



Inheritance properties of Krylov subspace methods for continuous-time algebraic Riccati equations

Liping Zhang^a, Hung-Yuan Fan^b, Eric King-wah Chu^{c,*}

^a Department of Mathematics, Zhejiang University of Technology, Hangzhou 310023, PR China

^b Department of Mathematics, National Taiwan Normal University, Taipei 116, PR China

^c School of Mathematics, 9 Rainforest Walk, Monash University, Victoria 3800, Australia

ARTICLE INFO

Article history:

Received 6 March 2019

Received in revised form 7 August 2019

MSC:

15A24

65F30

93C05

Keywords:

Continuous-time algebraic Riccati equation

Krylov subspace

LQR optimal control

Projection method

ABSTRACT

We investigate the theory behind the Krylov subspace methods for large-scale continuous-time algebraic Riccati equations. We show that the solvability of the projected algebraic Riccati equation need not be assumed but can be *inherited*. This study of inheritance properties is the first of its kind. We study the stabilizability and detectability of the control system, the stability of the associated Hamiltonian matrix and perturbation in terms of residuals. Special attention is paid to the stabilizing and positive semi-definite properties of approximate solutions. Illustrative numerical examples for the inheritance properties are presented.

© 2019 Elsevier B.V. All rights reserved.

1. Introduction

We investigate the theory behind the Krylov subspace methods for large-scale continuous-time algebraic Riccati equations (CAREs). Specifically, we consider the *inheritance* of solvability conditions and structures from a CARE by the corresponding projected equation. The inheritance properties are obviously important but have not been investigated previously.

1.1. Continuous-time algebraic Riccati equations

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{l \times n}$ with $m, l \leq n$. Consider the linear time-invariant control system $\{A, B, C\}$ in continuous-time (with B and C assumed to be full-rank):

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t).$$

With $H \equiv C^T C$ being positive semi-definite (p.s.d.) and R positive definite, the linear quadratic regulator (LQR) control minimizes the functional $J(x, u) \equiv \int_0^\infty [x(t)^T H x(t) + u(t)^T R u(t)] dt$. The optimal control $u(t) = -R^{-1} B^T X x(t)$ is expressed in terms of the unique stabilizing p.s.d. solution X of the CARE, where $G \equiv B R^{-1} B^T$ is p.s.d.:

$$C(X) \equiv A^T X + XA - XGX + H = 0. \quad (1)$$

* Corresponding author.

E-mail addresses: zhanglp@zjut.edu.cn (L. Zhang), hyfan@math.ntnu.edu.tw (H.-Y. Fan), eric.chu@monash.edu (E.K.-w. Chu).

The solution X is *stabilizing* when the closed-loop system matrix $A^c \equiv A - GX$ has all its eigenvalues possessing negative real parts. The CARE is solvable and yields a unique stabilizing p.s.d. solution X when the control system $\{A, B, C\}$ is stabilizable and detectable [1,2]. For the general background of the LQR problem or CAREs, consult [1–3].

1.2. Previous work

The solution of algebraic Riccati equations (AREs) is an active area of research due to its importance in optimal control and filtering problems. Dozens of methods for the problem have been proposed by control theorists and applied mathematicians [3–6]. Classical approaches utilize canonical forms, determinants and polynomial manipulation whereas state-of-the-art algorithms work in a numerically stable manner. A favourite approach formulates the CARE as a Hamiltonian eigenvalue problem defined by [7]:

$$\mathcal{H} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (A - GX), \quad \mathcal{H} \equiv \begin{bmatrix} A & -G \\ -H & -A^\top \end{bmatrix}, \quad (2)$$

and has been implemented in MATLAB (as the command `care`). The Newton–Kleinman method [8] is another favourite. On modern algorithms for AREs of moderate sizes, consult [3,7,9].

For control problems from PDEs [9–12] and the balancing-based model order reduction [13], large-scale CAREs or Lyapunov equations have to be solved. Solving such a CARE may involve the stable invariant subspace of \mathcal{H} , usually a prohibitively expensive exercise [14].

Benner et al. contributed much on large-scale CAREs [9,11,12,15,16]. They mainly built on Newton’s methods with the ADI for the associated Lyapunov equations. (The initialization of Newton’s method and the choice of parameters for the ADI are challenging. The structure-preserving doubling algorithm [5] has been adapted for large-scale CAREs [17].

The Krylov subspace methods (such as the block Krylov method [18,19], the extended block Arnoldi process [20] and the rational Krylov subspace methods [21,22]) are popular for large-scale CAREs. (See also the applications of the rational Krylov subspaces [23] to Lyapunov equations [24], evolutionary problems [25], large-scale dynamical systems [26], approximation of matrix functions [27,28] and eigenvalue problems [29,30].) They project the original equations onto certain subspaces and produce small projected CAREs (pCAREs). Various assumptions for the solvability of the pCARE can be found in [19–22,31]. Jbilou [18] assumed, for the increasing dimension d of the Krylov subspace, that the projected systems “ $\{H_d^\top, B_d\}$ is c-stabilizable and $\{H_d^\top, \tilde{C}_1\}$ is c-detectable. These conditions ensure that the (projected) matrix equation has a unique symmetric p.s.d. solution Y_d . If the preceding conditions are not satisfied we can use an implicit restart strategy to remove the unstable eigenvalues to obtain a c-stabilizable and c-detectable low-order model”. It is unclear how the solvability of the pCARE is affected.

There are obviously many questions we may ask on projection methods for CAREs. We do not know when the methods will be accurate *a priori* or which subspaces are *appropriate*. The only significant theoretical result seems to be the *a posteriori* bound $\|R_k\| \leq \sqrt{2} \cdot \|Y_k r_k\|$ on the residual $R_k \equiv \mathcal{C}(X_k)$, for the approximate solution $X_k = V_k Y_k V_k^\top$. As the Arnoldi residual r_k does not necessarily diminish with respect to k , we do not know when $\|R_k\|$ will be small or when the corresponding Krylov subspace method is convergent or accurate. We aim to ask some interesting questions on projection methods here. For example, when the *assumed* solvability of the pCARE is invalid, how should we proceed? We may not be able to ask or answer all questions in this first study of the inheritance properties. As stabilization is one of the main goal of LQR optimal control, we do not consider the assumption of stability for A (or the much stronger passiveness) is reasonable or desirable.

1.3. Main contributions

We have not proposed any new method, merely developed some theoretical results for the Krylov subspace methods. Also, numerical implementation and experimentation are not our main concern. In Section 2.2, we investigate the inheritance properties of the Krylov subspace methods for the CAREs. We are interested in when stabilizability, detectability and other structures of the original equations and control systems are passed on to, or inherited by, their projections.

We prove the solvability (for the unique stabilizing p.s.d. solution) of the CARE is *inherited* by the pCARE, when the system $\{A, B, C\}$ is bounded away from unstabilizability and undetectability (in the sense of Theorems 2.1 and 2.2 below). Other inheritance properties are presented in terms of the stability radius and perturbation theory. Note that the inheritance properties concern the flow of properties and structures from the CARE, through the projection, to the pCARE. While the solvability of a small pCARE may be tested easily, its link with the CARE and the Krylov subspace methods is more interesting.

1.4. Organization of paper

We consider Krylov subspace methods for CAREs in Section 2, elaborating on the inheritance properties. Accuracy of Krylov subspace methods is considered in Section 2.3, some numerical examples are presented in Section 3 and the study is concluded in Section 4.

Some notations are tabulated below, followed by a note on terminology:

Symbol	Description
$\ \cdot\ $	2-matrix norm on square real matrices
$(\cdot)^\top, (\cdot)^{-1}$	Transposes and inverses respectively, with the latter assuming invertibility implicitly; $(A^\top)^{-1}$ is abbreviated to $A^{-\top}$.
I	The identity matrix, occasionally with a subscript for its dimension.
$\sigma_{\max}(\cdot), \sigma_{\min}(\cdot)$	The maximum and minimum singular values respectively.
$\Lambda(\cdot)$	The spectrum.
\mathbb{C}_+	The closed right plane.
i	$\sqrt{-1}$.
$\mathbb{R}^{m \times n}, \mathbb{S}_n$	The sets of real $m \times n$ and real symmetric $n \times n$ matrices respectively.
$\mathcal{R}(M_1, \dots, M_k)$	The range or column space of the matrix $[M_1, \dots, M_k]$.

For a symmetric matrix, it is positive semi-definite (p.s.d.) when all its eigenvalues are non-negative. A symmetric matrix is stabilizing when the corresponding closed-loop system matrix is c-stable, i.e. with all eigenvalues having negative real parts. A matrix is numerically low-rank if it can be approximated accurately by a low-rank matrix of the same dimensions, with the error bounded in norm by a tolerance specified by the user.

2. Krylov subspace methods

Krylov subspace methods are applicable to large-scale CAREs when the solution X is *numerically* low-rank, which is so when H is low-rank [14]. Let $\mathcal{K}_k(\mathcal{M}, \mathcal{Z}) \equiv \mathcal{R}\{\mathcal{Z}, \mathcal{M}\mathcal{Z}, \dots, \mathcal{M}^{k-1}\mathcal{Z}\}$ be the Krylov subspace generated by the matrix \mathcal{M} and the initial block-vector \mathcal{Z} . Various Krylov subspaces have been applied to CAREs. For example, $\mathcal{K}_k(A^\top, C^\top)$ was applied in [18], $\mathcal{K}_k(A, C^\top)$ in [19] and $\mathcal{K}_k(A^\top, C^\top) \cup \mathcal{K}_k(A^{-\top}, C^\top)$ in [20]. Recently, the rational Krylov subspace [21,22,32]:

$$\mathcal{K}_k(A^\top, C^\top, s) \equiv \mathcal{R}\{C^\top, (A - s_1 I)^{-\top} C^\top, \dots, \Pi_{i=1}^{k-1} (A - s_i I)^{-\top} C^\top\} \quad (3)$$

with the shifts (or poles) $s_j \in \mathbb{C}_+$ in $s = [s_1, s_2, \dots, s_{k-1}]$, was applied to CAREs successfully. (See also the related [33]; for the optimal or adaptive selection of s , consult [21,25,28].) From the Arnoldi process with $V_0 \equiv C^\top$ (orthonormalized), $\mathcal{K}_k(A^\top, C^\top, s)$ is spanned by the columns of $V_k \in \mathbb{R}^{n \times n_k}$ ($k \geq 0$) which satisfies the Arnoldi relationship:

$$A^\top V_k = V_k \Phi_k^\top + v_{k+1} r_k^\top, \quad (4)$$

as in [23,31] and [21, Eqt. (5.1)], with $V_{k+1} \equiv [V_k, v_{k+1}]$, $V_{k+1}^\top V_{k+1} = I$ and $\Phi_k = V_k^\top A V_k$. We refer to r_k as the Arnoldi residual corresponding to the Krylov subspace. With the low-rank approximate solution $X_k \equiv V_k Y_k V_k^\top$, $Y_k^\top = Y_k \in \mathbb{R}^{n_k \times n_k}$, $G_{11} \equiv V_k^\top G V_k$ and $H_{11} \equiv V_k^\top H V_k$, the Galerkin condition $V_k^\top \mathcal{C}(X_k) V_k = 0$ implies the pCARE in Y_k :

$$\widehat{\mathcal{C}}(Y_k) \equiv \Phi_k^\top Y_k + Y_k \Phi_k - Y_k G_{11} Y_k + H_{11} = 0. \quad (5)$$

The Arnoldi residual r_k may *stagnate*, i.e., persist in norm and not diminish with respect to the increasing values of k . It plays an important part in the convergence and accuracy of Krylov subspace methods.

While an ARE generally has *many* solutions, we want only the unique stabilizing p.s.d. solution for the optimal control. For a symmetric *approximate* solution, we may require it to be stabilizing because stabilization of the closed-loop system is one of the main objectives of the LQR optimal control. As for a symmetric solution Y_k of the pCARE (5), its properties have no direct impact on $X_k = V_k Y_k V_k^\top$, although Y_k is p.s.d. iff X_k is so.

Individual solution methods are based on different assumptions, giving rise to approximate solutions with different properties. These will be elaborated further.

2.1. In-depth interpretation of projection methods

For the orthogonal $P \equiv [P_1, P_2] \in \mathbb{R}^{n \times n}$ with $P_1 \equiv V_k \in \mathbb{R}^{n \times n_k}$ and $Y_{ij} \equiv P_i^\top X P_j$ ($i, j = 1, 2$), the solution X of the CARE (1) satisfies

$$X = P \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^\top & Y_{22} \end{bmatrix} P^\top = \sum_{i,j=1}^2 P_i Y_{ij} P_j^\top,$$

where $Y_{21} = Y_{12}^\top$ and $Y_{ii}^\top = Y_{ii}$ from the symmetry of X . Obviously, Y_k in (5) is an approximation to Y_{11} . From the Arnoldi relationship (4), $A_{21} \equiv P_2^\top A P_1$ and $A_{22} \equiv P_2^\top A P_2$, we have

$$P^\top A P = \begin{bmatrix} P_1^\top A P_1 & P_1^\top A P_2 \\ P_2^\top A P_1 & P_2^\top A P_2 \end{bmatrix} = \begin{bmatrix} \Phi_k & r_k v_{k+1}^\top P_2 \\ A_{21} & A_{22} \end{bmatrix}. \quad (6)$$

For Krylov subspaces which satisfy $C^\top \subseteq \mathcal{R}(P_1)$, we have $H_{ij} \equiv P_i^\top H P_j = 0$ for $i, j = 1, 2$ except $H_{11} \equiv P_1^\top H P_1$ is p.s.d. Together with (6) and $G_{ij} \equiv P_i^\top G P_j$, we have

$$C(X) = PP^\top C(X)PP^\top = P \begin{bmatrix} \hat{C}(Y_{11}) - g_{11} + \hat{H}_{11} & \tilde{\Phi}_k Y_{12} + Y_{12} \tilde{A}_{22} - g_{12} + \hat{H}_{12} \\ * & \check{C}(Y_{22}) \end{bmatrix} P^\top \in \mathbb{S}_n,$$

with $*$ being the part retrievable from symmetry, $\tilde{\Phi}_k \equiv \Phi_k - G_{11}Y_{11}$, $\tilde{A}_{22} \equiv A_{22} - G_{22}Y_{22}$ and

$$\begin{aligned} \check{C}(Y_{22}) &\equiv A_{22}^\top Y_{22} + Y_{22} A_{22} - Y_{22} G_{22} Y_{22} - g_{22} + \hat{H}_{22}, \\ \hat{H}_{11} &\equiv A_{21}^\top Y_{12} + Y_{12} A_{21}, \quad g_{11} \equiv Y_{11} G_{12} Y_{12}^\top + Y_{12} G_{21} Y_{11} + Y_{12} G_{22} Y_{12}^\top; \\ \hat{H}_{12} &\equiv Y_{11} r_k v_{k+1}^\top P_2 + A_{21}^\top Y_{22}, \quad g_{12} \equiv Y_{11} G_{12} Y_{22} + Y_{12} G_{21} Y_{12}; \\ \hat{H}_{22} &\equiv P_2^\top v_{k+1} r_k^\top Y_{12} + Y_{12}^\top r_k v_{k+1}^\top P_2, \quad g_{22} \equiv Y_{12}^\top G_{11} Y_{12} + Y_{12}^\top G_{12} Y_{22} + Y_{22} G_{21} Y_{12}. \end{aligned}$$

We describe a Krylov subspace method and its approximate solution X_k as accurate when the residual $R_k \equiv C(X_k)$ is less than some small tolerance ε in norm. From [34, Theorem 2] when

$$4\|G\|\|\mathcal{L}^{-1}\|^2\|R_k\| < 1, \quad (7)$$

where the Lyapunov operator $\mathcal{L}(\cdot) \equiv (A - GX)^\top(\cdot) + (\cdot)(A - GX)$ is invertible in \mathbb{S}_n , we have

$$\|X_k - X\| \leq 2\|\mathcal{L}^{-1}\|\|R_k\| \leq 2\|\mathcal{L}^{-1}\|\varepsilon. \quad (8)$$

With $\|P_2^\top X\| = \|[Y_{12}^\top, Y_{22}]\|$, we also have

$$\max\{\|P_2^\top X\|, \|Y_k - Y_{11}\|\} \leq \left\| \begin{bmatrix} Y_{11} - Y_k & Y_{12} \\ Y_{12}^\top & Y_{22} \end{bmatrix} \right\| = \|X_k - X\| \leq 2\|\mathcal{L}^{-1}\|\varepsilon,$$

so X has a small $O(\varepsilon)$ component in span P_2 and Y_k approximates Y_{11} accurately. Further discussions can be found in Sections 2.2.4 and 2.3.

2.2. Inheritance of solvability and structures

Denote $B_i \equiv P_i^\top B$ and $C_i \equiv C P_i$ ($i = 1, 2$). We next study some solvability conditions and structures of the CARE (1) and its projection (5).

2.2.1. Stabilizability

We consider the stabilizability of the system $\{A, B\}$ and its projection $\{\Phi_k, B_1\}$. We modify the definition for the distance of the system $\{A, B\}$ from uncontrollability [35,36] to that from unstabilizability:

$$\begin{aligned} \tau(A, B) &\equiv \min\{\|[\delta A, \delta B]\| : \{A + \delta A, B + \delta B\} \text{ is not stabilizable}\} \\ &= \min_{\mu \in \mathbb{C}_+} \sigma_{\min}(A - \mu I, B). \end{aligned} \quad (9)$$

The numerical estimation of τ , required in the numerical examples in Section 4, is generally difficult and expensive [35–40], especially for large systems.

We have the following result on the inheritance of stabilizability:

Theorem 2.1 (Stabilizability). *Let $x_1 = \check{x}_1$ achieve the minimum in*

$$\min_{s \in \mathbb{D}_+} \min_{\|x_1\|=1} \|x_1^\top [sI - \Phi_k, B_1]\|. \quad (10)$$

Then we have

$$\tau(A, B) \leq \tau(\Phi_k, B_1) + \|\check{x}_1^\top r_k\|. \quad (11)$$

In particular, if $\tau(A, B) > \|\check{x}_1^\top r_k\|$, then $\{\Phi_k, B_1\}$ is stabilizable.

Proof. By the properties of singular values [41] on $P^\top [sI - A, B] \text{diag}(P, I_m)$ and (6), we have

$$\begin{aligned} \tau(A, B) &= \min_{s \in \mathbb{D}_+} \sigma_{\min}(sI - A, B) = \min_{s \in \mathbb{D}_+} \sigma_{\min} \left[\begin{array}{cc|c} sI - \Phi_k & B_1 & -r_k v_{k+1}^\top P_2 \\ \hline -A_{21} & B_2 & sI - A_{22} \end{array} \right] \\ &= \min_{s \in \mathbb{D}_+} \min_{\|[x_1^\top, x_2^\top]\|=1} \left\| [x_1^\top, x_2^\top] \begin{bmatrix} sI - \Phi_k & B_1 & -r_k v_{k+1}^\top P_2 \\ \hline -A_{21} & B_2 & sI - A_{22} \end{bmatrix} \right\| \\ &\leq \min_{s \in \mathbb{D}_+} \min_{\|[x_1^\top, x_2^\top]\|=1} \left\{ \left\| [x_1^\top, x_2^\top] \begin{bmatrix} sI - \Phi_k & B_1 & 0 \\ \hline -A_{21} & B_2 & sI - A_{22} \end{bmatrix} \right\| + \|x_1^\top r_k\| \right\}. \end{aligned}$$

$$\begin{aligned} &\leq \min_{s \in \mathbb{D}_+} \min_{\|x_1^\top, 0^\top\|=1} \left\| [x_1^\top, 0^\top] \begin{bmatrix} sI - \Phi_k & B_1 \\ -A_{21} & B_2 \end{bmatrix} \begin{bmatrix} 0 \\ sI - A_{22} \end{bmatrix} \right\| + \|\check{x}_1^\top r_k\| \\ &= \min_{s \in \mathbb{D}_+} \min_{\|x_1\|=1} \|x_1^\top [sI - \Phi_k, B_1]\| + \|\check{x}_1^\top r_k\| = \tau(\Phi_k, B_1) + \|\check{x}_1^\top r_k\|, \end{aligned}$$

with $x_1 = \check{x}_1$ satisfying (10). We have proven (11). \square

We speculate that \check{x}_1 is “top-heavy” (dominated by the first few components), so $\|\check{x}_1^\top r_k\|$ is diminishing for the stagnating “bottom-heavy” r_k . We have some supporting evidence on this from the numerical examples in Section 3; see also the heuristics on \check{x}_1 in the Appendix.

2.2.2. Detectability

We next consider the detectability of $\{A, C\}$ and its projection $\{\Phi_k, C_1\}$.

Theorem 2.2 (Detectability). *Let $y \equiv [y_1^\top, y_2^\top]^\top = [\tilde{y}_1^\top, \tilde{y}_2^\top]^\top$ achieve the minimum in*

$$\min_{s \in \mathbb{D}_+} \min_{\|y_1^\top, y_2^\top\|=1} \left\| \begin{bmatrix} sI - \Phi_k & 0 \\ -A_{21} & sI - A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|, \quad (12)$$

and $\check{y}_2 \equiv v_{k+1}^\top P_2 \tilde{y}_2$. Then we have

$$\tau(A^\top, C^\top) \leq \tau(\Phi_k^\top, C_1^\top) + \|r_k \check{y}_2\|. \quad (13)$$

In particular, if $\tau(A^\top, C^\top) > \|r_k \check{y}_2\|$, then $\{\Phi_k, C_1\}$ is detectable.

Proof. With (6) and $C_i \equiv CP_i$ ($i = 1, 2$), consider the distance of $\{A, C\}$ from undetectability:

$$\begin{aligned} \tau(A^\top, C^\top) &\equiv \min_{s \in \mathbb{D}_+} \sigma_{\min} \begin{bmatrix} sI - A \\ C \end{bmatrix} = \min_{s \in \mathbb{D}_+} \sigma_{\min} \begin{bmatrix} sI - \Phi_k & -r_k v_{k+1}^\top P_2 \\ -A_{21} & sI - A_{22} \end{bmatrix} \\ &= \min_{s \in \mathbb{D}_+} \min_{\|y_1^\top, y_2^\top\|=1} \left\| \begin{bmatrix} sI - \Phi_k & -r_k v_{k+1}^\top P_2 \\ -A_{21} & sI - A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \\ &\leq \min_{s \in \mathbb{D}_+} \min_{\|y_1^\top, y_2^\top\|=1} \left\{ \left\| \begin{bmatrix} sI - \Phi_k & 0 \\ -A_{21} & sI - A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| + \|r_k v_{k+1}^\top P_2 y_2\| \right\} \\ &\leq \min_{s \in \mathbb{D}_+} \min_{\|y_1^\top, y_2^\top\|=1} \left\{ \left\| \begin{bmatrix} sI - \Phi_k & 0 \\ -A_{21} & sI - A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \right\} + \|r_k \check{y}_2\| \end{aligned} \quad (14)$$

where $[\tilde{y}_1^\top, \tilde{y}_2^\top]^\top$ minimizing the first term in (14) (or satisfying (12)) and $\check{y}_2 \equiv v_{k+1}^\top P_2 \tilde{y}_2$. When $\tau(A^\top, C^\top) > \|r_k \check{y}_2\|$, (14) implies that $sI - A_{22}$ is nonsingular for $s \in \mathbb{D}_+$. Denote $\eta(s) \equiv -(sI - A_{22})^{-1} A_{21}$, $\Psi(s) \equiv \begin{bmatrix} I & 0 \\ \eta(s) & I \end{bmatrix}$ and $w \equiv \Psi(s)y$, we have

$$\begin{aligned} \tau(A^\top, C^\top) &\leq \min_{s \in \mathbb{C}_+} \min_{y \neq 0} \left\{ \left\| \begin{bmatrix} sI - \Phi_k & 0 \\ 0 & sI - A_{22} \end{bmatrix} \Psi(s)y \right\| \cdot \frac{1}{\|y\|} \right\} + \|r_k \check{y}_2\| \\ &= \min_{s \in \mathbb{C}_+} \min_{w \neq 0} \left\{ \left\| \begin{bmatrix} sI - \Phi_k & 0 \\ 0 & sI - A_{22} \end{bmatrix} w \right\| \cdot \|\Psi(s)^{-1} w\|^{-1} \right\} + \|r_k \check{y}_2\|. \end{aligned}$$

Restricting $w = [w_1^\top, w_2^\top]^\top$ by forcing $w_2 = 0$, we obtain

$$\begin{aligned} \tau(A^\top, C^\top) &\leq \min_{s \in \mathbb{C}_+} \min_{w_1 \neq 0} \left\{ \left\| \begin{bmatrix} sI - \Phi_k & 0 \\ 0 & sI - A_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ 0 \end{bmatrix} \right\| \cdot \left\| \Psi(s)^{-1} \begin{bmatrix} w_1 \\ 0 \end{bmatrix} \right\|^{-1} \right\} + \|r_k \check{y}_2\| \\ &= \min_{s \in \mathbb{C}_+} \min_{w_1 \neq 0} \left\{ \left\| \begin{bmatrix} sI - \Phi_k \\ C_1 \end{bmatrix} w_1 \right\| \cdot \left\| (sI - A_{22})^{-1} A_{21} w_1 \right\|^{-1} \right\} + \|r_k \check{y}_2\| \end{aligned}$$

$$\begin{aligned}
&\leq \min_{s \in \mathbb{C}_+} \min_{w_1 \neq 0} \left\{ \left\| \begin{bmatrix} sI - \Phi_k \\ C_1 \end{bmatrix} w_1 \right\| \cdot \frac{1}{\|w_1\|} \right\} + \|r_k \check{y}_2\| \\
&= \min_{s \in \mathbb{C}_+} \sigma_{\min} \begin{bmatrix} sI - \Phi_k \\ C_1 \end{bmatrix} + \|r_k \check{y}_2\| = \tau(\Phi_k^\top, C_1^\top) + \|r_k \check{y}_2\|.
\end{aligned} \tag{15}$$

We have proved the result. \square

We speculate that $\|\check{y}_2\|$ is diminishing for increasing k , thus fulfilling the condition in [Theorem 2.2](#). There is supporting evidence for this in [Section 3](#); see the heuristics in the [Appendix](#).

By [Theorems 2.1](#) and [2.2](#), when the CARE (1) is solvable with the unique stabilizing p.s.d. solution, the pCARE (5) inherits the unique solvability. The system $\{A, B, C\}$ is required to satisfy $\tau(A, B) > \|\check{x}_1^\top r_k\|$ and $\tau(A^\top, C^\top) > \|r_k \check{y}_2\|$, slightly stronger conditions than being merely stabilizable and detectable when the right-hand-sides are small.

If the pCARE (5) is stabilizable and detectable, its unique stabilizing p.s.d. solution Y_k exists. However, we cannot determine whether the approximate solution $X_k = P_1 Y_k P_1^\top$ for (1) (which is p.s.d.) is stabilizing. This requires further consideration as in [Sections 2.2.3](#) and [2.2.4](#).

2.2.3. Stability

We consider the stability of some Hamiltonian matrices and closed-loop system matrices in this section. Stabilizability and detectability are only sufficient conditions for the existence of the unique stabilizing p.s.d. solution to a CARE [\[1,2\]](#). We may consider directly the stability of the associated Hamiltonian matrix \mathcal{H} in (2), or the distance of its spectrum from the imaginary axis. An appropriate tool is the *stability radius* or *margin* [\[37–39,42–44\]](#):

$$\psi(M) \equiv \min \{ \|E\| : M + E \text{ is unstable} \} = \min_{\omega \in \mathbb{R}} \sigma_{\min}(M - \omega I).$$

It is well known that $\Lambda(\mathcal{H})$ is the union of the stable and anti-stable subspectra $\Lambda(A^c)$ and $\Lambda(-A^c)$ respectively [\[1,2\]](#). Therefore the stability radius $\psi(A^c) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(A^c - \omega I)$, the magnitude of the minimal perturbation to A^c which pushes it over to instability, is a good measure of solvability of the CARE (1). The estimation of ψ is nontrivial.

Recall when (7) holds, we have $\|X_k - X\| \leq 2\|\mathcal{L}^{-1}\| \|R_k\|$ from (8). With $c_1 \equiv 2\|\mathcal{L}^{-1}\| \|G\|$ and $A_k^c \equiv A - GX_k$, techniques similar to those in [Sections 2.2.1](#) and [2.2.2](#) produce

$$\psi(A^c) \leq \psi(A_k^c) + \|G(X_k - X)\| \leq \psi(A_k^c) + c_1 \|R_k\|. \tag{16}$$

Consequently from (16), if

$$\psi(A^c) > c_1 \|R_k\|, \tag{17}$$

then A_k^c is c-stable. Continuing further, with (6), $z = [z_1^\top, z_2^\top]^\top$ and $\Phi_k^c \equiv \Phi_k - G_{11}Y_k$, we have

$$\begin{aligned}
\psi(A_k^c) &= \psi \left(\begin{bmatrix} \Phi_k - G_{11}Y_k & r_k v_{k+1}^\top P_2 \\ A_{21} - P_2^\top G P_1 Y_k & A_{22} \end{bmatrix} \right) \\
&\leq \min_{\omega \in \mathbb{R}} \min_{\|z\|=1} \left\{ \left\| z^\top \begin{bmatrix} \Phi_k^c - \omega I & 0 \\ A_{21} - P_2^\top G P_1 Y_k & A_{22} - \omega I \end{bmatrix} \right\| + \|z_1^\top r_k\| \right\} \\
&\leq \min_{\omega \in \mathbb{R}} \min_{\|z_1\|=1} \left\{ \left\| [z_1^\top, 0] \begin{bmatrix} \Phi_k^c - \omega I & 0 \\ A_{21} - P_2^\top G P_1 Y_k & A_{22} - \omega I \end{bmatrix} \right\| + \|z_1^\top r_k\| \right\} \\
&= \min_{\omega \in \mathbb{R}} \min_{\|z_1\|=1} \|z_1^\top (\Phi_k^c - \omega I)\| + \|\check{z}_1^\top r_k\| = \psi(\Phi_k^c) + \|\check{z}_1^\top r_k\|,
\end{aligned} \tag{18}$$

where $z_1 = \check{z}_1$ optimizes the first term in (18). So when

$$\psi(A_k^c) > \|\check{z}_1^\top r_k\|, \tag{19}$$

we have $\psi(\Phi_k^c) > 0$ and the stabilizing property of X_k passes on to Y_k . If

$$\psi(A^c) > \|\check{z}_1^\top r_k\| + c_1 \|R_k\|, \tag{20}$$

(which implies (17)) then (16) and (18) imply that Y_k is stabilizing and Φ_k^c is c-stable. We summarize in the following theorem on ψ :

Theorem 2.3 (Stability). (a) Assume $4\|G\|\|\mathcal{L}^{-1}\|^2\|R_k\| < 1$ and $\psi(A^c) > \|\check{z}_1^\top r_k\| + c_1 \|R_k\|$, i.e., (7) and (20), (or only (19), that $\psi(A_k^c) > \|\check{z}_1^\top r_k\|$), and that a symmetric solution Y_k exists for the pCARE (5), then Φ_k^c is c-stable. The pCARE is equivalent to a Hamiltonian eigenvalue problem (as in [\[45\]](#)), which possesses a unique stabilizing p.s.d. solution Y_k .

(b) If $4\|G\|\|\mathcal{L}^{-1}\|^2\|R_k\| < 1$ and $\psi(A^c) > c_1 \|R_k\|$, i.e., (7) and (17) hold, then A_k^c is c-stable.

Proof. Assuming (19) (or alternatively (7) and (20)), (16) and (18) imply that Φ_k^c is c-stable. Recall that any stabilizing solution of an ARE is unique. Rewriting the pCARE (5) as

$\Phi_k^{c\top} Y_k + Y_k \Phi_k^c + Y_k G_{11} Y_k + H_{11} = 0$, properties of Lyapunov equations [1] imply Y_k is p.s.d. The pCARE (5) can be rearranged as the Hamiltonian eigenvalue problem:

$$\mathcal{H}_k \begin{bmatrix} I \\ Y_k \end{bmatrix} = \begin{bmatrix} I \\ Y_k \end{bmatrix} \Phi_k^c, \quad \mathcal{H}_k \equiv \begin{bmatrix} \Phi_k & -G_{11} \\ -H_{11} & -\Phi_k^\top \end{bmatrix}. \quad (21)$$

The solvability of the pCARE (5) follows from the c-stable invariant subspace of the Hamiltonian \mathcal{H}_k corresponding to Φ_k^c in (21). When (7) and (17) hold, (16) leads to the result in (b). \square

Thus Φ_k^c inherits the stability of A^c and Y_k is stabilizing when A^c (or A_k^c) is more than a distance of $\|\tilde{z}_1^\top r_k\| + c_1 \|R_k\|$ (or $\|\tilde{z}_1^\top r_k\|$) from instability. Note that the conditions in (17) for (b) are not explicitly depending on r_k and (19) for (b) is weaker than (20) for (a). We speculate that \tilde{z}_1 is top-heavy and $\|\tilde{z}_1^\top r_k\|$ is diminishing in (17) and (20) as k increases. Some supporting evidence has been observed in Section 3; see also the heuristics in the Appendix.

When Φ_k^c is c-stable, without $\{\Phi_k, B_1, C_1\}$ being stabilizable or detectable, the pCARE (5) is equivalent to the Hamiltonian eigenvalue problem (21), which can be solved by an eigen-solver as in [7]. It is presumptuous to assume that an arbitrary method is applicable to the pCARE (5).

2.2.4. Perturbation

We investigate the pCARE (5) via the related perturbed CAREs in X_k [21, Proposition 5.1]:

$$(A - P_1 r_k v_{k+1}^\top)^\top X_k + X_k (A - P_1 r_k v_{k+1}^\top) - X_k G X_k + H = 0, \quad (22)$$

with $R_k \equiv C(X_k) = v_{k+1} r_k^\top P_1^\top X_k + X_k P_1 r_k v_{k+1}^\top$. According to the Arnoldi process (4), we consider that the solution X_k is generated by the Krylov subspace method, i.e.

$$X_k \in \mathcal{K}_k((A - P_1 r_k v_{k+1}^\top)^\top, C^\top) = P_1 \cdot \mathcal{K}_k(\Phi_k^\top, P_1^\top C^\top).$$

Then the symmetric solution X_k has the form $X_k = P_1 Y_k P_1^\top$ and we can check that Y_k is a symmetric solution of (5). In [46, Lemma 2.2], the effects of perturbation on approximate solutions of CAREs have been studied. Here we perturb the CARE (1) to the perturbed CARE (22) and analyse the resulting error.

Theorem 2.4 (Perturbation). *Let X be the unique stabilizing p.s.d. solution to the CARE (1) with $A^c = A - GX$ c-stable and $X_k = P_1 Y_k P_1^\top$ be a symmetric solution to the perturbed CARE (22) generated by the Krylov subspace method. Recall that the Lyapunov operator $\mathcal{L}(\cdot) \equiv A^{c\top}(\cdot) + (\cdot)A^c$ in \mathbb{S}_n . If the residual $R_k \equiv C(X_k)$ satisfies*

$$4\|\mathcal{L}^{-1}\| \|\mathcal{L}^{-1}(R_k)\| \|G\| < 1, \quad (23)$$

then X_k is the unique symmetric solution to the perturbed CARE (22), $A_k^c = A - GX_k$ is c-stable and the error satisfies

$$\|X_k - X\| \leq \xi_* \equiv \frac{2\|\mathcal{L}^{-1}(R_k)\|}{1 + \sqrt{1 - 4\|\mathcal{L}^{-1}\| \|\mathcal{L}^{-1}(R_k)\| \|G\|}}. \quad (24)$$

Proof. From (22), we have $R_k \equiv C(X_k) = (V_k r_k v_{k+1}^\top)^\top X_k + X_k V_k r_k v_{k+1}^\top$. Consider the equation

$$\mathcal{L}(Z) = ZGZ + R_k \quad (25)$$

and the operator \mathcal{L} is invertible since $A^c = A - GX$ is c-stable. Under the condition (23) and

$$\begin{aligned} \|ZGZ\| &\leq \|Z\|^2 \|G\|, \\ \|ZGZ - \tilde{Z}G\tilde{Z}\| &\leq 2\|G\| \max\{\|Z\|, \|\tilde{Z}\|\} \|Z - \tilde{Z}\|, \end{aligned}$$

we obtain that Eq. (25) has unique solution Z_* and

$$\|Z_*\| \leq \frac{2\|\mathcal{L}^{-1}(R_k)\|}{1 + \sqrt{1 - 4\|\mathcal{L}^{-1}\| \|\mathcal{L}^{-1}(R_k)\| \|G\|}}$$

by [46, Lemma 2.2]. From (1), (22) and (25), $X_k = X + Z_*$ is the unique symmetric solution of (22) satisfying (24). Furthermore, depending on (24), [46, Corollary 2.5] and

$$2\|\mathcal{L}^{-1}\| \|G(X_k - X)\| \leq 2\|\mathcal{L}^{-1}\| \|G\| \xi_* \leq 4\|\mathcal{L}^{-1}\| \|\mathcal{L}^{-1}(R_k)\| \|G\| < 1,$$

we have $A - GX_k = A - GX - G(X_k - X) = A^c - G(X_k - X)$ is c-stable. \square

Under the condition of Theorem 2.4, $A_k^c = A - GX_k$ is c-stable, i.e., $\psi(A_k^c) > 0$. Combining with (19), Y_k is the unique stabilizing solution of (5).

Conditions (17) and (20) are dependent on the unknown X so are difficult to test, unlike (19). The tests associated with \tilde{x}_1 and \tilde{y}_2 respectively for stabilizability (in Theorem 2.1) and detectability (in Theorem 2.2) are *a priori*. Those involving \tilde{z}_1 and \tilde{z} for stability are *a posteriori*. Computationally, $\tilde{x}_1, \tilde{z}_1 \in \mathbb{R}^{n_k}$ are easier to estimate than the longer vectors $\tilde{y}_2 \in \mathbb{R}^{n-n_k}$ and $\tilde{z} \in \mathbb{R}^n$ (respectively in \tilde{y}_2 and \tilde{z}).

2.3. Accuracy

There are some recent results on the errors of (rational) Krylov subspace methods in [21], such as the error in X_k (Theorem 4.1), the perturbed CARE which X_k satisfies (Proposition 5.1) and the perturbation of the corresponding stable Hamiltonian invariant subspace (Proposition 6.1).

From the Arnoldi relationship (4) and $C^\top \in \text{span } P_1$, we have

$$\begin{aligned} P_2^\top C(X_k) &= P_2^\top (A^\top P_1 Y_k P_1^\top + P_1 Y_k P_1^\top A - P_1 Y_k P_1^\top G P_1 Y_k P_1^\top + H) \\ &= P_2^\top (P_1 \Phi_k^\top + v_{k+1} r_k^\top) Y_k P_1^\top = P_2^\top v_{k+1} r_k^\top Y_k P_1^\top. \end{aligned}$$

Together with the Galerkin condition $P_1^\top C(X_k) P_1 = 0$, the residual satisfies

$$R_k = P P^\top C(P_1 Y_k P_1^\top) P P^\top = P \begin{bmatrix} 0 & Y_k r_k v_{k+1}^\top P_2 \\ P_2^\top v_{k+1} r_k^\top Y_k & 0 \end{bmatrix} P^\top, \quad (26)$$

as in [21, Proposition 5.1]. When considering the singular values of the matrix on the right of (26), we obtain the result $\|R_k\| = \|Y_k r_k\|$ (a slightly better result than the inequality in [21]). This is revealing, especially when $\|r_k\|$ is large but $\|R_k\|$ small. Notice that Y_k is the coefficient matrix in $X_k = P_1 Y_k P_1^\top$ for the Krylov basis vectors in P_1 . Assume that the method is producing more accurate approximate solutions for increasing values of k , as the Krylov subspaces improve by adding less useful components. Intuitively, this suggests $(Y_k)_{ij} \rightarrow 0$ as i, j increase, thus the condition number $\kappa(Y_k) \equiv \sigma_{\max}(Y_k)/\sigma_{\min}(Y_k)$ grows as k increases. It indicates why $\|R_k\| = \|Y_k r_k\|$ can be small even when r_k stagnates and is significant in the last few components. We have observed these properties of Y_k and r_k numerically.

Unfortunately, the result in (26) at best enables the *a posteriori* computation of $\|R_k\|$. It does not foretell when the Krylov subspace method will be accurate.

3. Numerical examples

We present three examples to illustrate the inheritance properties of the Krylov subspace method with $\mathcal{K}_k(A^\top, C^\top, s)$ in (3). We choose a constant shift $s_j = \gamma > 0$ for all j as in [17], with γ chosen approximately as the average magnitude of the components of A . For more elaborate strategies for s , consult [21,25,28,47]. Furthermore, we add basis vectors from $\mathcal{K}_k((A - \gamma I)^\top, C^\top)$ which improve the efficiency of the computations while retaining the Arnoldi relationship (4), similar to the extended Krylov subspaces [20]. The performance of the Krylov subspace turns out to be acceptable, for our purpose in illustrating the inheritance properties.

Example 1 is from [18, Example 1], originally designed to show the difficulties from the stagnating r_k . Actually $\|r_k\|$ is small relative to the large stability radius. Examples 2 and 3 are artificially constructed. All examples are small enough that various quantities in the inheritance properties can be estimated. We compute using MATLAB Ver. R2015b on a Lenovo ThinkPad X1 Carbon 4th Signature Edition with an Intel Core i5 6200U CPU at 2.30 GHz and 8 GB RAM.

Estimating τ and ψ

To estimate τ and ψ , we use the eigenvalue optimization software Eigopt [48], which implements the methods in [49,50]. Accurate estimates are generally produced but the global optimum cannot be guaranteed. Similar results have been obtained using direct searches with the MATLAB command `fmincon`. For $\tau(A, B) \equiv \min_{\mu \in \mathbb{C}_+} \sigma_{\min}(A - \mu I, B)$ in (9), from

$$\sigma_{\min}(A, B) \geq \min_{\mu \in \mathbb{C}_+} \sigma_{\min}(A - \mu I, B) = \sigma_{\min}(A - \mu_{\text{opt}} I, B) \geq |\mu_{\text{opt}}| - \|(A, B)\|, \quad (27)$$

we obtain the bounds $|\mu_{\text{opt}}| \leq \|(A, B)\| + \sigma_{\min}(A, B) \leq 2\|(A, B)\|$. This leads to a smaller search region $\{\mu \in \mathbb{C}_+ : |\mu| \leq 2\|(A, B)\|\}$. For $\psi(M) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(M - i\omega I)$ from Section 2.2.3, techniques in (27) produce the search region $\{\omega \in \mathbb{R} : |\omega| \leq 2\|M\|\}$ instead of \mathbb{R} .

Stability test by Byers

For (19), we test whether $\psi(A_k^c) > \zeta$ holds with $\zeta = \|\tilde{z}_1^\top r_k\|$. Instead of computing ψ (a difficult task as mentioned before) and testing directly, we may apply [42, Theorem 1], that the Hamiltonian matrix

$$H_k(\zeta) = \begin{bmatrix} A_k^c & -\zeta I_n \\ \zeta I_n & -A_k^{c\top} \end{bmatrix} \quad (28)$$

has no purely imaginary eigenvalue iff $\psi(A_k^c) > \zeta$.

Table 1Example 1 (stabilizability and detectability) $\tau(A, B) = \tau(A^\top, C^\top) = 2.00$.

k	$\ r_k\ $	$\ \tilde{x}_1^\top r_k\ $	$\tau(\Phi_k, B_1)$	$c_3 + c_4$	NRes	$\ r_k \tilde{y}_2\ $	$\tau(\Phi_k^\top, C_1^\top)$	$c_7 + c_8$
1	1.0	0.4937	2.5321	3.0258	6.8142e-02	0.1218	2.6154	2.7371
2	1.5	0.2089	2.2384	2.4473	1.9000e-03	0.0476	2.2471	2.2947
3	1.5	0.0443	2.1792	2.2236	5.7914e-05	0.0288	2.1278	2.1566
4	1.5	0.0415	2.1451	2.1866	1.8888e-06	0.0184	2.0742	2.0926
5	1.5	0.2013	2.1170	2.3183	6.4307e-08	0.0129	2.0476	2.0605
6	1.5	0.1557	2.0763	2.2320	2.2558e-09	0.0097	2.0330	2.0427
7	1.5	0.1156	2.0491	2.1647	8.0907e-11	0.0076	2.0241	2.0317

Table 2Example 1 (stability radius) with $\zeta = \|\tilde{z}_1^\top r_k\|$ in H_k .

k	$\ r_k\ $	$\ \tilde{z}_1^\top r_k\ $	$\psi(\Phi_k^c)$	$c_3 + c_4$	$\psi(A_k^c)$	NRes	d_k
1	1.0	4.9366e-01	2.5384	3.0321	1.8617	6.8142e-02	1.9335
2	1.5	2.9737e-01	2.1972	2.4946	1.8517	1.9000e-03	1.9738
3	1.5	2.0364e-01	2.1068	2.3105	1.8516	5.7914e-05	1.9857
4	1.5	1.5364e-01	2.0745	2.2282	1.8515	1.8888e-06	1.9902
5	1.5	1.3574e-01	2.0513	2.1870	1.8515	6.4307e-08	1.9915
6	1.5	1.0833e-01	2.0358	2.1442	1.8515	2.2558e-09	1.9932
7	1.5	8.7261e-02	2.0254	2.1127	1.8515	8.0907e-11	1.9943

3.1. Example 1 (Jbilou [18, Example 1])

From [18], with $d = 0.5$ and $n = 1000$, we have $B = \text{rand}(n, 4)$ (with the seeding command `rng(0)`), $C = I_{8 \times n}$ (the first 8 rows of I_n), $R = I_m$ (as in other examples) and

$$A = - \begin{bmatrix} 4 & 1-d & 0 & \cdots & 0 & 1 \\ 1+d & 4 & 1-d & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & 1-d \\ 1 & 0 & \cdots & 0 & 1+d & 4 \end{bmatrix}.$$

We summarize the numerical results in Table 1, where $\|R_k\| = \|Y_k r_k\|$, the normalized residual is

$$\text{NRes} \equiv \frac{\|R_k\|}{2\|A^\top P_1 Y_k\| + \|Y_k G_{11} Y_k\| + \|H_{11}\|},$$

and “Rank” is the rank or width of P_1 . We terminate our computation when $\text{NRes} < \text{tol_ck}$.

The norm of the Arnoldi residual $\|r_k\| = 1.5$ stays constant for $k = 2, \dots, 7$ but we achieved the normalized residual NRes of $8.1\text{e-}11$ when $k = 7$ (and $8.8\text{e-}16$ when $k = 13$).

In Table 1 for each k , approximately 4 to 5 s were required for solving the CARE by the Krylov subspace method, including the estimation of \tilde{x}_1 , \tilde{y}_1 , $\tau(\Phi_k, B_1)$ and $\tau(\Phi_k^\top, C_1^\top)$. The estimation of $\tau(A, B) = 2.0001$ (at $\mu = -1.0491 \times 10^{-4}i$) and $\tau(A^\top, C^\top) = 2.0000$ (at $\mu = -1.2453 \times 10^{-3}i$) cost an additional 220 s.

On any row for a particular k , the entry in column 5 (or the sum of the entries in columns 3 and 4) should be greater than or equal to $\tau(A, B)$ as in (11). Similarly, the entry in column 9 should be no less than $\tau(A^\top, C^\top)$ as in (13). Both hold for Examples 1–3 as shown in Tables 1, 3 and 5 respectively. For Theorems 2.1 and 2.2, we require respectively $\tau(A, B) > \|\tilde{x}_1^\top r_k\|$ (in column 3) and $\tau(A^\top, C^\top) > \|r_k \tilde{y}_2\|$ (in column 7), which hold for all k . As r_k stagnates, NRes diminishes quickly, and $\|\tilde{x}_1^\top r_k\|$ and $\|r_k \tilde{y}_2\|$ decrease. We have observed the bottom-heavy behaviour of r_k and the top-heavy (or diminishing) behaviour of \tilde{x}_1 (or \tilde{y}_2), as suggested in Section 2.

The Hamiltonian matrices for the original and projected CAREs have no eigenvalues near the imaginary axis. The distances between the stable subspectrum of the Hamiltonian matrices to the imaginary axis fall respectively in [1.9937, 6.6567] and [2.0484, 6.6550].

As $\tau(A, B)$, $\tau(A^\top, C^\top) \approx 2 > \|r_k\| \approx 1.5$ ($k \geq 2$), the inheritance of stabilizability and detectability follows from Theorems 2.1 and 2.2. We have estimated a large stability radius $\psi(A_k^c) > 1.8 > \|r_k\| \approx 1.5$ ($k \geq 2$; see Table 2), indicating an “easy” CARE. The ranks of P_1 , thus the execution times required, are small. From the numerical results, $\|r_k\|$ is actually small relative to $\tau(A, B)$, $\tau(A^\top, C^\top)$ and $\psi(A_k^c)$ for moderate values of k . The diminishing R_k yields accurate approximate solutions X_k .

In Table 2, we illustrate the stabilizing property of X_k . To check (18), we display quantities defined in Section 2.2 and d_k which is the minimum absolute value of the real parts of the eigenvalues of H_k in (28). On any row, the 5th entry should

Table 3Example 2 (stabilizability and detectability) $\tau(A, B) = 0.4233$, $\tau(A^\top, C^\top) = 0.4198$.

k	$\ r_k\ $	$\ \tilde{x}_1^\top r_k\ $	$\tau(\Phi_k, B_1)$	$c_3 + c_4$	NRes	$\ r_k \tilde{y}_2\ $	$\tau(\Phi_k^\top, C_1^\top)$	$c_7 + c_8$	$\ X - X_k\ /\ X\ $
1	0.1991	0.1962	0.7850	0.9811	7.3013e-03	0.0214	1.8402	1.8616	1.2810e-01
2	0.2391	0.1300	0.4688	0.5988	1.1276e-04	0.0067	0.4794	0.4861	2.8119e-03
3	0.2259	0.0470	0.4574	0.5044	2.1826e-06	0.0130	0.4597	0.4727	7.3092e-05
4	0.2310	0.0224	0.4502	0.4726	4.0205e-08	0.0144	0.4555	0.4699	1.7515e-06
5	0.1372	0.0116	0.4415	0.4531	2.0645e-10	0.0013	0.4511	0.4524	1.3699e-08

Table 4

Example 2 (stability radius).

k	$\ r_k\ $	$\ \tilde{z}_1^\top r_k\ $	$\psi(\Phi_k^c)$	$c_3 + c_4$	$\psi(A_k^c)$	NRes	d_k
1	0.1991	1.9630e-01	0.9466	1.1429	0.3498	7.3013e-03	4.5002e-01
2	0.2391	1.2977e-01	0.4677	0.5975	0.3386	1.1276e-04	4.8193e-01
3	0.2259	3.0297e-02	0.4572	0.4875	0.3385	2.1826e-06	5.0191e-01
4	0.2310	2.1032e-02	0.4509	0.4720	0.3385	4.0205e-08	5.0249e-01
5	0.1372	1.5206e-02	0.4421	0.4573	0.3385	2.0645e-10	5.0275e-01

Table 5Example 3 (stabilizability and detectability) $\tau(A, B) = \tau(A^\top, C^\top) = 0.0110$.

k	$\ r_k\ $	$\ \tilde{x}_1^\top r_k\ $	$\tau(\Phi_k, B_1)$	$c_3 + c_4$	NRes	$\ r_k \tilde{y}_2\ $	$\tau(\Phi_k^\top, C_1^\top)$	$c_7 + c_8$	$\ X - X_k\ /\ X\ $
1	1.0000	1.0000e+00	2.2361	3.2361	1.1180e-01	9.7117e-01	2.2361	3.2072	4.9373e-01
2	0.9998	1.8370e-01	0.1371	0.3208	1.9595e-02	1.7314e-01	0.1371	0.3102	1.6418e-01
3	0.9976	3.4579e-02	0.0167	0.0513	4.1449e-03	3.0597e-02	0.0167	0.0473	5.1835e-02
4	0.9923	8.1170e-03	0.0112	0.0193	8.7056e-04	6.6748e-03	0.0112	0.0179	1.3300e-02
5	0.9841	9.6034e-04	0.0110	0.0119	1.6076e-04	7.3198e-04	0.0110	0.0117	2.6949e-03
6	0.9727	5.0796e-05	0.0110	0.0110	2.4785e-05	3.5746e-05	0.0110	0.0110	4.3399e-04
7	0.9575	1.4210e-06	0.0110	0.0110	3.1125e-06	9.1824e-07	0.0110	0.0110	5.5443e-05
8	0.9376	2.2997e-08	0.0110	0.0110	3.1295e-07	1.3552e-08	0.0110	0.0110	5.5759e-06
9	0.9123	2.2555e-10	0.0110	0.0110	2.4795e-08	1.2017e-10	0.0110	0.0110	4.3661e-07
10	0.8803	1.3719e-12	0.0110	0.0110	1.5204e-09	6.5385e-13	0.0110	0.0110	2.6226e-08
11	0.8405	5.1926e-15	0.0110	0.0110	7.0545e-11	2.1553e-15	0.0110	0.0110	1.1843e-09

be no less than the 6th. Condition (19) is verified directly, as well as by [42, Theorem 1] with $d_k \neq 0$, which indicates no purely imaginary eigenvalues for H_k in (28). The diminishing nature of $\|\tilde{z}_1^\top r_k\|$ and the top-heaviness of \tilde{z}_1 have been observed. The estimation of $\psi(\Phi_k^c)$ and \tilde{z}_1 required 0.14 to 0.54 s for each k .

3.2. Example 2 [20, Example 4.1]

The matrix A is obtained from the central finite difference discretization of

$$L(u) = \Delta u - 10y \frac{\partial u}{\partial x} - 2x \frac{\partial u}{\partial y} - (y^2 - x^2)u,$$

on the unit square $[0, 1] \times [0, 1]$ with homogeneous Dirichlet boundary conditions. The dimension of A is $n = n_0^2$, where n_0 is the number of inner grid points in each spatial direction. The entries of B and C are uniformly distributed on $[0, 1]$ (with the seeding command `rand('state', 0)`). For the small example with $n = 36$ and $m = l = 2$, we are able to estimate $\tau(A, B)$, $\tau(A^\top, C^\top)$ and other quantities in Tables 3 and 4. For the verification of (11) and (13), we estimated that $\tau(A, B) = 0.4233$ at $\mu = (2.0399 - 2.6186i) \times 10^{-8}$ and $\tau(A^\top, C^\top) = 0.4198$ at $\mu = (2.0310 - 2.1754i) \times 10^{-8}$.

We illustrate the inheritance of stabilizability and detectability in Table 3 and the stabilizing property of X_k in Table 4, which exhibit similar behaviour as in Tables 1 and 2 respectively for Example 1. The stagnation of r_k , the convergence of NRes, and the inequalities (11), (13) and (18) are evident. We have also observed the bottom-heavy nature of r_k and the top-heaviness of \tilde{x}_1 and \tilde{z}_1 , as well as the diminishing behaviour of \tilde{y}_2 .

3.3. Example 3

We construct this small example to illustrate the inheritance properties, where $A = \text{tridiag}(1, -2, 1)$ and $B = C^\top = e_1$, the first column of I_n . For $n = 36$ in Table 5, we estimate that $\tau(A, B) = \tau(A^\top, C^\top) = 0.0110$, both at $\mu = 5.3508 \times 10^{-7} + 5.1238 \times 10^{-10}i$.

Tables 5 and 6 for Example 3 exhibit similar behaviour as in Tables 1 and 2 respectively for Example 1. The stagnation of r_k , the convergence of NRes, and the inequalities (11), (13) and (18) are evident. We have also observed the bottom-heavy nature of r_k and the top-heaviness of \tilde{x}_1 and \tilde{z}_1 , as well as the diminishing behaviour of \tilde{y}_2 . However, conditions (19) for Theorem 2.1 are violated for $k \leq 3$. Coincidentally the stability tests in Table 6 failed also for $k \leq 3$, with some negligible d_k in the last column.

Table 6
Example 3 (stability radius).

k	$\ r_k\ $	$\ \tilde{z}_1^\top r_k\ $	$\psi(\Phi_k^c)$	$c_3 + c_4$	$\psi(A_k^c)$	NRes	d_k
1	1.0000	1.0000e+00	2.2361	3.2361	0.0073	1.1180e-01	2.7756e-17
2	0.9998	1.6353e-01	0.0744	0.2379	0.0076	1.9595e-02	3.4694e-18
3	0.9976	2.2605e-02	0.0099	0.0325	0.0074	4.1449e-03	1.9949e-16
4	0.9923	4.5954e-03	0.0075	0.0121	0.0074	8.7056e-04	5.7948e-03
5	0.9841	4.4250e-04	0.0074	0.0078	0.0074	1.6076e-04	7.3816e-03
6	0.9727	1.5932e-05	0.0074	0.0074	0.0074	2.4785e-05	7.3949e-03
7	0.9575	7.1173e-07	0.0074	0.0074	0.0074	3.1125e-06	7.3949e-03
8	0.9376	2.6699e-08	0.0074	0.0074	0.0074	3.1295e-07	7.3949e-03
9	0.9123	2.7331e-09	0.0074	0.0074	0.0074	2.4795e-08	7.3949e-03
10	0.8803	1.7268e-10	0.0074	0.0074	0.0074	1.5204e-09	7.3949e-03
11	0.8405	8.3688e-12	0.0074	0.0074	0.0074	7.0545e-11	7.3949e-03

4. Conclusions

A novel analysis of the Krylov subspace methods has been presented. The inheritance properties of stabilizability and detectability, and other structures of CAREs, have been investigated. One result from the perturbation theory depends explicitly on R_k but not r_k .

We have presented some numerical results for a particular Krylov subspace. A comprehensive comparison of different Krylov subspaces is a worthwhile but large project for the future. We would like to further illustrate numerically the inheritance properties associated with AREs. However, the estimation of the distances to unstabilizability and undetectability or the stability radius is difficult, especially for large-scale problems. Studies of inheritance properties for other types of Riccati equations and related linear matrix equations are also possible.

We have been motivated by the assumptions of solvability of the pCARE in existing literature, and we explore other possibilities. Applicability of the results in the paper is dependent on the particular applications and the preferences of the users. Not all possibilities have been explored, many problems are left unsolved and much more has to be done. We have barely scratched the surface of the theory behind the Krylov subspace methods for CAREs.

Acknowledgements

Part of the work occurred when the first and third authors visited the School of Mathematical Sciences at Fudan University. The first and the second authors were supported by the National Natural Science Foundation, China (Grant 11601484) and the Ministry of Science and Technology, Taiwan (Grant #MOST-108-2115-M-003-004), respectively.

Appendix. Heuristics on \tilde{x}_1 , \tilde{y}_2 and \tilde{z}_1

Motivated by the diminishing $\|R_k\| = \|Y_k r_k\|$ in Section 2.3, possibly the result of the “top-heaviness” of Y_k and the “bottom-heaviness” of the stagnating r_k , the speculation on the smallness of $\|\tilde{x}_1^\top r_k\|$, $\|r_k \tilde{y}_2\|$ and $\|\tilde{z}_1^\top r_k\|$ may be understood with some heuristics.

For \tilde{x}_1 in Theorem 2.1, recall the link between the projection methods and model order reduction as suggested in [21, Section 3]. In (11), let $k' < k$ and

$$[\Phi_k, B_1] = \begin{bmatrix} \Phi_{k'} & \Phi_k^{(12)} & B_1^{(1)} \\ \Phi_k^{(21)} & \Phi_k^{(22)} & B_1^{(2)} \end{bmatrix}. \quad (29)$$

With $\tilde{x}_1 = [\tilde{x}_1^{(1)\top}, \tilde{x}_1^{(2)\top}]^\top$ and $\tilde{\mu} \in \mathbb{C}_+$ optimizing the first term in (11), consider

$$\begin{aligned} \tau(\Phi_k, B_1) &= \min_{\mu \in \mathbb{C}_+} \min_{\|x_1\|=1} \|x_1^\top [\mu I - \Phi_k, B_1]\| \\ &= \min_{\mu \in \mathbb{C}_+} \min_{\|x_1\|=1} \left\| x_1^\top \begin{bmatrix} \mu I - \Phi_{k'} & -\Phi_k^{(12)} & B_1^{(1)} \\ -\Phi_k^{(21)} & \mu I - \Phi_k^{(22)} & B_1^{(2)} \end{bmatrix} \right\| \\ &= \left\| \tilde{x}_1^{(1)\top} [\tilde{\mu} I - \Phi_{k'}, -\Phi_k^{(12)}, B_1^{(1)}] + \tilde{x}_1^{(2)\top} [-\Phi_k^{(21)}, \tilde{\mu} I - \Phi_k^{(22)}, B_1^{(2)}] \right\|. \end{aligned} \quad (30)$$

Assume that $[-\Phi_k^{(21)}, \tilde{\mu} I - \Phi_k^{(22)}, B_1^{(2)}]$ in (30) is full-rank, which is generically true. When the essence of $\{A, B\}$ is represented adequately by the reduced system $\{\Phi_{k'}, B_1^{(1)}\}$ for a large enough k' , the stabilizability of $\{\Phi_k, B_1\}$ is represented accurately by that of $\{\Phi_{k'}, B_1^{(1)}\}$. The distance from unstabilizability $\tau(\Phi_k, B_1)$ is converging with respect to k and will not be influenced much by $\Phi_k^{(21)}$, $\Phi_k^{(22)}$ or $B_1^{(2)}$. This holds iff $\tilde{x}_1^{(2)\top} [-\Phi_k^{(21)}, \tilde{\mu} I - \Phi_k^{(22)}, B_1^{(2)}] \approx 0$ in (30) or $\tilde{x}_1^{(2)} \approx 0$. So \tilde{x}_1 is top-heavy and $\|\tilde{x}_1^\top r_k\|$ is small for the stagnating r_k in Theorem 2.1.

For \check{z}_1 in (18), a similar heuristic may apply for $\|\check{z}_1^\top r_k\|$ in (19). Assume that the (i, j) -component in Y_k is diminishing as i, j increase, thus $\mathcal{E} = Y_k - \begin{bmatrix} Y_{k'} & 0 \\ 0 & 0 \end{bmatrix}$ is negligible for a large enough k' . In (18), with $k' < k$, $\check{\omega} \in \mathbb{R}$ and Φ_k partitioned as in (29), we have

$$\begin{aligned} \Phi_k^c - i\check{\omega}I &= \Phi_k - Y_k G_{11} - i\check{\omega}I \\ &= \begin{bmatrix} \Phi_{k'}^{(12)} & \Phi_k^{(12)} \\ \Phi_k^{(21)} & \Phi_k^{(22)} \end{bmatrix} - \left\{ \begin{bmatrix} Y_{k'} & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{E} \right\} \begin{bmatrix} G_{k'}^{(12)} & G_k^{(12)} \\ G_k^{(21)} & G_k^{(22)} \end{bmatrix} - i\check{\omega}I \\ &= \begin{bmatrix} \Phi_{k'}^{(12)} - Y_{k'} G_{k'}^{(12)} - i\check{\omega}I & \Phi_k^{(12)} - Y_{k'} G_k^{(12)} \\ \Phi_k^{(21)} & \Phi_k^{(22)} - i\check{\omega}I \end{bmatrix} + O(\|\mathcal{E}\|) \\ &= \begin{bmatrix} \Phi_{k'}^c - i\check{\omega}I & \Phi_k^{(12)} - Y_{k'} G_k^{(12)} \\ \Phi_k^{(21)} & \Phi_k^{(22)} - i\check{\omega}I \end{bmatrix} + O(\|\mathcal{E}\|). \end{aligned}$$

With $\check{z}_1 = [\check{z}_1^{(1)\top}, \check{z}_1^{(2)\top}]^\top$ being the left singular vector in (18), analogous to (30), we have

$$\psi(\Phi_k^c) = \left\| \check{z}_1^{(1)\top} [\Phi_{k'}^c - i\check{\omega}I, \Phi_k^{(12)} - Y_{k'} G_k^{(12)}] + \check{z}_1^{(2)\top} [\Phi_k^{(21)}, \Phi_k^{(22)} - i\check{\omega}I] \right\|.$$

Assume that $[\tilde{\Phi}_k^{(21)}, \tilde{\Phi}_k^{(22)}]$ is full-rank, which is generically true, and $\|\mathcal{E}\|$ is negligible. The stability radius $\psi(\Phi_k^c)$ is approximately equal to $\psi(\Phi_{k'}^c)$, which is true iff $\check{z}_1^{(2)\top} [\Phi_k^{(21)}, \Phi_k^{(22)} - i\check{\omega}I] \approx 0$ or $\check{z}_1^{(2)} \approx 0$. So \check{z}_1 is top-heavy and $\|\check{z}_1^\top r_k\|$ diminishes in (19) and (20).

For \check{y}_2 in (13), considering Φ_k and A_{22} respectively as representations of the reduced and complementary subsystems, \check{y}_2 is part of the singular vector associated with A_{22} corresponding to the minimum singular value in the first term of (14). In a successful Krylov subspace method, the system $\{A, C\}$ is represented increasingly better by $\{\Phi_k, C_1\}$ for increasing values of k , with A_{22} having little influence. In (14), with a generically nonsingular $\tilde{\mu}I - A_{22}$, $(\tilde{\mu}I - A_{22})\check{y}_2 \approx 0$ or $\check{y}_2 \approx 0$. Thus $\check{y}_2 \equiv v_{k+1}^\top P_2 \check{y}_2$ diminishes in norm in (13).

From numerical experiments, we have supporting evidence for the top-heaviness of \check{x}_1 , \check{z}_1 and \check{z} and the diminishing nature of $\|\check{x}_1^\top r_k\|$, $\|r_k \check{y}_2\|$ and $\|\check{z}_1^\top r_k\|$, with respect to increasing k .

References

- [1] P. Lancaster, L. Rodman, *Algebraic Riccati Equations*, Clarendon Press, Oxford, 1995.
- [2] V.L. Mehrmann, The Autonomous Linear Quadratic Control Problem, in: *Lecture Notes in Control and Information Sciences*, vol. 163, Springer Verlag, Berlin, 1991.
- [3] D.A. Bini, B. Iannazzo, B. Meini, *Numerical Solution of Algebraic Riccati Equations*, SIAM, Philadelphia, 2012.
- [4] R. Byers, Solving the algebraic Riccati equation with the matrix sign function, *Linear Algebra Appl.* 85 (1987) 267–279.
- [5] E.K.-W. Chu, H.-Y. Fan, W.-W. Lin, A structure-preserving doubling algorithm for continuous-time algebraic Riccati equations, *Linear Algebra Appl.* 396 (2005) 55–80.
- [6] J.D. Gardiner, A stabilized matrix sign function algorithm for solving algebraic Riccati equations, *SIAM J. Sci. Comput.* 18 (1997) 1393–1411.
- [7] A.J. Laub, A schur method for solving algebraic Riccati equation, *IEEE Trans. Automat. Control* AC-24 (1979) 913–921.
- [8] D. Kleinman, On an iterative technique for Riccati equation computations, *IEEE Trans. Automat. Control* 13 (1968) 114–115.
- [9] P. Benner, J.-R. Li, T. Penzl, Numerical solution of large Lyapunov equations, Riccati equations and linear-quadratic control problems, *Numer. Linear Algorithms Appl.* 15 (2008) 755–777.
- [10] E. Bänsch, P. Benner, Stabilization of incompressible flow problems by Riccati-based feedback, in: G. Leugering, S. Engell, A. Griewank, M. Hinze, R. Rannacher, V. Schulz, M. Ulbrich (Eds.), *Constrained Optimization and Optimal Control for Partial Differential Equations*, in: *International Series of Numerical Mathematics*, vol. 160, Birkhäuser, Basel, 2011, pp. 5–20.
- [11] P. Benner, J. Saak, A galerkin-Newton-ADI method for solving large-scale algebraic Riccati equations, DFG Priority Programme 1253 Optimization with Partial Differential Equations, 2010, Preprint SPP1253-090, available at www.am.uni-erlangen.de/home/spp1253/wiki/images/2/28/Preprint-SPP1253-090.pdf.
- [12] P. Benner, J. Saak, Numerical solution of large and sparse continuous time algebraic matrix Riccati and Lyapunov equations: A state of the art survey, *GAMM-Mitt.* 6 (2013) 32–52.
- [13] E.A. Jonckheere, L.M. Silverman, A new set of invariants for linear systems — application to reduced order compensator design, *IEEE Trans. Automat. Control* 28 (1983) 953–964.
- [14] P. Benner, Z. Bujanović, On the solution of large-scale algebraic Riccati equations by using low-dimensional invariant subspaces, *Linear Algebra Appl.* 488 (2016) 430–459.
- [15] P. Benner, Z. Bujanović, P. Kürschner, J. Saak, RADI: A low-rank ADI-type algorithm for large-scale algebraic Riccati equations, *Numer. Math.* 138 (2018) 301–330.
- [16] J. Saak, M. Köhler, p. Benner, M-M.E.S.S.-1.0.1 — The matrix equations sparse solvers library, 2016, <https://doi.org/10.5281/zenodo.50575>; see also: <https://www.mpi-magdeburg.mpg.de/projects/mess>.
- [17] T. Li, E.K.-W. Chu, W.-W. Lin, P.C.-Y. Weng, Solving large-scale continuous-time algebraic Riccati equations by doubling, *J. Comput. Appl. Math.* 237 (2013) 373–383.
- [18] K. Jbilou, Block Krylov subspace methods for large algebraic Riccati equations, *Numer. Algorithms* 34 (2003) 339–353.
- [19] K. Jbilou, An Arnoldi based algorithm for large algebraic Riccati equations, *Appl. Math. Lett.* 19 (2006) 437–444.
- [20] M. Heyouni, K. Jbilou, An extended block Arnoldi algorithm for large-scale solutions of the continuous-time algebraic Riccati equations, *Electron. Trans. Numer. Anal.* 33 (2009) 53–62.
- [21] V. Simoncini, Analysis of the rational Krylov subspace projection method for large-scale algebraic Riccati equations, *SIAM J. Matrix Anal. Appl.* 37 (2016) 1655–1674.

- [22] V. Simoncini, D.B. Szyld, M. Monsalve, On two numerical methods for the solution of large-scale algebraic Riccati equations, *IMA J. Numer. Anal.* 34 (2014) 904–920.
- [23] K. Deckers, A. Bultheel, Rational Krylov Sequences and Orthogonal Rational Functions, Tech. report TW499, Department of Computer Science, KU Leuven, Leuven, Belgium, 2007.
- [24] V. Druskin, L. Knizhnerman, V. Simoncini, Analysis of the rational Krylov subspace and ADI methods for solving the Lyapunov equation, *SIAM J. Numer. Anal.* 49 (2011) 1875–1898.
- [25] V. Druskin, C. Lieberman, M. Zaslavsky, On adaptive choice of shifts in rational Krylov subspace reduction of evolutionary problems, *SIAM J. Sci. Comput.* 32 (2010) 2485–2496.
- [26] V. Druskin, V. Simoncini, Adaptive rational Krylov subspaces for large-scale dynamical systems, *Syst. Control Lett.* 60 (2011) 546–560.
- [27] S. Güttel, Rational krylov approximation of matrix functions: numerical methods and optimal pole selection, *GAMM-Mitt.* 36 (2013) 8–31.
- [28] S. Güttel, L. Knizhnerman, Automated parameter selection for rational Arnoldi approximation of Markov functions, *Proc. Appl. Math. Mech.* 11 (2011) 15–18.
- [29] K.H.A. Olsson, A. Ruhe, Rational Krylov for eigenvalue computation and model order reduction, *BIT* 46 (2006) 99–111.
- [30] A. Ruhe, Rational Krylov sequence methods for eigenvalue computation, *Linear Algebra Appl.* 58 (1984) 391–405.
- [31] Y. Lin, V. Simoncini, A new subspace iteration method for the algebraic Riccati equation, *Numer. Linear Algebra Appl.* 22 (2015) 26–47.
- [32] Y. Lin, V. Simoncini, Minimal residual methods for large scale Lyapunov equations, *Appl. Numer. Math.* 72 (2013) 52–71.
- [33] A. Massoudi, M.R. Opmeer, T. Reis, Analysis of an iteration method for the algebraic Riccati equation, *SIAM J. Matrix Anal. Appl.* 37 (2016) 624–648.
- [34] C. Kenney, A.J. Laub, M. Wette, Error bounds for Newton refinement of solutions to algebraic Riccati equations, *Math. Control Signals Systems* 3 (1990) 211–224.
- [35] M. Gu, E. Mengi, M.L. Overton, J. Xia, J. Zhu, Fast methods for estimating the distance to uncontrollability, *SIAM J. Matrix Anal. Appl.* 28 (2006) 477–502.
- [36] C.C. Paige, Properties of numerical algorithms relating to computing controllability, *IEEE Trans. Automat. Control* 26 (1981) 130–138.
- [37] G. Hu, F.J. Davison, Real controllability/stabilizability radius of LTI systems, *IEEE Trans. Automat. Control* 49 (2004) 254–257.
- [38] F. Kangal, K. Meerbergen, E. Mengi, W. Michiels, A subspace method for large scale eigenvalue optimization, *SIAM J. Matrix Anal. Appl.* 39 (2018) 48–82.
- [39] C. Kenney, A. Laub, Controllability and stability radii for companion form systems, *Math. Control Signals Systems* 1 (1988) 239–256.
- [40] D. Kressner, Deflation in Krylov subspace methods and distance to uncontrollability, *Ann. Univ. Ferrara Sezione 7: Sci. Mat.* 53 (2007) 309–318.
- [41] G.H. Golub, C.F. Van Loan, *Matrix Computations*, third ed., Johns Hopkins University Press, Baltimore, MD, 1996.
- [42] R. Byers, A bisection method for measuring the distance of a stable matrix to the unstable matrices, *SIAM J. Sci. Stat. Comput.* 9 (1988) 875–881.
- [43] D. Kressner, Subspace methods for computing the pseudospectral abscissa and the stability radius, *SIAM J. Matrix Anal. Appl.* 35 (2014) 292–313.
- [44] C. Van Loan, How near is a stable matrix to an unstable matrix?, *Contemp. Math.* 47 (1985) 465–478.
- [45] P. Benner, T. Mitchell, Faster and more accurate computation of the H_∞ norm via optimization, *SIAM J. Sci. Comput.* 10 (2018) A3609–A3635.
- [46] J.-G. Sun, Residual bounds of approximate solutions of the algebraic Riccati equations, *Numer. Math.* 76 (1997) 249–263.
- [47] P. Benner, P. Kürschner, J. Saak, Self-generating and efficient shift parameters in ADI methods for large Lyapunov and Sylvester equations, *Electron. Trans. Numer. Anal.* 43 (2014) 142–162.
- [48] E. Mengi, *Eigopt: Software for eigenvalue optimization*, 2014, available at <http://home.ku.edu.tr/~eMengi/software/eigopt.html>.
- [49] L. Breiman, A. Cutler, A deterministic algorithm for global optimization, *Math. Program.* 58 (1993) 179–199.
- [50] E. Mengi, E.A. Yildirim, M. Kilic, Numerical optimization of eigenvalues of Hermitian matrix functions, *SIAM J. Matrix. Anal. Appl.* 35 (2014) 699–724.