. We employed the present scheme in Eqs. (53)–(57) for cases $p = 3$ and $p = 5$ in our simulations. Table 5 lists the conservative invariants Q^n and E^n at various times $T = 10, 20, 30$ and 40, where $h = 0.1$ and $\tau = 0.25$, and numerical solutions are shown in Figs. 6 and 7 for $p = 3$ and $p = 5$, respectively. Results imply that the present 2D scheme is conservative for mass and energy. In particular, we plotted the numerical solutions along the middle plane at $y = (\beta - \alpha)/2$ at $T = 0, 10, 20, 30, 40, 50$ in Fig. 8, which were obtained based on the present scheme using $\tau = 0.25$ and two different spacial grid sizes $h = 0.5$ and $h = 0.1$, respectively. Fig. 8 shows clearly that the shapes of the soliton wave at different times are almost identical to each other. Furthermore, Fig. 8 shows that the solitary wave travels at the same speed under two different grid sizes, implying again that our scheme is grid-independent.

5. Conclusion

We have presented a new fourth-order accurate and conservative stable finite difference scheme for solving a generalized Rosenau–KdV equation in both 1D and 2D. This method shows the second- and fourth-order accuracies in time and space, respectively. The solvability, conservation, stability, mass and energy conservations, and convergence with $O(\tau^2 + h^4)$ have been analysed theoretically. In addition, the results of numerical experiments support the theoretical analysis for convergence rate, and confirm that the obtained scheme gives the well resolution for the generalized Rosenau–KdV equation.

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