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Numerical solution of two-dimensional nonlinear Schrödinger equation using a new two-grid finite element method ☆

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ABSTRACT

A new two-grid finite element scheme is presented for two-dimensional nonlinear Schrödinger equation. One Newton iteration is applied on the fine grid to linearize the nonlinear system using the coarse-grid solution as the initial guess, and furthermore one more linear system on the coarse space is solved. The error estimations of the two-grid solution in the L^2 and H^1 norm are given. It is shown, both theoretically and numerically, that the coarse space can be extremely coarse and two-grid algorithm still achieves optimal approximation as long as the mesh sizes satisfy $H = O(h^{\frac{1}{3}})$.

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1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most important equations of mathematical physics. It has been used extensively in underwater acoustics, quantum mechanics, plasma physics, bimolecular dynamics, nonlinear optics, electromagnetic wave propagation and design of optoelectronic devices. In this paper, we consider the following initial–boundary value problem of NLS equation in two dimension:

$$i\frac{\partial u}{\partial t} - \Delta u + |u|^2 u = 0, \quad (x, y, t) \in \Omega \times (0, T], \tag{1.1}$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \bar{\Omega}, \tag{1.2}$$

$$u(x, y, t) = 0, \quad (x, y) \in \Gamma, \quad 0 \le t \le T, \tag{1.3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, Ω is $[x_l, x_r] \times [y_l, y_r]$, Γ is the boundary of Ω, $u(x, y, t)$ is a complex function, $u_0(x, y)$, is prescribed smooth complex function, and $i^2 = -1$.

Computing the inner product of Eq. (1.1) with u and $\frac{\partial u}{\partial t}$, and then taking the imaginary part and the real part, respectively, the two conservative laws are obtained as follows

$$Q(t) = \|u(\cdot, t)\| = \|u_0\| = Q(0), \tag{1.4}$$

$$E(t) = \|u(\cdot, t)\|_{H^1}^2 + \frac{1}{2}\|u(\cdot, t)\|_{L^4}^4 = E(0). \tag{1.5}$$

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Zhang et al. found that the nonconservative schemes may easily show nonlinear blow-up when studying for NLS equation, so they presented a conservative difference scheme in [1]. Moreover, extensive mathematical and numerical studies have been carried out for the NLS equations in the literature [2–12]. In recent years, much attention has been paid to the finite difference schemes and finite element methods for two-dimensional Schrödinger equations [2,10,13–21].

The two-grid method is presented and analyzed in the work of Xu [22,23] for Galerkin procedures applied to nonlinear elliptic equations. It is a simple but effective algorithm that has been applied to many types of problems, such as nonlinear parabolic equations [24–27], eigenvalue problems [28], incompressible miscible displacement problems [29–32], the Schrödinger equations [10–12,20,21,33]. Jin et al. [10] successfully extended the two-grid finite element method to solve coupled partial differential equations such as linear Schrödinger equation, where the equations on fine grid are decoupled, so that the computational complexity of solving Schrödinger equation is comparable to solving two decoupled Poisson equations on the same fine grid. Wu [11,12] constructed two-grid mixed finite element schemes for nonlinear Schrödinger equations, where a linear and indefinite (the typical nature of mixed finite element) discretization systems are solved on the fine grid. Numerical experiments using these schemes have been shown to be efficient, however, no rigorous error analysis has been conducted. Recently, we [20] extended the two-grid approach given in [10] to time-dependent nonlinear Schrödinger equation, the main idea of our algorithm is to solve a nonlinear coupled system in the very coarse grid and then solve the decoupled linear system on the fine grid. The purpose of this paper is to make a further refinement in the aforementioned process by solving one more linear equation on the coarse space. The additional correction step (which needs very little extra work) improves the accuracy of the algorithm up to one order. It is shown, both theoretically and numerically, that coarse grid can be coarser and the two-grid algorithm still achieves optimal approximation as long as the mesh sizes satisfy $H = O(h^{\frac{1}{3}})$. Under certain assumptions, the two-grid algorithm can retain the same order of approximation accuracy as the Newton iterative algorithm and with improving the computational efficiency.

The remainder of this paper is organized as follows. We present the weak formulation of our model, in Section 2. In Section 3, a simple analysis of the finite element solutions of the model is made. Our main algorithm and its convergence analysis will be advocated in Section 4. In the last Section, numerical results will be discussed.

2. Weak formulation of the problem and preliminaries

Now, $W^{m,p}(\Omega)$ denotes the complex Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha \phi\|_{L^p}^p$. Set $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty}$ and $\|\cdot\| = \|\cdot\|_{0,2}$, and $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ consisting of functions with vanishing trace on $\partial\Omega$. Throughout this paper C denotes a positive constant independent of h, but may have different values at different places.

Let $(\cdot,\cdot)$ be the inner product on $L^2(\Omega) = H^0(\Omega)$ defined by

$$(u, v) = \int_\Omega u(x)\bar{v}(x)dx, \quad \text{for } u, v \in L^2(\Omega),$$

and $\|\cdot\|$ denote the associated L^2 norm.

For any complex-valued function $v = v_1 + iv_2$, let

$$\|v\|_{m,p} = (\|v_1\|_{m,p}^p + \|v_2\|_{m,p}^p)^{\frac{1}{p}},$$
$$\|v\|_{m,\infty} = \|v_1\|_{m,\infty} + \|v_2\|_{m,\infty}.$$

In this paper, we assume that u the solution of (1.1) is sufficiently smooth. Let $\mathcal{T}_h$ be a family of regularity finite element partitions of Ω. Let the corresponding finite element space $M_h \subset H_0^1(\Omega)$. We assume that M_h satisfies the following approximation property: there exists an integer $l \ge 1$ such that

$$\inf_{z_h \in M_h} (\|z - z_h\|_{L^2(\Omega)} + h\|z - z_h\|_{H^1(\Omega)}) \le Ch^{l+1}\|z\|_{H^{l+1}(\Omega)}, \quad \text{for} \quad z \in H^{l+1}(\Omega) \cap H_0^1(\Omega). \tag{2.1}$$

The weak solution of problem (1.1)–(1.3) is defined as follows: Find $u \in H_0^1$ such that

$$i(\frac{\partial u}{\partial t}, \chi) + (\nabla u, \nabla\chi) + (|u|^2 u, \chi) = 0, \quad \text{for } \forall\ \chi \in H_0^1. \tag{2.2}$$

Let $\Delta t = T/N$ and $t^n = n\Delta t$. Define

$$u^n = u(., t^n),\ \partial_t u^n = \frac{u^n - u^{n-1}}{\Delta t} \text{ and } u_h^{n-\frac{1}{2}} = \frac{u_h^{n-1} + u_h^n}{2}.$$

The full discrete procedure of (1.1)–(1.3) at $t = t^n$ is given by the finding of u_h^n such that $u_h(x, 0) = u_h^0$.

$$i(\partial_t u_h^n, \chi) + (\nabla u_h^{n-\frac{1}{2}}, \nabla\chi) + (|u_h^{n-\frac{1}{2}}|^2 u_h^{n-\frac{1}{2}}, \chi) = 0, \quad \text{for } \forall\ \chi \in H_0^1. \tag{2.3}$$

Now, Similar to the proof of Proposition 3.1 in [9], we introduce an existence uniqueness lemma.

Lemma 2.1. *Suppose that* $\|u_h^0\|^2 < \frac{1}{4}$, *then, there exists a unique solution* u_h^n *of the scheme* (2.3) *that satisfies the two conservative laws*

$$Q^n = \|u_h^n\| = \|u_h^{n-1}\| = \cdots = \|u_h^0\|. \tag{2.4}$$

$$E^n = \|u_h^n\|_{H^1}^2 + \frac{1}{2}\|u_h^n u_h^{n+\frac{1}{2}}\|_{L^4}^2 = \cdots = \|u_h^0\|_{H^1}^2 + \frac{1}{2}\|u_h^0 u_h^{\frac{1}{2}}\|_{L^4}^2. \tag{2.5}$$

Lemma 2.2 (*Gronwall's Inequality [28]*)**.** *Suppose that the discrete mesh function* $\{w_n, n = 1, 2, \ldots, N\}$, $N\Delta t = T$, *satisfies the inequality*

$$w_n \le A + \sum_{k=1}^{n} B_k w_k \Delta t, \quad 0 \le n\Delta t \le T,$$

where A *and* B_k $(k = 1, 2, \ldots, N)$ *are nonnegative constants. Then,*

$$\|w_n\|_\infty \le Ae^{2\sum_{k=1}^{N} B_k \Delta t}, \tag{2.6}$$

where Δt *is sufficiently small, such that* $\Delta t(\max_{k=1,2,\ldots,N} B_k) \le \frac{1}{2}$.

3. L^p Error estimate of finite element method

In this section, we demonstrate the L^p error estimate of the finite element solution for u variable. Now, we define the elliptic projection operator $P_h : H_0^1 \longrightarrow M_h$ by

$$(\nabla(P_h u - u), \nabla\chi) + \lambda(P_h u - u, \chi) = 0, \quad \chi \in M_h. \tag{3.1}$$

First, we derive a useful Lemma by using an elliptic problem.

Lemma 3.1. *Suppose that* $P_h u^n$ *is the elliptic projection at* $t = t^n$ *with* $1 \le n \le N$ *defined in* (3.1), *then, for* $2 \le q < \infty$, *we have*

$$\|u^n - P_h u^n\|_{L^p} \le C\|u\|_{W^{l+1,p}} h^{l+1}. \tag{3.2}$$

Proof. Define elliptic operator L as

$$Lu^n = -\nabla \cdot (\nabla u^n) + \lambda u^n, \tag{3.3}$$

and corresponding bilinear form

$$A(u^n, \chi) = (\nabla u^n, \nabla\chi) + (\lambda u^n, \chi), \tag{3.4}$$

where λ is chosen to guarantee the coercivity of the bilinear form $A(u, \chi)$ such that

$$A(u^n, u^n) \ge \gamma\|u^n\|_1^2,$$

where γ is a positive constant. From the formulation (3.1), we know that $P_h u^n$ is in fact the finite element solution of the elliptic problem. In order to prove (3.2), we use a duality argument by considering the auxiliary problem:

$$L\omega = sgn(u^n - P_h u^n)|u^n - P_h u^n|^{p-1}, \ \text{in } \Omega, \tag{3.5}$$

$$\omega = 0, \ \text{on } \partial\Omega. \tag{3.6}$$

is uniquely solvable for $\omega \in L^p$ and holds the following regularity:

$$\|\omega\|_{2,q} \le C\|L\omega\|_q = C\|u^n - P_h u^n\|_p^{p-1}. \tag{3.7}$$

By using (3.1), *Hölder's* inequality, (2.1) and (3.7), we obtain

$$\begin{aligned}
\|u^n - P_h u^n\|_{L^p}^p &= A(u^n - P_h u^n, \omega) \\
&= A(u^n - P_h u^n, \omega - I_h\omega) \\
&\le C\|u^n - P_h u^n\|_{1,p}|\omega - I_h\omega|_{1,q} \\
&\le Ch\|u^n - P_h u^n\|_{1,p}\|\omega\|_{2,q} \\
&\le Ch\|u^n - P_h u^n\|_{1,p}\|u^n - P_h u^n\|_p^{p-1},
\end{aligned}$$

where I_h is an interpolation operator.

From Theorem 8.53 of [34] and the approximation property, we obtain

$$\|u^n - P_h u^n\|_{1,p} \le \inf_{\forall u_h \in M_h} \|u^n - u_h^n\|_{1,p} \le Ch^l\|u\|_{W^{l+1,p}}.$$

Moreover, we have

$$\|u^n - P_h u^n\|_{L^p} \le Ch\|u^n - P_h u^n\|_{1,p} \le Ch^{l+1}\|u\|_{W^{l+1,p}}. \quad \square \tag{3.8}$$

Differentiation in time of (3.1) leads to the additional estimate

$$\|\partial_t(u - P_h u)\|_{L^p} \le M(\|u\|_{l+1} + \|\frac{\partial u}{\partial t}\|_{H^{l+1}})h^{l+1}. \tag{3.9}$$

Now, we will obtain the L^2 error estimate and the H^1 error estimate between the elliptic projection solution and the finite element solution.

Lemma 3.2. *Suppose that $P_h u^n$ is the elliptic projection at $t = t^n$ with $1 \le n \le m \le N$ defined in (3.1)and u_h^n is the Galerkin solution that satisfies Eq. (2.3), then, for $2 \le q < \infty$, we have*

$$\|P_h u^m - u_h^m\| + h\|P_h u^m - u_h^m\|_1 \le C(h^{l+1} + (\Delta t)^2). \tag{3.10}$$

Proof. Adding (3.1) and (2.2), and subtracting (2.3) at $t = t^{n-\frac{1}{2}}$, we obtain

$$\begin{aligned} i(\frac{\partial u^{n-\frac{1}{2}}}{\partial t} - \partial_t u_h^n, \chi) + (\nabla(P_h u^{n-\frac{1}{2}} - u_h^{n-\frac{1}{2}}), \nabla\chi) = &-\lambda(P_h u^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}, \chi) - \frac{1}{8}(|u^{n-1} + u^n|^2 \\ &(u^n + u^{n-1}) - |u_h^n + u_h^{n-1}|^2(u_h^n + u_h^{n-1}), \chi), \end{aligned} \tag{3.11}$$

Let $\rho^n = P_h u^n - u_h^n$, $\eta = u - P_h u$ and choosing $\chi = \rho^{n-\frac{1}{2}}$, then, we rewrite (3.11) as

$$\begin{aligned} i(\partial_t \rho^n, \rho^{n-\frac{1}{2}}) + (\nabla\rho^{n-\frac{1}{2}}, \nabla\rho^{n-\frac{1}{2}}) = &i(\frac{\partial u^{n-\frac{1}{2}}}{\partial t} - \frac{u^n - u^{n-1}}{\Delta t}, \rho^{n-\frac{1}{2}}) - i(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \rho^{n-\frac{1}{2}}) \\ &+ \lambda(\eta^{n-\frac{1}{2}}, \rho^{n-\frac{1}{2}}) - (f(u^{n-\frac{1}{2}}) - f(u_h^{n-\frac{1}{2}}), \rho^{n-\frac{1}{2}}), \end{aligned} \tag{3.12}$$

where $f(u) = |u|^2 u$.

Taking the imaginary part, we obtain

$$\frac{\|\rho^n\|^2 - \|\rho^{n-1}\|^2}{2\Delta t} = (F_1 + F_2 + F_3 + F_4, \rho^{n-\frac{1}{2}}). \tag{3.13}$$

For each term on the right-hand side of (3.13), we bound them as follows:

$$|(F_1, \rho^{n-\frac{1}{2}})| = Re(\frac{\partial u^{n-\frac{1}{2}}}{\partial t} - \partial_t u^n, \rho^{n-\frac{1}{2}}) \le C\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \|\rho^{n-\frac{1}{2}}\|^2, \tag{3.14}$$

$$\begin{aligned} |(F_2, \rho^{n-\frac{1}{2}})| =& Re|(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \rho^{n-\frac{1}{2}})| \le C\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} \|\frac{\partial \eta}{\partial t}\| dt \cdot \|\rho^{n-\frac{1}{2}}\| \\ \le& C\frac{1}{\Delta t}\|\frac{\partial \eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} + \|\rho^{n-\frac{1}{2}}\|^2, \end{aligned} \tag{3.15}$$

$$|(F_3, \rho^{n-\frac{1}{2}})| = Im|\lambda(\eta^{n-\frac{1}{2}}, \rho^{n-\frac{1}{2}})| \le |\lambda|\|\eta^{n-\frac{1}{2}}\|^2 + \|\rho^{n-\frac{1}{2}}\|^2 \le Ch^{2l+2} + \|\rho^{n-\frac{1}{2}}\|^2, \tag{3.16}$$

$$\begin{aligned} |(F_4, \rho^{n-\frac{1}{2}})| =& Im|-\frac{1}{8}(|u^{n-1} + u^n|^2(u^n + u^{n-1}) - |u_h^n + u_h^{n-1}|^2(u_h^n + u_h^{n-1}), \rho^{n-\frac{1}{2}})| \\ \le& (|u^{n-\frac{1}{2}}|^2(\rho^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}) + [|u^{n-\frac{1}{2}}|^2 - |u_h^{n-\frac{1}{2}}|^2]u_h^{n-\frac{1}{2}}, \rho^{n-\frac{1}{2}}) \\ \le& C((\rho^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}) + [|\rho^{n-\frac{1}{2}}| + |\eta^{n-\frac{1}{2}}|], \rho^{n-\frac{1}{2}}) \\ \le& C((\|\rho^{n-\frac{1}{2}}\|)^2 + h^{2l+2}), \end{aligned} \tag{3.17}$$

where we make the induction hypothesis that

$$\|u_h^n\|_\infty \le C. \tag{3.18}$$

Then, from (3.13)–(3.17), we have

$$\begin{aligned} \frac{\|\rho^n\|^2 - \|\rho^{n-1}\|^2}{2\Delta t} \le& C[\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \frac{1}{\Delta t}\|\frac{\partial \eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ \|\rho^n\|^2 + \|\rho^{n-1}\|^2 + h^{2l+2}], \end{aligned} \tag{3.19}$$

and multiplying $2\Delta t$ on both sides of the above inequality and summing from $n = 1$ to m and noting that $R_h u_0 = u_0$, we have

$$\|\rho^m\|^2 \le C[h^{2l+2} + (\Delta t)^4] + C\Delta t\sum_{n=1}^{m}\|\rho^n\|^2. \tag{3.20}$$

According to Lemma 2.2, when Δt is small enough, it follows that

$$\|\rho^m\|^2 \le C[h^{2l+2} + (\Delta t)^4]. \tag{3.21}$$

Note that (3.2) and (3.21) imply that the induction hypothesis (3.18) holds for small h, so that the entire argument is validated.

Now, we shall analyze the H^1 error estimation between the elliptic projection solution and the finite element solution. Choosing $\chi = \partial_t \rho^n$ and taking the real part for (3.11), then, we have

$$\frac{\|\nabla\rho^n\|^2 - \|\nabla\rho^{n-1}\|^2}{2\Delta t} = (N_1 + N_2 + N_3 + N_4, \partial_t\rho^n). \tag{3.22}$$

For each term on the right-hand side of (3.22), we bound them as follows:

$$|(N_1, \partial_t\rho^n)| = Im(\frac{\partial u^{n-\frac{1}{2}}}{\partial t} - \partial_t u^n, \partial_t\rho^n) \le C\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \varepsilon\|\partial_t\rho^n\|^2, \tag{3.23}$$

$$\begin{aligned} |(N_2, \partial_t\rho^n)| &= Im|(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \partial_t\rho^n)| \le C\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\|\frac{\partial\eta}{\partial t}\|dt \cdot \|\partial_t\rho^n\| \\ &\le C\frac{1}{\Delta t}\|\frac{\partial\eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} + \varepsilon\|\partial_t\rho^n\|^2, \end{aligned} \tag{3.24}$$

$$|(N_3, \partial_t\rho^n)| = Re|\lambda(\eta^{n-\frac{1}{2}}, \partial_t\rho^n)| \le |\lambda|\|\eta^{n-\frac{1}{2}}\|^2 + \|\rho^{n-\frac{1}{2}}\|^2 \le Ch^{2l+2} + \varepsilon\|\partial_t\rho^n\|^2, \tag{3.25}$$

$$\begin{aligned} |(N_4, \partial_t\rho^n)| &= Re|-\frac{1}{8}(|u^{n-1}+u^n|^2(u^n+u^{n-1}) - |u_h^n + u_h^{n-1}|^2(u_h^n + u_h^{n-1}), \partial_t\rho^n)| \\ &\le (|u^{n-\frac{1}{2}}|^2(\rho^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}) + [|u^{n-\frac{1}{2}}|^2 - |u_h^{n-\frac{1}{2}}|^2]u_h^{n-\frac{1}{2}}, \partial_t\rho^n) \\ &\le C((\rho^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}) + [|\rho^{n-\frac{1}{2}}| + |\eta^{n-\frac{1}{2}}|], \partial_t\rho^n) \\ &\le C(h^{2l+2} + (\Delta t)^4 + \varepsilon\|\partial_t\rho^n\|^2), \end{aligned} \tag{3.26}$$

where we make the induction hypothesis that$\|u_h^n\|_\infty \le C$ and (3.21). Then, from (3.22)–(3.26), we have

$$\begin{aligned} \frac{\|\nabla\rho^n\|^2 - \|\nabla\rho^{n-1}\|^2}{2\Delta t} \le& C[\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \frac{1}{\Delta t}\|\frac{\partial\eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ 4\varepsilon\|\partial_t\rho^n\|^2 + h^{2l+2} + (\Delta t)^4]. \end{aligned} \tag{3.27}$$

Choosing $\chi = \partial_t\rho^n$ and taking the imaginary part for (3.11), then, a similar estimate may be obtained for $\|\partial_t\rho^n\|$.

$$\begin{aligned} \|\partial_t\rho^n\|^2 \le& C[\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \frac{1}{\Delta t}\|\frac{\partial\eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ 4\varepsilon\|\partial_t\rho^n\|^2 + h^{2l+2} + (\Delta t)^4]. \end{aligned} \tag{3.28}$$

Choosing $\chi = \rho^n$ for (3.1) at $t = t^n$, we have

$$\begin{aligned} 0 = (\nabla\eta^n, \nabla\rho^n) + \lambda(\eta^n, \rho^n) &\le C(\|\nabla\eta^n\|\|\nabla\rho^n\| + \|\eta^n\|\|\rho^n\|) \\ &\le (h^{2l} + (\Delta t)^4 + \|\nabla\rho^n\|^2). \end{aligned} \tag{3.29}$$

Then, from (3.28)–(3.29), we have

$$\begin{aligned} \frac{\|\nabla\rho^n\|^2 - \|\nabla\rho^{n-1}\|^2}{2\Delta t} + (1-8\varepsilon)\|\partial_t\rho^n\|^2 \le& C[\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \frac{1}{\Delta t}\|\frac{\partial\eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ \|\nabla\rho^n\|^2 + h^{2l} + (\Delta t)^4], \end{aligned} \tag{3.30}$$

and multiplying $2\Delta t$ on both sides of the above inequality and summing from $n = 1$ to m and noting that $R_h u_0 = u_0$, and assuming ε is small enough, we have

$$\|\nabla\rho^m\|^2 \le C[h^{2l} + (\Delta t)^4] + C\Delta t\sum_{m=1}^{m}\|\nabla\rho^n\|^2. \tag{3.31}$$

According to Lemma 2.2, when Δt is small enough, it follows that

$$\|\nabla\rho^m\|^2 \le C[h^{2l} + (\Delta t)^4]. \tag{3.32}$$

Moreover, we have

$$\|\rho^m\|_1^2 = \|\nabla\rho^m\|^2 + \|\rho^m\|^2 \le C[h^{2l} + (\Delta t)^4]. \quad \square \tag{3.33}$$

4. Two-grid algorithm and error estimate

In this section, we will present the two-grid algorithm and analyze the convergence accuracy of schemes. The fundamental ingredient in this scheme is another finite element space $M_H(\subset M_h)$ $(h < H < 1)$ defined on a coarser quasi-uniform triangulation or rectangulation of Ω. For $n \geq 1$, let in (2.3) $U_h^n = V_h^n + iW_h^n$, $V_h^n, W_h^n \in RM_h$, where RM_h denotes the space of the real-valued elements of M_h. Now, we give the two-grid algorithm which based on one Newton iteration on the fine grid and a further coarse grid correction. The two-grid algorithm mainly contains three steps as follows.

Step1: On the coarse grid $\mathcal{T}_H$, we solve the following nonlinear coupling system for $v_H^n, w_H^n \in RM_H$

$$(\partial_t v_H^n, \chi) = -(\nabla w_H^{n-\frac{1}{2}}, \nabla\chi) - ([(v_H^{n-\frac{1}{2}})^2 + (w_H^{n-\frac{1}{2}})^2]w_H^{n-\frac{1}{2}}, \chi), \quad \text{for } \forall\ \chi \in RM_H, \tag{4.1}$$

$$(\partial_t w_H^n, \chi) = (\nabla v_H^{n-\frac{1}{2}}, \nabla\chi) + ([(v_H^{n-\frac{1}{2}})^2 + (w_H^{n-\frac{1}{2}})^2]v_H^{n-\frac{1}{2}}, \chi), \quad \text{for } \forall\ \chi \in RM_H, \tag{4.2}$$

Step2: On the fine grid $\mathcal{T}_h$, we compute $V_h^n, W_h^n \in RM_h$ to satisfy the following linear decoupling system

$$\begin{aligned}(\partial_t V_h^n, \varphi) =& -\frac{1}{2}(\nabla(W_h^n + W_h^{n-1}), \nabla\varphi) - \frac{1}{4}((v_H^n + V_h^{n-1})(w_H^n + W_h^{n-1})(V_h^n - v_H^n), \varphi)\\ &- \frac{1}{8}([(v_H^n + V_h^{n-1})^2 + 3(w_H^n + W_h^{n-1})^2](W_h^n - w_H^n), \varphi)\\ &- \frac{1}{8}([(v_H^n + V_h^{n-1})^2 + (w_H^n + W_h^{n-1})^2](w_H^n + W_h^{n-1}), \varphi), \quad \text{for } \forall\ \varphi \in RM_h,\end{aligned} \tag{4.3}$$

$$\begin{aligned}(\partial_t W_h^n, \varphi) =& \frac{1}{2}(\nabla(V_h^n + V_h^{n-1}), \nabla\varphi) + \frac{1}{4}((w_H^n + W_h^{n-1})(v_H^n + V_h^{n-1})(W_h^n - w_H^n), \varphi)\\ &+ \frac{1}{8}([(w_H^n + W_h^{n-1})^2 + 3(v_H^n + V_h^{n-1})^2](V_h^n - v_H^n), \varphi)\\ &+ \frac{1}{8}([(w_H^n + W_h^{n-1})^2 + (v_H^n + V_h^{n-1})^2](v_H^n + V_h^{n-1}), \varphi), \quad \text{for } \forall\ \varphi \in RM_h.\end{aligned} \tag{4.4}$$

Step3: On the coarse grid $\mathcal{T}_H$, we solve the following linear decoupling system for $e_H^n, \bar{e}_H^n \in RM_H$

$$\begin{aligned}(\partial_t e_H^n, \chi) =& -\frac{1}{2}(\nabla(\bar{e}_H^n + \bar{e}_H^{n-1}), \nabla\chi) - \frac{1}{4}((v_H^n + V_h^{n-1})(w_H^n + W_h^{n-1})e_H^n, \chi)\\ &- \frac{1}{8}([(v_H^n + V_h^{n-1})^2 + 3(w_H^n + W_h^{n-1})^2]\bar{e}_H^n, \chi)\\ &- \frac{1}{8}((w_H^n + W_h^{n-1})(V_h^n - v_H^n)^2, \chi) - \frac{3}{8}((w_H^n + W_h^{n-1})(W_h^n - w_H^n)^2, \chi)\\ &- \frac{1}{4}((v_H^n + V_h^{n-1})(W_h^n - w_H^n)(V_h^n - v_H^n), \chi), \quad \text{for } \forall\ \chi \in RM_H,\end{aligned} \tag{4.5}$$

$$\begin{aligned}(\partial_t \bar{e}_H^n, \chi) =& \frac{1}{2}(\nabla(e_h^n + e_h^{n-1}), \nabla\chi) + \frac{1}{4}((w_H^n + W_h^{n-1})(v_H^n + V_h^{n-1})\bar{e}_H^n, \chi)\\ &+ \frac{1}{8}([(w_H^n + W_h^{n-1})^2 + 3(v_H^n + V_h^{n-1})^2]e_H^n, \chi)\\ &+ \frac{1}{8}((v_H^n + V_h^{n-1})(V_h^n - v_H^n)^2, \chi) + \frac{3}{8}((v_H^n + V_h^{n-1})(W_h^n - w_H^n)^2, \chi)\\ &+ \frac{1}{4}((w_H^n + W_h^{n-1})(W_h^n - w_H^n)(V_h^n - v_H^n), \chi), \quad \text{for } \forall\ \chi \in RM_H,\end{aligned} \tag{4.6}$$

set

$$\tilde{v}_h^n = V_h^n + e_H^n \in RM_h,$$
$$\tilde{w}_h^n = W_h^n + \bar{e}_H^n \in RM_h.$$

For simplify, define K so that for $g : R^2 \to R,\ g(x, y) = (x^2 + y^2)y$, there holds

$$max|\partial_1 g(x, y)| + max|\partial_2 g(x, y)| \le K,$$

where we set $\partial_1 g = g_x,\ \partial_2 g = g_y$.

From Lemmas 3.1, 3.2 and (3.8), we easily have the following result.

Lemma 4.1. *Suppose that u^n is the exact solution that satisfies Eq. (2.2) at $t = t^n$, and $u_H^n = v_H^n + iw_H^n$ is the two-grid solution that satisfies Eqs. (4.1) and (4.2), and choosing $R_H v_0 = v_H^0$, $R_H w_0 = w_H^0$, respectively, then, for $1 \le n \le N$, we have*

$$\|u^n - u_H^n\| + H\|u^n - u_H^n\|_1 \le C((\Delta t)^2 + H^{l+1}). \tag{4.7}$$

Now, we start to analyze the error estimation of two-grid solution of step 2. In the following analysis, we have the following result.

Theorem 4.1. *Suppose that u^n is the exact solution that satisfies Eq. (2.2) at $t = t^n$, and $U_h^n = V_h^n + iW_h^n$ is the two-grid solution that satisfies Eqs. (4.3) and (4.4). If we select $R_h u_0 = U_h^0$, $\Delta t = o(H^2)$, then, for $1 \le n \le m \le N$, we have*

$$\|u^m - U_h^m\| \le C(h^{l+1} + H^{2l+2} + (\Delta t)^2). \tag{4.8}$$

Proof. We use the elliptic projection as an intermediate variable to assist the proof. Adding (2.2) and (3.1), then, the equation can be rewritten as a system of two equations for $v^{n-\frac{1}{2}}$ and $w^{n-\frac{1}{2}}$, and subtracting (4.3) at $t = t^{n-\frac{1}{2}}$ from the first equation of the system, we obtain the following relationship.

$$\begin{aligned}(\frac{\partial v^{n-\frac{1}{2}}}{\partial t} - \partial_t V_h^n, \varphi) =& -(\nabla(P_h w^{n-\frac{1}{2}} - W_h^{n-\frac{1}{2}}), \nabla\varphi) - \lambda(P_h w^{n-\frac{1}{2}} - w^{n-\frac{1}{2}}, \varphi)\\ &- (g(v^{n-\frac{1}{2}}, w^{n-\frac{1}{2}}) - g(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})\\ &- \frac{1}{2}\partial_1 g(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})(V_h^n - v_H^n)\\ &- \frac{1}{2}\partial_2 g(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})(W_h^n - w_H^n), \varphi),\end{aligned} \tag{4.9}$$

where $\bar v_H^{n-\frac{1}{2}} = \frac{v_H^n + V_h^{n-1}}{2},\ \bar w_H^{n-\frac{1}{2}} = \frac{w_H^n + W_h^{n-1}}{2}$.

Let $\xi^n = P_h v^n - V_h^n$, $\zeta^n = P_h w^n - W_h^n$, $\eta_1^{n-\frac{1}{2}} = v^{n-\frac{1}{2}} - P_h v^{n-\frac{1}{2}}$, $\eta_2^{n-\frac{1}{2}} = w^{n-\frac{1}{2}} - P_h w^{n-\frac{1}{2}}$, then we have

$$\begin{aligned}(\partial_t \xi^n, \varphi) + (\nabla\zeta^{n-\frac{1}{2}}, \nabla\varphi) =& -(\frac{\partial v^{n-\frac{1}{2}}}{\partial t} - \partial_t v^n, \varphi) - (\partial_t \eta_1^n, \varphi) + \lambda(\eta_2^{n-\frac{1}{2}}, \varphi)\\ &- \frac{1}{2}(\partial_1 g(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})(v^n - V_h^n)\\ &+ \partial_2 g(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})(w^n - W_h^n), \varphi)\\ &+ \frac{1}{8}(\partial_1^2 g(v^*, w^*)(v^n - v_H^n)^2 + \partial_2^2 g(v^*, w^*)(w^n - w_H^n)^2\\ &+ 2\partial_1\partial_2 g(v^*, w^*)(v^n - v_H^n)(w^n - w_H^n), \varphi)\\ =&(Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \varphi),\end{aligned} \tag{4.10}$$

where we use two order Taylor'expansion at a point $(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})$ for $g(v^{n-\frac{1}{2}}, w^{n-\frac{1}{2}})$. Let $\varphi = \xi^{n-\frac{1}{2}}$, then we have

$$|(Q_1, \xi^{n-\frac{1}{2}})| \le K\|\frac{\partial^3 v}{\partial t^3}\|^2_{L^2(t^{n-1}, t^n, L^2)}(\Delta t)^3 + \|\xi^{n-\frac{1}{2}}\|^2, \tag{4.11}$$

$$|(Q_2, \xi^{n-\frac{1}{2}})| \le \frac{K}{\Delta t}\|\frac{\partial \eta_1}{\partial t}\|^2_{L^2(t^{n-1}, t^n, L^2)} + \|\xi^{n-\frac{1}{2}}\|^2, \tag{4.12}$$

$$|(Q_3, \xi^{n-\frac{1}{2}})| \le C\|\eta_2^{n-\frac{1}{2}}\|^2 + \|\xi^{n-\frac{1}{2}}\|^2, \tag{4.13}$$

$$\begin{aligned}|(Q_4, \xi^{n-\frac{1}{2}})| &\le C|(\partial_1 g(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})(v^n - V_h^n) + \partial_2 g(\bar v_H^{n-\frac{1}{2}}, \bar w_H^{n-\frac{1}{2}})(w^n - W_h^n), \xi^{n-\frac{1}{2}})|\\ &\le K(\|v^n - V_h^n\| + \|w^n - W_h^n\|)\|\xi^{n-\frac{1}{2}}\|\\ &\le C(\|v^n - P_h v^n + P_h v^n - V_h^n\| + \|w^n - P_h w^n + P_h w^n - W_h^n\|)\cdot\|\xi^{n-\frac{1}{2}}\|\\ &\le C(h^{2l+2} + (\Delta t)^4 + \|\xi^n\|^2 + \|\zeta^n\|^2 + \|\xi^{n-\frac{1}{2}}\|^2),\end{aligned} \tag{4.14}$$

$$\begin{aligned}|(Q_5, \xi^{n-\frac{1}{2}})| &\le C|(\partial_1^2 g(v^*, w^*)(v^n - v_H^n)^2 + \partial_2^2 g(v^*, w^*)(w^n - w_H^n)^2 \\ &+ 2\partial_1\partial_2 g(v^*, w^*)(v^n - v_H^n)(w^n - w_H^n), \xi^{n-\frac{1}{2}})| \\ &\le C(\|v^n - v_H^n\|_4^4 + \|w^n - w_H^n\|_4^4 + \|\xi^{n-\frac{1}{2}}\|^2) \\ &\le C(\|v^n - P_h v^n + P_h v^n - v_H^n\|_4^4 + \|w^n - P_h w^n + P_h w^n - w_H^n\|_4^4 + \|\xi^{n-\frac{1}{2}}\|^2) \\ &\le C(h^{2l+2} + H^{-2}H^{4l+4} + (\Delta t)^4 + \|\xi^{n-\frac{1}{2}}\|^2),\end{aligned} \tag{4.15}$$

where the proofs of Q_1, Q_2, Q_3 are similar to that of Lemma 3.2.

Then, from (4.10)–(4.15), we have

$$\begin{aligned}\frac{\|\xi^n\|^2 - \|\xi^{n-1}\|}{2\Delta t} + (\nabla\zeta^{n-\frac{1}{2}}, \nabla\xi^{n-\frac{1}{2}}) \le& C(\|\frac{\partial^3 v}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \frac{1}{\Delta t}\|\frac{\partial \eta_1}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^n\|^2 \\ &+ \|\zeta^n\|^2 + \|\zeta^{n-1}\|^2 + h^{2l+2} + H^{-2}H^{4l+4}),\end{aligned} \tag{4.16}$$

In an entirely analogous manner a similar estimate may be obtained for $R_h w^n - W_h^n$.

$$\begin{aligned}\frac{\|\zeta^n\|^2 - \|\zeta^{n-1}\|}{2\Delta t} - (\nabla\xi^{n-\frac{1}{2}}, \nabla\zeta^{n-\frac{1}{2}}) \le& C(\|\frac{\partial^3 w}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \frac{1}{\Delta t}\|\frac{\partial \eta_2}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^n\|^2 \\ &+ \|\zeta^n\|^2 + \|\zeta^{n-1}\|^2 + h^{2l+2} + H^{-2}H^{4l+4}),\end{aligned} \tag{4.17}$$

Adding (4.16) and (4.17), we obtain

$$\begin{aligned}\frac{\|\mu^n\|^2 - \|\mu^{n-1}\|}{2\Delta t} \le& C(\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3 + \frac{1}{\Delta t}\|\frac{\partial \eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ \|\mu^n\|^2 + \|\mu^{n-1}\|^2 + h^{2l+2} + H^{-2}H^{4l+4}),\end{aligned} \tag{4.18}$$

where $\|\mu^n\| = \|\xi^n\| + \|\zeta^n\|$.

Multiplying $2\Delta t$ on both sides of this inequality and summing from $n = 1$ to $n = m \le N$, we obtain the error inequality

$$\begin{aligned}\|\mu^m\|^2 \le& C((\Delta t)^4 \sum_{n=1}^m \|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)} + \sum_{n=1}^m \|\frac{\partial \eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ &+ h^{2l+2} + H^{4l+4}) + \Delta t \sum_{n=1}^m \|\mu^n\|^2 \\ \le& C(h^{2l+2} + H^{4l+4} + (\Delta t)^4) + \Delta t \sum_{n=1}^m \|\mu^n\|^2,\end{aligned} \tag{4.19}$$

By using discrete Gronwall inequality, we have

$$\|\mu^m\|^2 \le C(h^{2l+2} + H^{4l+4} + (\Delta t)^4). \tag{4.20}$$

Then, we have

$$\|\mu^m\| \le C(h^{l+1} + H^{2l+2} + (\Delta t)^2). \tag{4.21}$$

From Lemma 3.1 and (4.21), the theorem can derive by using the triangle inequality. □

Remark 1. As in the argument of Lemma 3.2 and Theorem 4.1, we can also derive H^1 error estimate between the exact solution and two-grid solution of step 2: $\|u^n - U_h^n\|_1 \le C(h^l + H^{2l} + (\Delta t)^2)$.

Now, we can prove the following main theorem.

Theorem 4.2. *Suppose that u^n is the exact solution that satisfies Eq. (2.2) at $t = t^n$, and $\tilde{u}_h^n = \tilde{v}_h^n + i\tilde{w}_h^n$ is the two-grid solution that satisfies Eqs. (4.5) and (4.6). If we set $R_h u_0 = \tilde{u}_h^0$, $\Delta t = o(H^4)$, then, for $1 \le n \le m \le N$, we have*

$$\|u^m - \tilde{u}_h^m\| \le C(h^{l+1} + H^{3l+3} + (\Delta t)^2). \tag{4.22}$$

Proof. Adding (2.2) and (3.1), then, the equation can be rewritten as a system of two equations for $v^{n-\frac{1}{2}}$ and $w^{n-\frac{1}{2}}$, and subtracting (4.3),(4.5) at $t = t^{n-\frac{1}{2}}$ from the first equation of the system, we get the following relationship.

$$\begin{aligned}(\frac{\partial v^{n-\frac{1}{2}}}{\partial t} - \partial_t \tilde{v}_h^n, \chi) =& -(\nabla(P_h w^{n-\frac{1}{2}} - \tilde{w}_h^{n-\frac{1}{2}}), \nabla\chi) - \lambda(P_h w^{n-\frac{1}{2}} - w^{n-\frac{1}{2}}, \chi)\\ &- (g(v^{n-\frac{1}{2}}, w^{n-\frac{1}{2}}) - g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}}) - \frac{1}{2}\partial_1 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(\tilde{v}_h^n - v_H^n)\\ &- \frac{1}{2}\partial_2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(\tilde{w}_h^n - w_H^n), \chi) + \frac{1}{8}(\partial_1^2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(V_h^n - v_H^n)^2\\ &+ \partial_2^2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(W_h^n - w_H^n)^2\\ &+ 2\partial_1\partial_2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(V_h^n - v_H^n)(W_h^n - w_H^n), \chi),\end{aligned} \tag{4.23}$$

where $\bar{v}_H^{n-\frac{1}{2}} = \frac{v_H^n + V_h^{n-1}}{2}$, $\bar{w}_H^{n-\frac{1}{2}} = \frac{w_H^n + W_h^{n-1}}{2}$.

Let $\xi^n = P_h v^n - \tilde{v}_h^n$, $\zeta^n = P_h w^n - \tilde{w}_h^n$, $\eta_1^{n-\frac{1}{2}} = v^{n-\frac{1}{2}} - P_h v^{n-\frac{1}{2}}$, $\eta_2^{n-\frac{1}{2}} = w^{n-\frac{1}{2}} - P_h w^{n-\frac{1}{2}}$, then we have

$$\begin{aligned}(\partial_t \xi^n, \chi) + (\nabla\zeta^{n-\frac{1}{2}}, \nabla\chi) =& -(\frac{\partial v^{n-\frac{1}{2}}}{\partial t} - \partial_t v^n, \chi) - (\partial_t \eta_1^n, \chi) + \lambda(\eta_2^{n-\frac{1}{2}}, \chi)\\ &- \frac{1}{2}(\partial_1 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(v^n - \tilde{v}_h^n) + \partial_2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(w^n - \tilde{w}_h^n), \chi)\\ &+ \frac{1}{8}(\partial_1^2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(v^n - V_h^n)[(V_h^n - v_H^n) + (v^n - v_H^n)]\\ &+ \partial_2^2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(w^n - W_h^n)[(W_h^n - w_H^n) + (w^n - w_H^n)]\\ &+ 2\partial_1\partial_2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})[(v^n - V_h^n)(w^n - w_H^n)\\ &+ (w^n - W_h^n)(v^n - V_H^n) - (v^n - V_h^n)(w^n - W_h^n)], \chi)\\ &+ \frac{1}{48}(\partial_1^3 g(v^*, w^*)(v^n - v_H^n)^3 + \partial_2^3 g(v^*, w^*)(w^n - w_H^n)^3\\ &+ 3\partial_1^2\partial_2 g(v^*, w^*)(v^n - v_H^n)^2(w^n - w_H^n)\\ &+ 3\partial_1\partial_2^2 g(v^*, w^*)(v^n - v_H^n)(w^n - w_H^n)^2, \chi)\\ =&(M_1 + M_2 + M_3 + M_4 + M_5 + M_6, \chi),\end{aligned} \tag{4.24}$$

where we use three order Taylor'expansion at a point $(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})$ for $g(v^{n-\frac{1}{2}}, w^{n-\frac{1}{2}})$.

Let $\chi = \xi^{n-\frac{1}{2}}$, then we have

$$\begin{aligned}|(M_1, \xi^{n-\frac{1}{2}})| &\le K\|\frac{\partial^3 v}{\partial t^3}\|^2_{L^2(t^{n-1}, t^n, L^2)}(\Delta t)^3 + \|\xi^{n-\frac{1}{2}}\|^2,\\ |(M_2, \xi^{n-\frac{1}{2}})| &\le \frac{K}{\Delta t}\|\frac{\partial\eta_1}{\partial t}\|^2_{L^2(t^{n-1}, t^n, L^2)} + \|\xi^{n-\frac{1}{2}}\|^2,\\ |(M_3, \xi^{n-\frac{1}{2}})| &\le C\|\eta_2^{n-\frac{1}{2}}\|^2 + \|\xi^{n-\frac{1}{2}}\|^2,\end{aligned}$$

where the proofs of M_1, M_2, M_3 are similar to that of Theorem 4.1.

$$\begin{aligned}|(M_4, \xi^{n-\frac{1}{2}})| &\le C|(\partial_1 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(v_h^n - \tilde{v}_h^n) + \partial_2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(w^n - \tilde{w}_h^n), \xi^{n-\frac{1}{2}})|\\ &\le K(\|v^n - \tilde{v}_h^n\| + \|w^n - \tilde{w}_h^n\|)\|\xi^{n-\frac{1}{2}}\|\\ &\le C(\|v^n - P_h v^n + P_h v^n - \tilde{v}_h^n\| + \|w^n - P_h w^n + P_h w^n - \tilde{w}_h^n\|)\cdot\|\xi^{n-\frac{1}{2}}\|\\ &\le C(h^{2l+2} + (\Delta t)^4 + \|\xi^n\|^2 + \|\zeta^n\|^2 + \|\xi^{n-\frac{1}{2}}\|^2).\\ |(M_5, \xi^{n-\frac{1}{2}})| &\le C|(\partial_1^2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(v^n - V_h^n)[(V_h^n - v_H^n) + (v^n - v_H^n)]\\ &+ \partial_2^2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})(w^n - W_h^n)[(W_h^n - w_H^n) + (w^n - w_H^n)]\\ &+ 2\partial_1\partial_2 g(\bar{v}_H^{n-\frac{1}{2}}, \bar{w}_H^{n-\frac{1}{2}})[(v^n - V_h^n)(w^n - w_H^n)\\ &+ (w^n - W_h^n)(v^n - v_H^n) - (v^n - V_h^n)(w^n - W_h^n)], \xi^{n-\frac{1}{2}}\ \|^2)|\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{4.25}$$

Table 1
Error and CPU time of Newton iterative method.

H	h	$\|v^n - \tilde{v}_h^n\|$	Rate	$\|v^n - \tilde{v}_h^n\|_1$	Rate	$\|w^n - \tilde{w}_h^n\|$	Rate	$\|w^n - \tilde{w}_h^n\|_1$	Rate	CPU time
$\frac{1}{2}$	$\frac{1}{8}$	6.0926e−1		1.7244		2.2697e−2		1.8530e−1		9.44 s
$\frac{1}{4}$	$\frac{1}{64}$	9.6175e−4	2	2.1700e−1	1	3.5859e−4	2	2.1515e−2	1	818.33 s
$\frac{1}{8}$	$\frac{1}{512}$	1.5208e−5	2	2.7125e−1	1	5.4801e−6	2	2.6893e−3	1	343680.86 s

Table 2
Error and CPU time of two-grid method.

H	h	$\|v^n - \tilde{v}_h^n\|$	Rate	$\|v^n - \tilde{v}_h^n\|_1$	Rate	$\|w^n - \tilde{w}_h^n\|$	Rate	$\|w^n - \tilde{w}_h^n\|_1$	Rate	CPU time
$\frac{1}{2}$	$\frac{1}{8}$	6.0926e−1		1.7244		2.2697e−2		1.8530e−1		7.98 s
$\frac{1}{4}$	$\frac{1}{64}$	9.6175e−4	2	2.1700e−1	1	3.5859e−4	2	2.1515e−2	1	485.87 s
$\frac{1}{8}$	$\frac{1}{512}$	1.5208e−5	2	2.7125e−1	1	5.4801e−6	2	2.6893e−3	1	194060.54 s

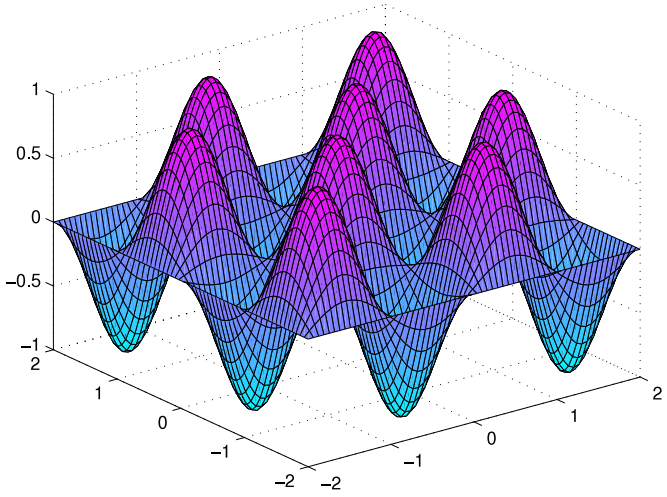


Fig. 1. Real part of v t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

As for I_1, I_2, I_3, it follows from Lemma 4.1 and Theorem 4.1 that

$$\begin{aligned}|(I_1, \xi^{n-\frac{1}{2}})| &\le C(\|v^n - \tilde{v}_h^n\|_4(\|v^n - \tilde{v}_h^n\|_4 + 2\|v^n - v_H^n\|_4)\|\xi^{n-\frac{1}{2}}\|)\\ &\le CH^{-2}(h^{4l+4} + H^{8l+8} + (\Delta t)^8) + \|\xi^{n-\frac{1}{2}}\|^2,\end{aligned} \tag{4.26}$$

$$\begin{aligned}|(I_2, \xi^{n-\frac{1}{2}})| &\le C(\|w^n - W_h^n\|_4(\|w^n - W_h^n\|_4 + 2\|w^n - w_H^n\|_4)\|\xi^{n-\frac{1}{2}}\|)\\ &\le CH^{-2}(h^{4l+4} + H^{8l+8} + (\Delta t)^8) + \|\xi^{n-\frac{1}{2}}\|^2,\end{aligned} \tag{4.27}$$

$$\begin{aligned}|(I_3, \xi^{n-\frac{1}{2}})| &\le C(\|v^n - V_h^n\|_4\|w^n - w_H^n\|_4 + \|w^n - W_h^n\|_4\|v^n - v_H^n\|_4\\ &\quad + \|v^n - V_h^n\|_4\|w^n - W_h^n\|_4)\|\xi^{n-\frac{1}{2}}\|\\ &\le CH^{-2}(h^{4l+4} + H^{8l+8} + (\Delta t)^8) + \|\xi^{n-\frac{1}{2}}\|^2.\end{aligned} \tag{4.28}$$

It follows from (4.26)–(4.28) that

$$|(M_5, \xi^{n-\frac{1}{2}})| \le CH^{-2}(h^{4l+4} + H^{8l+8} + (\Delta t)^8) + 3\|\xi^{n-\frac{1}{2}}\|^2. \tag{4.29}$$

$$\begin{aligned}|(M_6, \xi^{n-\frac{1}{2}})| &\le C|(\partial_1^3 g(v^*, w^*)(v^n - v_H^n)^3 + \partial_2^3 g(v^*, w^*)(w^n - w_H^n)^3\\ &\quad + 3\partial_1^2\partial_2 g(v^*, w^*)(v^n - v_H^n)^2(w^n - w_H^n)\\ &\quad + 3\partial_1\partial_2^2 g(v^*, w^*)(v^n - v_H^n)(w^n - w_H^n)^2, \xi^{n-\frac{1}{2}})|\\ &\le C(\|v^n - v_H^n\|_6^6 + \|v^n - v_H^n\|_6^4\|w^n - w_H^n\|_6^2\\ &\quad + \|v^n - v_H^n\|_6^2\|w^n - w_H^n\|_6^4 + \|w^n - w_H^n\|_6^6 + \|\xi^{n-\frac{1}{2}}\|^2)\\ &\le CH^{-4}(H^{6l+6} + (\Delta t)^{12}) + \|\xi^{n-\frac{1}{2}}\|^2,\end{aligned} \tag{4.30}$$

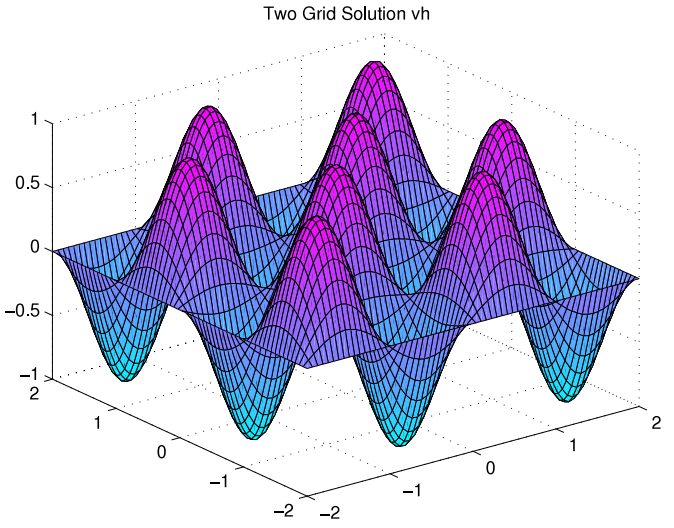


Fig. 2. Real part of $\tilde{v}$ t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

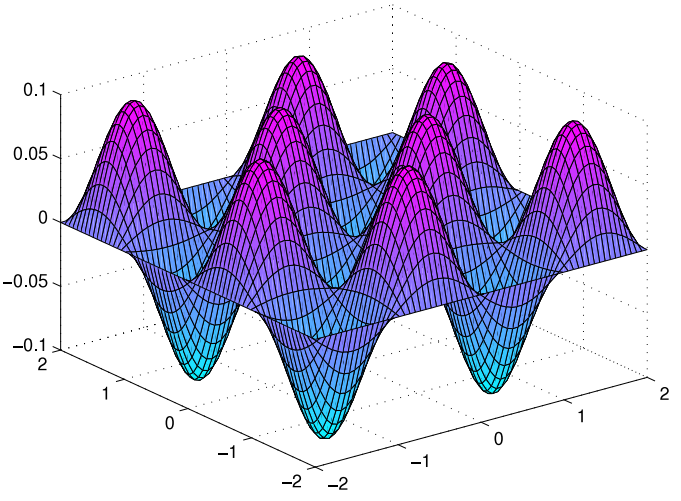


Fig. 3. Imaginary part of $\tilde{w}$ t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

Then, from (4.24)–(4.30), we have

$$\begin{aligned}\frac{\|\xi^n\|^2-\|\xi^{n-1}\|}{2\Delta t}+(\nabla\zeta^{n-\frac{1}{2}},\nabla\xi^{n-\frac{1}{2}})\leq& C(\|\frac{\partial^3 v}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3+\frac{1}{\Delta t}\|\frac{\partial\eta_1}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)}\\&+\|\xi^n\|^2+\|\xi^{n-1}\|^2+\|\eta^{n-1}\|^2+\|\eta^n\|^2\\&+\|\zeta^n\|^2+\|\zeta^{n-1}\|^2+h^{2l+2}+H^{-4}H^{6l+6}),\end{aligned}\tag{4.31}$$

In an entirely analogous manner a similar estimate may be obtained for $R_h w^n-\tilde{w}_h^n$.

$$\begin{aligned}\frac{\|\zeta^n\|^2-\|\zeta^{n-1}\|}{2\Delta t}-(\nabla\xi^{n-\frac{1}{2}},\nabla\zeta^{n-\frac{1}{2}})\leq& C(\|\frac{\partial^3 w}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3+\frac{1}{\Delta t}\|\frac{\partial\eta_2}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)}\\&+\|\xi^n\|^2+\|\xi^{n-1}\|^2+\|\eta^{n-1}\|^2+\|\eta^n\|^2\\&+\|\zeta^n\|^2+\|\zeta^{n-1}\|^2+h^{2l+2}+H^{-4}H^{6l+6}),\end{aligned}\tag{4.32}$$

Adding (4.31) and (4.32), we obtain

$$\begin{aligned}\frac{\|\mu^n\|^2-\|\mu^{n-1}\|}{2\Delta t}\leq& C(\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)}(\Delta t)^3+\frac{1}{\Delta t}\|\frac{\partial\eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)}\\&+\|\mu^n\|^2+\|\mu^{n-1}\|^2+h^{2l+2}+H^{-4}H^{6l+6}),\end{aligned}\tag{4.33}$$

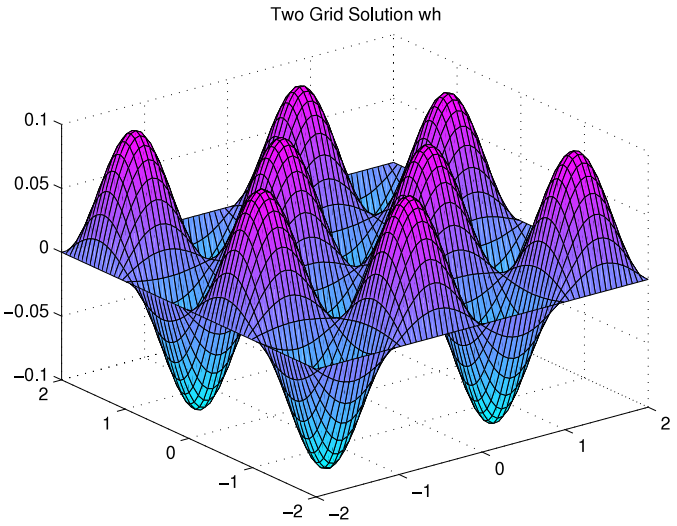


Fig. 4. Imaginary part of $\tilde{w}$ t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

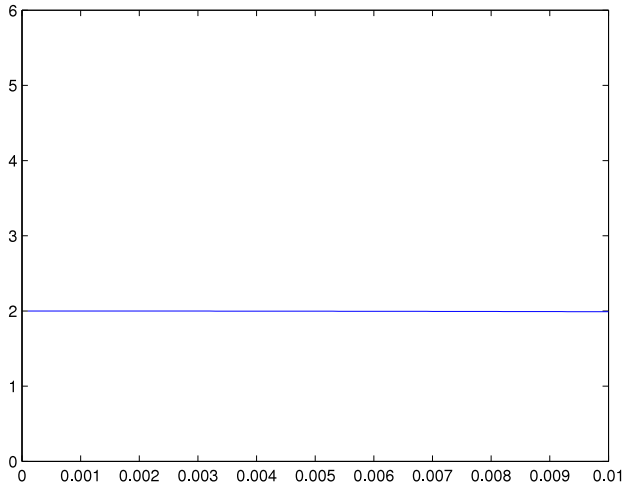


Fig. 5. Charge conservation of $\tilde{u}$ t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

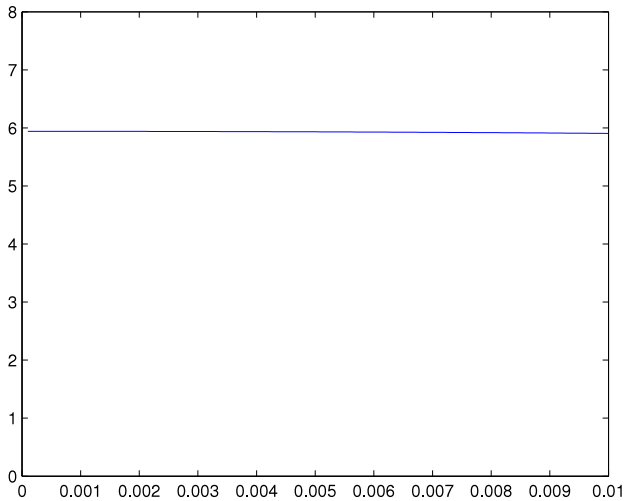


Fig. 6. Energy conservation of $\tilde{u}$ t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

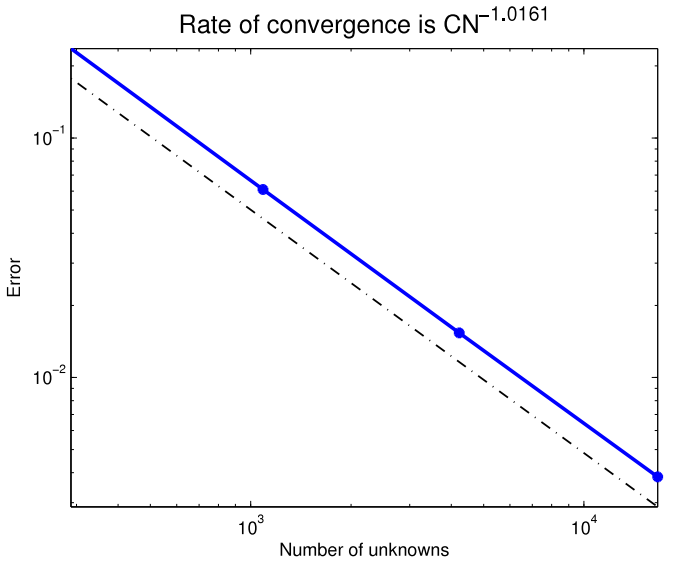


Fig. 7. Order of two-grid solution v t = 1.0e−2, $\tau = 1.0e-4$.

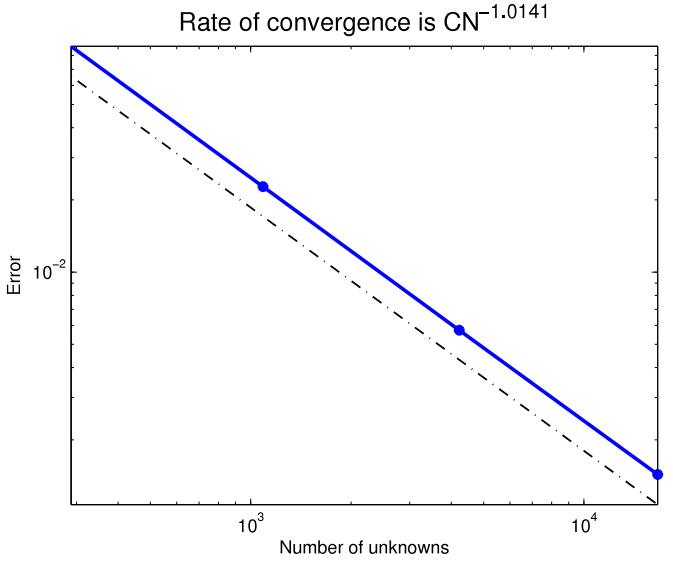


Fig. 8. Order of two-grid solution w t = 1.0e−2, $\tau = 1.0e-4$.

Multiplying $2\Delta t$ on both sides of the above inequality and summing from $n = 1$ to $n = m \leq N$, we obtain the error inequality

$$\begin{aligned}\|\mu^m\|^2 \leq & C((\Delta t)^4 \sum_{n=1}^{m} \|\frac{\partial^3 u}{\partial t^3}\|^2_{L^2(t^{n-1},t^n,L^2)} + \sum_{n=1}^{m} \|\frac{\partial \eta}{\partial t}\|^2_{L^2(t^{n-1},t^n,L^2)} \\ & + h^{2l+2} + H^{6l+6}) + \Delta t \sum_{n=1}^{m} \|\mu^n\|^2 \\ \leq & C(h^{2l+2} + H^{6l+6} + (\Delta t)^4) + \Delta t \sum_{n=1}^{m} \|\mu^n\|^2,\end{aligned} \tag{4.34}$$

By using discrete Gronwall inequality, we have

$$\|\mu^m\|^2 \leq C(h^{2l+2} + H^{6l+6} + (\Delta t)^4). \tag{4.35}$$

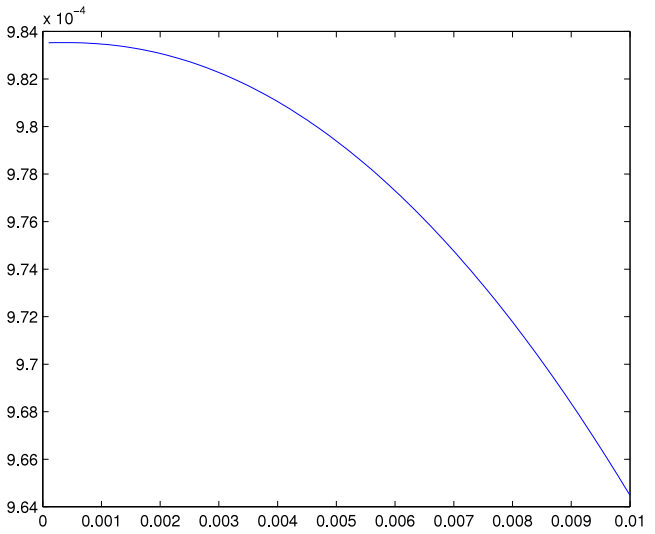


Fig. 9. Error of v t = [0,0.01], $h = 1/64$, $\tau = 1.0e-4$.

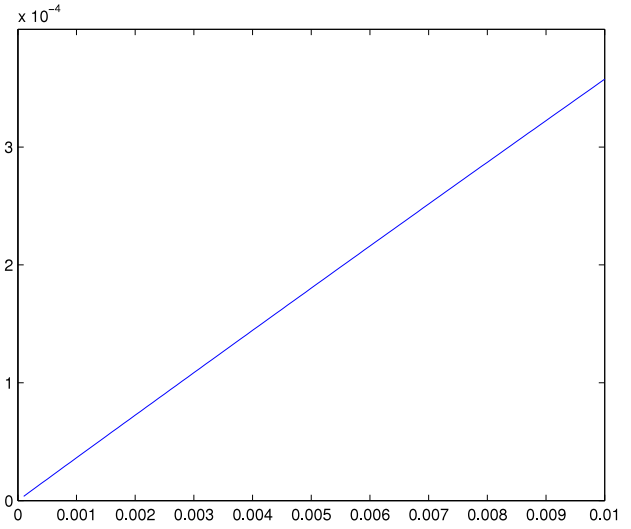


Fig. 10. Error of w t = [0,0.01], $h = 1/64$, $\tau = 1.0e-4$.

Then we have

$$\|\mu^m\| \le C(h^{l+1} + H^{3l+3} + (\Delta t)^2). \tag{4.36}$$

From Lemma 3.1 and (4.36), the theorem can derive by using the triangle inequality. □

Remark 2. As in the argument of Lemma 3.2 and Theorem 4.2, we can also derive H^1 error estimate between the exact solution and two-grid solution of step 3: $\|u^n - \tilde{u}_h^n\|_1 \le C(h^l + H^{3l} + (\Delta t)^2)$.

5. Numerical experiments

In this section, we construct two examples to illustrate the efficiency of the algorithm discussed in Section 4. In these cases, we use piecewise linear continuous functions for the numerical solutions.

Example 1.

$$i\frac{\partial u}{\partial t} - \Delta u + |u|^2 u = (3\pi^2 + \sin^2(\pi x)\sin^2(\pi y))u, \quad (x, y) \in [-2, 2] \times [-2, 2], \quad t \in [0, 0.01], \tag{5.1}$$

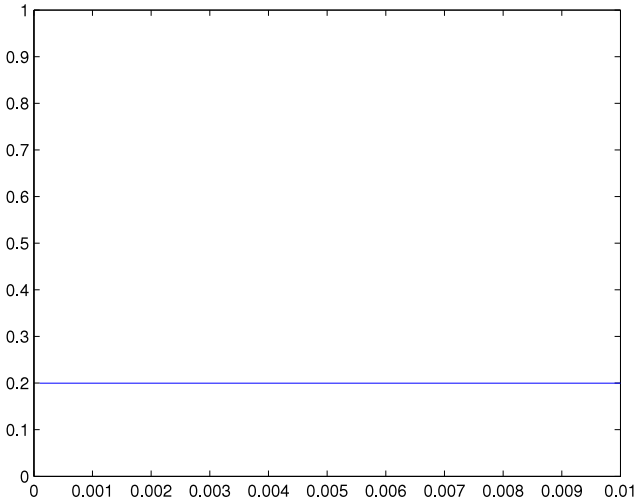


Fig. 11. Charge conservation of ũ t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

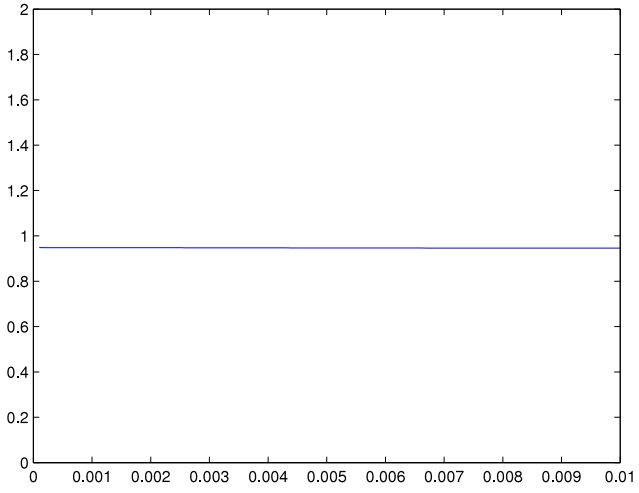


Fig. 12. Energy conservation of ũ t = 1.0e−2, $h = 1/64$, $\tau = 1.0e-4$.

$$u(x, y, t) = 0, \quad (x, y) \in \Gamma, \quad t \in [0, 0.01], \tag{5.2}$$

$$u(x, y, 0) = \sin(\pi x)\sin(\pi y), \quad (x, y) \in [-2, 2] \times [-2, 2]. \tag{5.3}$$

The exact solution of the equation is

$$u(x, y, t) = \sin(\pi x)\sin(\pi y)e^{-\pi^2 it}. \tag{5.4}$$

From Tables 1 and 2 for $\|v - \tilde{v}_h\|$, $\|v - \tilde{v}_h\|_1$, $\|w - \tilde{w}_h\|$ and $\|w - \tilde{w}_h\|_1$, respectively, the two-grid method retains the same order of accuracy as the Newton iterative method, but the two-grid method has much lower time cost than the Newton iterative method. Moreover, we can find that the convergence order of the error for $\|v - \tilde{v}_h\|$, $\|v - \tilde{v}_h\|_1$, $\|w - \tilde{w}_h\|$ and $\|w - \tilde{w}_h\|_1$ are about 2,1,2 and 1 order accuracy, respectively, as it supports the theoretical results($l = k = 0$).

As shown in Figs. 1–4, we can see that display good agreement between the exact solution and the two-grid solution. Figs. 5 and 6 show that the charge conservation and energy conservation of the scheme (2.3) is very good. From Figs. 7 and 8, we know the two-grid algorithm achieves optimal approximation when the mesh sizes between the coarse grid and fine grid satisfy $H = O(h^{\frac{1}{3}})$. In Figs. 9 and 10, we can observe that error of two-grid solution V_h, W_h, varied with time.

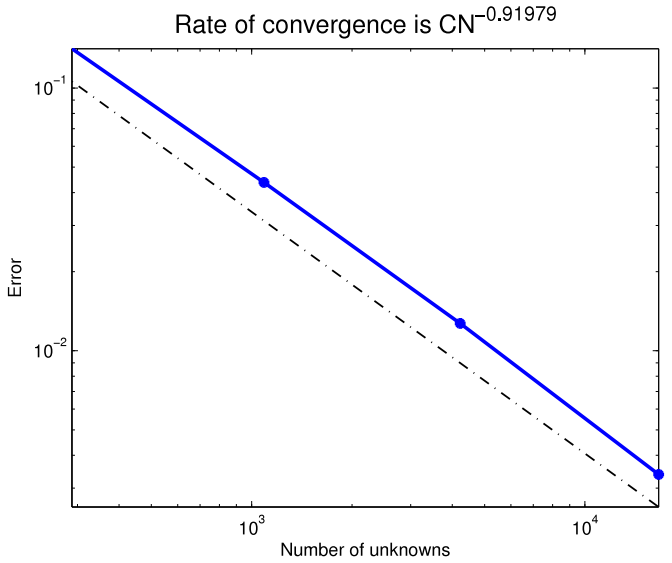


Fig. 13. Order of two-grid solution v t = 1.0e−2, $\tau = 1.0e-4$.

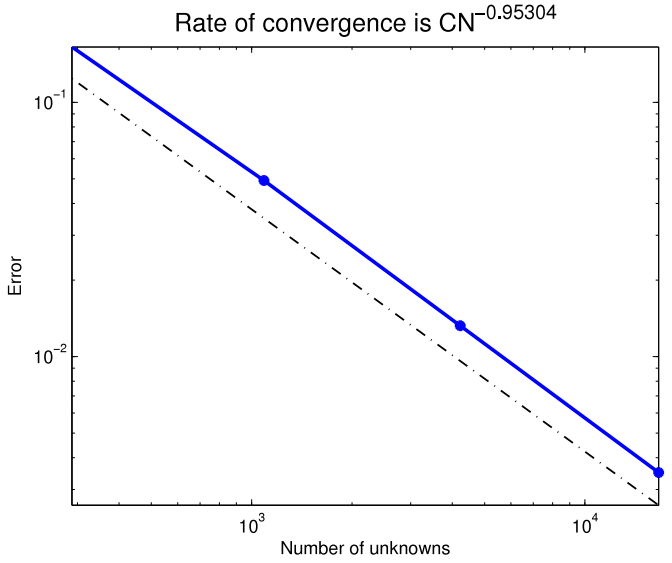


Fig. 14. Order of two-grid solution w t = 1.0e−2, $\tau = 1.0e-4$.

Example 2.

$$
\begin{aligned}
&i\frac{\partial u}{\partial t} - \Delta u + |u|^2 u = (201 - 200(\tanh(10x))^2 - 200(\tanh(10y))^2 + (sech(10x)sech(10y))^2)u,\\
&\quad (x, y) \in [-2, 2] \times [-2, 2], \quad t \in [0, 0.01],\\
&\qquad u(x, y, t) = 0, \quad (x, y) \in \Gamma, \quad t \in [0, 0.01],\\
&\qquad u(x, y, 0) = sech(10x)sech(10y), \quad (x, y) \in [-2, 2] \times [-2, 2].
\end{aligned}
$$

The exact solution of the equation is

$$u(x, y, t) = sech(10x)sech(10y)e^{-it}. \tag{5.5}$$

From Tables 3 and 4 for $\|v - \tilde{v}_h\|$, $\|v - \tilde{v}_h\|_1$, $\|w - \tilde{w}_h\|$ and $\|w - \tilde{w}_h\|_1$, respectively, the two-grid method retains the same order of accuracy as the Newton iterative method, but the two-grid method has much lower time cost than the Newton iterative method. Moreover, we can find that the convergence order of the error for $\|v - \tilde{v}_h\|$, $\|v - \tilde{v}_h\|_1$, $\|w - \tilde{w}_h\|$ and $\|w - \tilde{w}_h\|_1$ are about 2,1,2 and 1 order accuracy, respectively, as it supports the theoretical results($l = k = 0$).

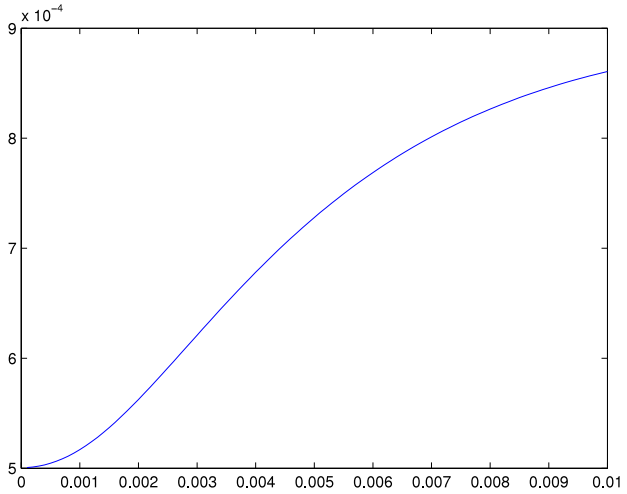


Fig. 15. Error of v t = [0,0.01], $h = 1/64$, $\tau = 1.0e-4$.

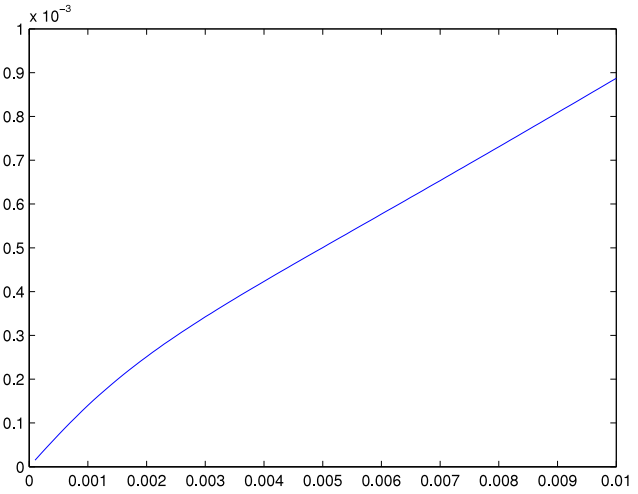


Fig. 16. Error of w t = [0,0.01], $h = 1/64$, $\tau = 1.0e-4$.

Table 3
Error and CPU time of Newton iterative method.

H	h	$\|v^n - \tilde{v}_h^n\|$	Rate	$\|v^n - \tilde{v}_h^n\|_1$	Rate	$\|w^n - \tilde{w}_h^n\|$	Rate	$\|w^n - \tilde{w}_h^n\|_1$	Rate	CPU time
$\frac{1}{2}$	$\frac{1}{8}$	4.3727e−2		7.5143e−1		4.9251e−2		4.4649e−1		9.45 s
$\frac{1}{4}$	$\frac{1}{64}$	8.6126e−4	2	1.0587e−1	1	8.8858e−4	2	5.5812e−2	1	780.20 s
$\frac{1}{8}$	$\frac{1}{512}$	1.3521e−5	2	1.3233e−2	1	1.3939e−5	2	6.9765e−3	1	307680.56 s

Table 4
Error and CPU time of two-grid method.

H	h	$\|v^n - \tilde{v}_h^n\|$	Rate	$\|v^n - \tilde{v}_h^n\|_1$	Rate	$\|w^n - \tilde{w}_h^n\|$	Rate	$\|w^n - \tilde{w}_h^n\|_1$	Rate	CPU time
$\frac{1}{2}$	$\frac{1}{8}$	4.3727e−2		7.5143e−1		4.9251e−2		4.4649e−1		8.19 s
$\frac{1}{4}$	$\frac{1}{64}$	8.5837e−4	2	1.0587e−1	1	8.8223e−4	2	5.5812e−2	1	491.23 s
$\frac{1}{8}$	$\frac{1}{512}$	1.3410e−5	2	1.3233e−2	1	1.3663e−5	2	6.9765e−3	1	172528.24 s

As shown in Figs. 11–12, the charge conservation and energy conservation of the scheme (2.3) is very good. From Figs. 13–14, we know the two-grid algorithm achieves optimal approximation when the mesh sizes between the coarse

grid and fine grid satisfy $H = O(h^{\frac{1}{3}})$. In Figs. 15–16, we can observe that error of two-grid solution $\tilde{v}_h$, $\tilde{w}_h$, varied with time.

6. Conclusion

In this paper, a new two-grid finite element scheme is presented for two-dimensional nonlinear Schrödinger equation. One Newton iteration is applied on the fine grid to linearize the nonlinear system using the coarse-grid solution as the initial guess, and furthermore one more linear system on the coarse space is solved. The error estimations of the two-grid solution in the L^2 and H^1 norm are given. It is shown that coarse space can be extremely coarse and the two-grid algorithm achieves optimal approximation as long as the mesh sizes satisfy $H = O(h^{\frac{1}{3}})$. Finally, numerical results are carried out to confirm the theoretical analysis.

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