JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 95 (1998) 13–27

An approach to solutions of systems of linear partial differential equations with applications

H.I. Abdel-Gawad *

Department of Mathematics, Faculty of Science, Cairo University, Egypt

Received 10 February 1997; received in revised form 10 February 1998

Abstract

Initial and boundary value problems governed by a system of linear partial differential equations can be solved by using the classical methods. This holds in solving problems which are governed by a unique system of equations over the whole region of interest. But if a problem is governed by a given system of equations over a region and by another system over the complementary one, classical methods may fail in treating this problem. A typical problem is that of evaluating the time-dependent electric field in the conductive half-space (the substratum) as a model in geophysical prospecting. The electric field in the air above the substratum is time independent. This problem has been solved numerically. Here, we solve it analytically. We proceed by presenting an approach for finding the solutions of systems of linear partial differential equations. Eigen operators and fractional powers of matrices of operators have been introduced. The formal solutions obtained are adequate for studying initial and boundary value problems whose solutions are anharmonic ones. They are used to solve the above-mentioned problem. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Geophysical prospecting; System of partial differential equations; A method of solution

1. Introduction

The theory of fractional calculus has been developed remarkably in the last decade. The use of fractional calculus in the applications has the advantage of easily tackling problems with complicated boundary conditions: For example in problems with discontinuous boundary conditions at the surface which separates two media [3]. Also, this holds in solving the problems where the boundary surface assumes an arbitrary geometry [4]. A third example is concerning the problems which have to be solved over a domain Ω and they are described by a set of partial differential equations PDE on $\Omega_1 \subset \Omega$ and by another set on $\Omega_1 \backslash \Omega^c$. In this case, the use of fractional operators is more feasible. A typical example is the problem of evaluating the electric field in the conductive half-space (the

* E-mail: abdelgawad@fac.sci.cairo.uni.eun.eg.

PII S0377-0427(98)00061-2

substratum) as a model in geophysical prospecting. The details of this problem is given in Section 4. In what follows we shall develop a method for solving a system of PDE.

In a set of papers [1, 2] the author has developed an approach for solving PDE in the form

$$\partial_t^n f(x,t) = \hat{L}(x,t) f(x,t) + S(x,t), \tag{1.1}$$

where $\hat{L}(x,t) = \sum_{i=1}^m a_i(x,t)\partial_x^i$ and S is a source function.

In these papers, we have used fractional power operators to obtain the formal solution of (1.1). The present work may be considered as a continuation of the work done in [1, 2]. Here, we shall introduce the notions of eigenoperators and fractional power of a matrix of operators.

Now, we consider the system of PDE

$$\partial_t^n u - \hat{M} u = \boldsymbol{S} \quad \text{in } \Omega, \tag{1.2}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)^T$, $\boldsymbol{S} = (S_1, S_2, \ldots, S_m)^T$, $u_j = u_j(\boldsymbol{x}, t)$, $S_j = S_j(\boldsymbol{x}, t)$; $\boldsymbol{x}(x_1, x_2, x_3)$; $j = 1, 2, \ldots, m$ and $\Omega = \Re^3 \times (0, T)$. In (1.2)

$$\hat{M} = (\hat{M}_{ij}), \quad \hat{M}_{ij} = \sum_{k,l,s} a_{ij}^{kls} \partial_{x_1}^k \partial_{x_2}^l \partial_{x_3}^s,$$

where a_{ij}^{kls} are constants and $i, j = 1, 2, \ldots, m$.

We solve (1.2) when $n = 1$ and $n \geqslant 2$ separately for initial value problems

$$\partial_t^r \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{f}_r(\boldsymbol{x}), \quad r = 0, 1, \ldots, n-1, \tag{1.3}$$

where $\boldsymbol{f}_j(\boldsymbol{x}) = (f_{jl}(\boldsymbol{x}), \ldots, f_{jm}(\boldsymbol{x}))^T$ will be assumed to belong to $C^\infty \cap L_1$ on $\Re^3$. This condition is necessary to continue in finding the solution of (1.2).

2. Solution of the Eq. (1.2) when $n = 1$

In this case Eq. (1.2) becomes

$$(\partial_t - \hat{M})\boldsymbol{u} = \boldsymbol{S} \quad \text{in } \Omega, \tag{2.1}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{f}_0(\boldsymbol{x}). \tag{2.2}$$

The formal solution of (2.1–2) is [1, 2].

$$\boldsymbol{u} = \mathrm{e}^{t\hat{M}} \hat{f}_0(\boldsymbol{x}) + \int_0^t \mathrm{e}^{(t-t_1)\hat{M}} \boldsymbol{S}(\boldsymbol{x}, t_1)\,\mathrm{d}t_1. \tag{2.3}$$

One can easily verify that (2.3) satisfies (2.1–2). To continue with (2.3), we assume that $\boldsymbol{f}_0, \boldsymbol{S} \in C^\infty(\Re^3)$ and that partial derivatives of all orders of $\boldsymbol{f}_0$ and $\boldsymbol{S}$ belong to $L_1(\Re^3)$. Equivalently, $\boldsymbol{f}_0$ and $\boldsymbol{S}$ are assumed to belong to the Sobolov space H_1^∞. We remark that the commutator $[\hat{M}_{ij}, \hat{M}_{kl}]$ vanishes. Now, we give the following definition:

Definition 1. The operator $\hat{L}$ is an eigenoperator of the matrix of operators M defined on $H_1^\infty(\Omega)$ such that $\hat{\boldsymbol{M}}\boldsymbol{f} \neq \boldsymbol{0}$ $\forall \boldsymbol{f} \in H_1^\infty$ if

$$|\hat{\boldsymbol{M}} - \hat{L}I| = \hat{\boldsymbol{0}}, \tag{2.4}$$

where I is the identity matrix. We shall show that $\hat{L}$ preserves the properties of $(\hat{M}_{ij})$ namely $[\hat{L}_k, \hat{L}_s] = 0$ and $\hat{L}_k : L_1(\Re^3) \mapsto (\Re^3)$ for all $k, s = 1, 2, \ldots, m$.

To this end, we proceed by the following lemma:

Lemma 2. *Under the above assumptions* $\hat{M}_{ij} f_0, \hat{M}_{ij} \boldsymbol{S} \in L_1(\Re^3)$ *and then*

$$\left(\prod_{k=1}^{m} \hat{M}_{i_k j_k} \text{ and } \hat{M}_{ij}^k \right) : L_1(\Re^3) \mapsto L_1(\Re)^3, \quad k \in N.$$

We define fractional power of $\hat{M}_{ij}$ namely $\hat{M}_{ij}^{\beta}, 0 < \beta < 1$ as $\hat{M}_{ij}^{\beta} = (\hat{M}_{ij}^{-\beta})^{-1}$ where

$$\hat{M}_{ij}^{-\beta} = \frac{1}{\Gamma(\beta)} \int \mathrm{e}^{-\lambda \hat{M}_{ij}} \lambda^{\beta - 1} \, \mathrm{d}\lambda, \tag{2.5}$$

where, we bear in mind that the integral in (2.5) converges [7, pp. 5–21]. That is the eigenvalues of $(\hat{M}_{ij})$ (or their real parts) are all positive. In (2.5), the exponential operator is defined by

$$\mathrm{e}^{-\lambda \hat{M}_{ij}} = \frac{1}{2\pi i} \int_{\Gamma} \mathrm{e}^{-\lambda z} (Z\hat{I} - \hat{M}_{ij})^{-1} \, \mathrm{d}z, \tag{2.6}$$

where $Z\hat{I} - \hat{M}_{ij}$ is the resolvant operator. The contour Γ is chosen with holes about the eigen values of $\hat{M}_{ij}$ if they are discrete. But if they are continuous and positive, then Γ is taken with a semi-circle enclosing the positive eigenvalues of $\hat{M}_{ij}$.

Lemma 3. *Under the conditions in Lemma* 1,

$$(\exp -\lambda \hat{M}_{ij} \text{ and } \hat{M}_{ij}^{\beta}) : L_1(\Re^3) \mapsto L_1(\Re^3), \quad 0 < \beta < 1.$$

The proof of this lemma follows from (2.5–6).

Theorem 4. *If a matrix of operators* $(\hat{M}_{ij}) : L_1(\Re^3) \mapsto L_1(\Re^3)$, *then so will be its eigenoperators* $\hat{L}_k, s$.

Proof. Any eigenoperator $\hat{L}$ of $(\hat{M}_{ij})$ is functional in $\hat{M}_{ij}$. By analogy to (2.5) it can be expressed as

$$\hat{L} = \hat{G}_0(\hat{M}_{ij}) + \int_0^{\infty} \hat{G}_1(\hat{M}_{ij}, \lambda) \, \mathrm{d}\lambda, \tag{2.7}$$

where $\hat{G}_0$ is linear in $\hat{M}_{ij}$ and G_1 is expressed as a power series in $\prod_{k=1} \hat{M}_{i_k j_k}$. The rest of the proof follows from (2.7) and Lemmas 2 and 3. Justification of formula (2.7) can be done by induction on $m = 2, 3, \ldots$.

As a direct corollary of this theorem, it follows that $\hat{L}^{-1} : L_1(\Re^3) \mapsto L_1(\Re^3)$. We notice that any two eigenoperators commute.

We return to (2.3) and distinguish two cases:

(a) When the eigenoperators of the matrix $(\hat{M}_{ij})$ are all distinct. In this case (2.3) becomes [9, pp. 155–165] (see Appendix)

$$\boldsymbol{u} = \sum_{i=1}^{m} \hat{H}_i \left\{ e^{t\hat{L}_i} \boldsymbol{f}_0(\boldsymbol{x}) + \int_0^t e^{(t-t_1)\hat{L}_i} \boldsymbol{S}(\boldsymbol{x}, t_1)\, dt_1 \right\}, \tag{2.8}$$

$$\hat{H}_i = \prod_{j \neq i}^{m} (\hat{M} - \hat{L}_j I)(\hat{L}_i - \hat{L}_j)^{-1}. \tag{2.9}$$

By introducing the Fourier transform of $\boldsymbol{f}_0$ and $\boldsymbol{S}$, one can easily find that [1, 2]

$$\boldsymbol{u} = \sum_{i=1}^{m} \int_D \hat{H} \left\{ e^{tL_i} \tilde{\boldsymbol{f}}_0(\boldsymbol{p}) + \int_0^t e^{(t-t_1)L_i} \tilde{\boldsymbol{S}}(\boldsymbol{p}, t_1)\, dt_1 \right\} e^{i\boldsymbol{p}.\boldsymbol{x}} \cdot \frac{d\boldsymbol{p}}{(2\pi)^3}. \tag{2.10}$$

In (2.11) $(\hat{L}_i, \hat{M}) e^{i\boldsymbol{p}.\boldsymbol{x}} = L_i((p_1, p_2, p_3), M(p_1, p_2, p_3)) e^{i\boldsymbol{p}.\boldsymbol{x}}$.

The domain D of the integral in (2.10) is determined such that $L_i(p_1, p_2, p_3)$ are positive for all $i = 1, \ldots, m$.

It is important to notice that if all the eigenoperators of $\hat{M}$ can be expressed in terms of $\hat{G}_0$ only (cf. (2.7) then the condition on $\boldsymbol{f}_0$ and $\boldsymbol{S}$ is relaxed to $\boldsymbol{f}_0, \boldsymbol{S} \in C^k(\Re^3) \cap L_1(\Re^3)$, where k is the maximum order in $(\hat{M}_{ij})$.

(b) When the eigenoperators of $\hat{M}$ are not all distinct. We assume that, for simplicity, only one eigenoperator, namely $\hat{L}_1$, is degenerate with degree of degeneracy k. The remaining eigenoperators are $\hat{L}_{k+1}, \ldots, \hat{L}_m$. In this case, we have

$$e^{t\hat{M}} = e^{t\hat{L}_1} \sum_{r=1}^{k} t^{r-1} \hat{\boldsymbol{B}}_r + \sum_{j=k+1}^{m} e^{t\hat{L}_i} \hat{\boldsymbol{K}}_j, \tag{2.11}$$

where $\hat{\boldsymbol{K}}_j$ and $\hat{\boldsymbol{B}}_r$ are given by

$$\hat{\boldsymbol{K}}_j = (\hat{\boldsymbol{M}} - \hat{L}_1 T)(\hat{L}_1 - \hat{L}_j)^{-k} \prod_{i \neq j}^{m} (\hat{\boldsymbol{M}} - \hat{L}_j I)(\hat{L}_i - \hat{L}_j)^{-1}, \tag{2.12}$$

$$\hat{\boldsymbol{B}}_1 = \hat{I} - \sum_{j=k+1}^{m} \hat{\boldsymbol{K}}_j, \tag{2.13}$$

$$1!\hat{\boldsymbol{B}}_2 = (\hat{\boldsymbol{M}} - \hat{L}_1 I) + \sum_{j=k+1}^{m} (\hat{L}_1 - \hat{L}_j) \hat{\boldsymbol{K}}_j, \tag{2.14}$$

$$\vdots$$

$$(k-1)!\hat{\boldsymbol{B}}_k = (\hat{\boldsymbol{M}} - \hat{L}_1 I)^{k-1} + (-1)^k \sum_{j=k+1}^{m} (\hat{L}_1 - \hat{L}_j)^{k-1} \hat{\boldsymbol{K}}_j. \tag{2.15}$$

When substituting (2.11–15) into (2.3), we obtain formulae similar to (2.8–10).

3. Solution of Eq. (1.2) when $n \geqslant 2$

First, we consider the case $n=2$ and $\boldsymbol{S}=0$, so that Eq. (1.2) becomes

$$(\partial_t^2 - \hat{\boldsymbol{M}})\boldsymbol{u} = 0, \tag{3.1}$$

we rewrite (3.1) in the form

$$(\partial_t - \hat{\boldsymbol{M}}^{1/2})(\partial_t + \hat{\boldsymbol{M}}^{1/2})\boldsymbol{u} = 0. \tag{3.2}$$

In (3.2) $\hat{\boldsymbol{M}}^{1/2} = \hat{\boldsymbol{M}}(\hat{\boldsymbol{M}}^{1/2})$, where

$$\hat{\boldsymbol{M}}^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-\lambda \hat{\boldsymbol{M}}} \lambda^{1/2}\, \mathrm{d}\lambda. \tag{3.3}$$

Again, we bear in mind that the integral in (3.3) converges and $\exp(-\lambda \hat{M})$ is defined as in Section 2.

Here, we shall assume that the eigenvalues of the eigenoperators are all positive. By this assumption, the convergence of the integral in (3.3) is guaranteed.

By using (3.3), the results of Section 2, and by assuming that the eigenoperators of $\hat{M}$ are distinct, we find (see also [9, pp. 155–165])

$$\hat{\boldsymbol{M}}^{-1/2} = \sum_{i=1}^{m} \hat{L}_i^{-1/2} \hat{\boldsymbol{H}}_i, \tag{3.4}$$

where $\hat{\boldsymbol{H}}_i$ is given by (2.9). Now, we have

$$\hat{\boldsymbol{M}}^{-1/2} = \hat{\boldsymbol{M}}(\hat{\boldsymbol{M}}^{-1/2}) = \sum_{i=1}^{m} [\hat{L}^{-1/2}(\hat{\boldsymbol{M}} - \hat{L}_i I)\hat{\boldsymbol{H}}_i + \hat{L}_i^{1/2} \hat{\boldsymbol{H}}_i]. \tag{3.5}$$

By using the Cayley–Hamilton theorem in the linear algebra [9], the first term in (3.5) vanishes.

In the previous section, we have shown that $\hat{L}_i$ assumes the same properties as $(\hat{M}_{ij})$. By a similar way, we can show that $\hat{L}_i^{1/2}$ assumes the same properties as $\hat{L}_i$. Now, the solution of (3.2) is

$$\boldsymbol{u}_h = \mathrm{e}^{t\hat{M}^{1/2}} \boldsymbol{\psi}_0 + \mathrm{e}^{-t\hat{M}^{1/2}} \boldsymbol{\psi}_1. \tag{3.6}$$

To continue with (3.6), we distinguish two cases:

(a) When the eigenoperators of $\hat{\boldsymbol{M}}$ are all distinct. In this case, we use the following theorem:

Theorem 5. *If A is a matrix of real numbers (a_{ij}), $i,j = 1,2,\ldots,m$ and its eigenvalues λ_i are all positive and distinct, then the eigenvalues of $A^{1/2}$ are $\lambda_i^{1/2}$, $i = 1,2,\ldots,m$.*

The proof of this theorem is direct when $m=2$. This is done by using the result $A^{1/2} = (\sqrt{\lambda_1} + \sqrt{\lambda_2})(A + \sqrt{\lambda_1 \lambda_2} I)$. The generalization for all $m \in N$ is immediate.

As a corollary of this theorem the eigenvalues of $A^{1/n}$ are $\lambda_i^{1/n}$, $i = 1,2,\ldots,m$.

We remark that, Theorem 4 holds for a matrix of operators $\hat{M}$ if the eigenvalues of its eigenoperators are all positive. By using Theorem 4 and the results of the previous section (3.6) it becomes

$$u_h = \sum_{k=0}^{n} \sum_{i=1}^{m} e^{b_k L_i^{1/2}} \hat{H}_i \psi_k, \tag{3.7}$$

where $\hat{H}_i$ is defined by (2.9) and b_k satisfies $b_k^2 = 1$.

Thus, for $n \geqslant 2$ we have the following theorem:

Theorem 6. *The solution of the Eq.* $(\partial_t^n - \hat{\boldsymbol{M}})\boldsymbol{u} = 0$ *for initial functions which are in* $C^\infty(\Re^3) \cap L_1(\Re^3)$ *is*

$$\boldsymbol{u} = \sum_{k=0}^{n} \sum_{i=1}^{m} \int_D e^{b_k t L_i^{1/n}} \hat{\boldsymbol{H}}_i \tilde{\psi}_k e^{ip.x} \frac{d\boldsymbol{p}}{(2\pi)^3}, \quad b_k^n = 1, \tag{3.8}$$

where D is determined such that the eigenvalues L_i, s are all positive. The proof of this theorem follows from above and the mathematical induction.

We notice that the arbitrary functions ψ_k are determined in terms of the initial functions. So that they are also in $C^\infty(R^3) \cap L_1(R^3)$ and then (3.8) holds in view of the definition of fractional operator and the work in [1, 2].

We mention that the case $\boldsymbol{S} \neq \boldsymbol{0}$ can be treated as in Theorem 2 of [2]

(b) When the eigenoperators of $\hat{\boldsymbol{M}}$ are not distinct, we show that Theorem 4 also holds in this case. The continuation in the derivation is done as in (a).

In the next section, we give an application to the approach developed here. We study the problem of evaluating the electric field in the conductive half-space as a model in geophysical prospecting.

4. Applications

A formulation of Maxwell's equations in a conductive medium is used in mining and petroleum prospecting [5, 6, 8]. The aim is to identify quantities specifying the electromagnetic properties of the substratum from the measuring electric field at the surface. The technique consists of applying an electric current flowing between two electrodes located at the surface and of measuring the induced field in the whole space, which is composed of the substratum and the air above.

The electric field in a conductive medium is governed by the equation

$$\left[\frac{1}{c^2}\partial_t^2 + \mu\sigma^*\partial_t + \text{curl}(\text{curl})\right]\boldsymbol{E} + \mu\partial_t \boldsymbol{J}_s^* = \boldsymbol{0} \quad \text{in } \Re^3 \times (0,\infty), \tag{4.1}$$

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0^*(\boldsymbol{x}). \tag{4.2}$$

Eq. (4.1) arises from coupling Maxwell's equations and Ohm's law. In (4.1), $\boldsymbol{E}(\boldsymbol{x},t)$ is the electric field, $\boldsymbol{J}_s^*(\boldsymbol{x},t)$ is the source current and $\sigma^*(\boldsymbol{x})$ is the electric conductivity. We notice that

$$\sigma^*(\boldsymbol{x}) = \begin{cases} 0, & z > 0, \\ \sigma_0(\boldsymbol{x}), & z < 0 \end{cases} \tag{4.3}$$

and μ is the magnetic permeability. In (4.1), one finds that the first term in bracket describes a propagative effect with time scale as $\tau_0 \sim L/c$, where L is a characteristic length for the electric field to vary significantly. That is $\boldsymbol{E}$ vanishes merely for $|\boldsymbol{x}|>L$. The second term describes a diffusive effect. The corresponding time scale is $\tau_1 \sim L\mu\sigma$. Thus $\tau_0 \ll \tau_1$ and consequently diffusive effect persists for long time. Accordingly, we can neglect the first term in (4.1). By writing

$$(\boldsymbol{E}, \boldsymbol{E}_0^*, \boldsymbol{J}_s^*) = \begin{cases} (\boldsymbol{E}^+, 0, 0), & z>0, \\ (\boldsymbol{E}^-, \boldsymbol{E}_0, \boldsymbol{J}), & z<0. \end{cases} \tag{4.4}$$

It has been shown that problem (4.1,2) is equivalent to the problem (see [5] Eq. (3.9)])

$$\operatorname{Curl}\operatorname{Curl}\boldsymbol{E}^+ = 0, \tag{4.5}$$

$$(\mu\sigma_0\partial_t + \operatorname{curl}(\operatorname{curl}))\boldsymbol{E}^- = -\mu\frac{\partial \boldsymbol{J}}{\partial t}, \tag{4.6}$$

$$\boldsymbol{E}^-(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}), \tag{4.7}$$

$$\boldsymbol{E}^+(z=0) = \boldsymbol{E}^-(z=0), \tag{4.8}$$

$$\frac{\partial \boldsymbol{E}^+}{\partial z}(z=0) = \frac{\partial \boldsymbol{E}^-}{\partial z}(z=0). \tag{4.9}$$

We notice that the last two equations in (3.9) in [6] are replaced here by (4.8,9). The solution of Eq. (4.6) can be written in the form [4]

$$\boldsymbol{E}^- = \boldsymbol{E}_{\mathrm{IV}}^- + \boldsymbol{E}_{\mathrm{BV}}^- + \tfrac{1}{2}(\boldsymbol{E}_s^{*-} + \boldsymbol{E}_s^{**-}), \tag{4.10}$$

where $\boldsymbol{E}_{\mathrm{IV}}^-$ ($\boldsymbol{E}_{\mathrm{BV}}^-$) is the part of the solution which corresponds to the initial (boundary) value problem. The third term in (4.10) corresponds to the presence of the source term in (4.6). It is decomposed into two parts $\boldsymbol{E}_s^{*-}$ and $\boldsymbol{E}_s^{**-}$. They may be interpreted as due to mixing of the presence of a source and the initial (or boundary) conditions. To go further in the calculations, first, we assume that σ_0 is const and $\mu\sigma_0 = \sigma$ and set $\boldsymbol{E}^- \to \boldsymbol{E}^- - \boldsymbol{E}_0(\boldsymbol{x})$, then (4.6) becomes

$$(\mu\sigma_0\partial_t + \operatorname{curl}(\operatorname{curl}))\boldsymbol{E}^- = -\mu\frac{\partial \boldsymbol{J}^*}{\partial t}, \tag{4.11}$$

where

$$\boldsymbol{J}^*(\boldsymbol{x}, t) = \boldsymbol{J}(x, t) - \frac{t}{\mu}\hat{M}\boldsymbol{E}_0(\boldsymbol{x}), \quad \hat{M} = (\hat{M}_{ij}) = \operatorname{curl}(\operatorname{curl}). \tag{4.12}$$

where $\hat{M}_{11} = \hat{M}_{xx} = -(\partial_y^2 + \partial_z^2)$ and $\hat{M}_{xy} = \partial_x\partial_y, \ldots$, etc. The initial condition (4.7) is then the zero condition. Thus, we have

$$\boldsymbol{E}_{\mathrm{IV}} = 0$$

and $\boldsymbol{E}_s^{*-}$ is given by

$$\boldsymbol{E}_s^{*-} = -\frac{\mu}{\sigma}\int_0^t \mathrm{e}^{(t-t_1)\hat{M}}\partial_{t_1}\boldsymbol{J}^*(\boldsymbol{x}, t_1)\,\mathrm{d}t_1, \tag{4.13}$$

In order to use the approach developed here, we determine the eigenoperators of the matrix of operators $\hat{M}$ which are $\hat{\mathrm{O}}, -\nabla^2, -\nabla^2$ and ∇^2 is the Laplacian in $\Re^3$. Thus, the eigenoperators are degenerate and we make use of (4.12–16) to obtain

$$\mathrm{e}^{t\hat{M}} = \left\{ (\nabla^2)^{-2}(\hat{\boldsymbol{M}} + \nabla^2 I)^2 + \mathrm{e}^{t\nabla^2/\sigma} \left\{ \frac{t}{\sigma}(\hat{\boldsymbol{M}} + \nabla^2 I) \right.\right.$$
$$\left.\left. -(\nabla^2)^{-1}(\hat{\boldsymbol{M}} + \nabla^2 I)^2 \frac{t}{\sigma} + \hat{I} - (-\nabla^2)^{-2}(\hat{\boldsymbol{M}} + \nabla^2 I)^2 \right\}\right\}. \tag{4.14}$$

By a direct calculation, one finds that

$$(\hat{\boldsymbol{M}} + \nabla^2 I)^2 = \nabla^2(\hat{\boldsymbol{M}} + \nabla^2 I). \tag{4.15}$$

When using (4.15) into (4.14) and after a set of manipulations, (4.14) simplifies to

$$\mathrm{e}^{t\hat{\boldsymbol{M}}} = [\hat{I} + \hat{\boldsymbol{M}}(\nabla^2)^{-1}(\hat{I} - \mathrm{e}^{t\nabla^2/\sigma})]. \tag{4.16}$$

Finally, we have

$$\boldsymbol{E}_s^{*-} = -\frac{\mu}{\sigma}\int_0^t [\hat{I} + \hat{\boldsymbol{M}}(\nabla^2)^{-1}(\hat{I} - \mathrm{e}^{(t-t_1)\nabla^2/\sigma})]\partial_{t_1}\boldsymbol{J}^*(\boldsymbol{x}, t_1)\,\mathrm{d}t_1. \tag{4.17}$$

To find $\boldsymbol{E}_{\mathrm{BV}}^-$, we notice that the boundary conditions (4.8,9) are given at $z=0$. It is convenient to rewrite (4.6) as

$$[A\partial_z^2 - \hat{\boldsymbol{B}}\partial_z - \hat{C}] = \boldsymbol{E}_{\mathrm{BV}}^- = 0, \tag{4.18}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\boldsymbol{B}} = \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 1 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix},$$
$$\hat{\boldsymbol{C}} = \begin{pmatrix} \sigma\partial_t - \partial_y^2 & \partial_x\partial_y & 0 \\ \partial_y\partial_x & \sigma\partial_t - \partial_x^2 & 0 \\ 0 & 0 & \sigma\partial_t - \nabla_z^2 \end{pmatrix}, \tag{4.19}$$

where $\nabla_z^2 = \nabla^2 - \partial_z^2$.

We remark that the matrix A is singular and not semisimple as its eigenvalues are 1,1 and 0. In this case, it is necessary to restrict the vector $\boldsymbol{E}$ to a subspace of $\Re^3$. Indeed, we solve (4.18) for E_1 and E_2. The component E_3 is given accordingly by

$$E_3 = -\hat{C}_z^{-1}\,\partial z(\partial x\, E_1 + \partial y\, E_2), \tag{4.20}$$

where $\hat{C}_z = \sigma\partial_t - \nabla_z^2$ and $\hat{C}_z^{-1}$ is defined as in (2.5) when $\beta = 1$. The components E_1 and E_2 are governed by the following equation:

$$\begin{pmatrix} 1 + \hat{C}_z^{-1}\partial^2 x & \hat{C}_z^{-1}\partial_{xy}^2 \\ \hat{C}_z^{-1}\partial_{yx}^2 & 1 + \hat{C}_z^{-1}\partial_y^2 \end{pmatrix} \partial^2 z\, \boldsymbol{E}_{\mathrm{BV}}^{*-} = \begin{pmatrix} \sigma\partial_t - \partial^2 y & \partial_{xy}^2 \\ \partial_{yx}^2 & \sigma\partial_t - \partial_x^2 \end{pmatrix} \boldsymbol{E}_{\mathrm{BV}}^{*-}, \tag{4.21}$$

where $\boldsymbol{E}^*_{\mathrm{BV}} = (E_1, E_2)$. We operate by the inverse of the matrix of operators in the left-hand side of (4.21) to obtain

$$\partial_z^2 \boldsymbol{E}^{*-}_{\mathrm{BV}} = \hat{C}_z \boldsymbol{E}^{*-}_{\mathrm{BV}}. \tag{4.22}$$

We notice that the inverse of a matrix of operators can be defined also as in (2.5) when $\beta = 1$.

Eq. (4.22) solves to

$$\boldsymbol{E}^{*-}_{\mathrm{BV}} = \mathrm{e}^{z\hat{C}_z^{1/2}} \boldsymbol{\phi}_0^*(x,y,t) + \mathrm{e}^{-z\hat{C}_z^{1/2}} \boldsymbol{\phi}_1^*(x,y,t). \tag{4.23}$$

For the reason of finiteness, we set $\boldsymbol{\phi}_1^* = 0$. Finally, we have

$$\boldsymbol{E}^-_{\mathrm{BV}} = \boldsymbol{K}_0(t) - \boldsymbol{K}_0(0), \tag{4.24}$$

$$\boldsymbol{K}_0(t) = \mathrm{e}^{z\hat{C}_z^{1/2}} \boldsymbol{\phi}_0(x,y,t), \tag{4.25}$$

where $\boldsymbol{\phi}_0 = (\phi_{01}, \phi_{02}, \phi_{03})$, $\boldsymbol{E}^-_{\mathrm{BV}} = (E_1, E_2, E_3)$ and by using (4.20), we find

$$\phi_{03} = -\hat{C}_z^{-1} \partial_z (\partial_x \phi_{01} + \partial_y \phi_{02}). \tag{4.26}$$

We remark that Eq. (4.24) satisfies the zero initial condition.

In a similar way, we have

$$\boldsymbol{E}^{**-}_s = \boldsymbol{K}_1(t) - \boldsymbol{K}_1(0), \tag{4.27}$$

$$\boldsymbol{K}_1(t) = -\mu \int_0^z \left(\int_0^{z_1} \mathrm{e}^{(z-2z_1+z_2)\hat{C}_z^{1/2}} \partial_t \boldsymbol{J}^*(x,y,z_2,t)\, \mathrm{d}z_1 \right) \mathrm{d}z_2. \tag{4.28}$$

We remark, also, that Eq. (4.27) satisfies the zero initial condition.

Thus, the solution of (4.6) is given by (4.10), (4.18), (4.24)–(4.28). The solution of (4.5) can be obtained as by (4.25) but $\sigma = 0$, namely,

$$\boldsymbol{E}^+ = \mathrm{e}^{z(\nabla_z^2)^{1/2}} \boldsymbol{\phi}(x_1, x_2). \tag{4.29}$$

To continue with the calculations in view of (4.18), we assume that $\boldsymbol{J}$ and its partial derivatives up to second order are in $L_1(\Re^3)$. Also, we assume that $\boldsymbol{E}_0 \in L_1(\Re^3)$. The Fourier transforms of $\boldsymbol{E}_0, \boldsymbol{J}$ and $\boldsymbol{E}$ are introduced in view of (4.4). Our aim now is to find the arbitrary vector functions $\boldsymbol{\phi}_0$ and $\boldsymbol{\phi}$ by using the conditions (4.7–9). The use of these conditions gives rise to

$$s^{-1}\tilde{\boldsymbol{\phi}}(p_1,p_2) = \tilde{\boldsymbol{\phi}}_0(p_1,p_2,s) - \frac{\mu s}{2\sigma} \int_{-\infty}^{\infty} \tilde{H}(p_1,p_2,p_3,s) \tilde{\boldsymbol{J}}^*(\boldsymbol{p},s)\, \mathrm{d}p_3, \tag{4.30}$$

$$p_\perp s^{-1} \tilde{\boldsymbol{\phi}} = C_3^{1/2} \tilde{\boldsymbol{\phi}} - \mathrm{i}\frac{\mu s}{2\sigma} \int_{-\infty}^{\infty} \tilde{H} \tilde{\boldsymbol{J}}^* p_3\, \mathrm{d}\tilde{p}_3, \tag{4.31}$$

and $\tilde{H}$ is given by

$$\tilde{H} = [s(\sigma s + p^2)]^{-1} \begin{pmatrix} p_1^2 & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & p_2^2 & p_2 p_3 \\ p_3 p_1 & p_3 p_2 & p_3^2 \end{pmatrix}, \tag{4.32}$$

where $p = |\boldsymbol{p}|$, $\boldsymbol{p} = (p_1, p_2, p_3)$, $p_\perp^2 = p_1^2 + p_2^2$, $\mathrm{d}\tilde{p}_3 = \mathrm{d}p_3/2\pi$.

Equations (4.29–32) determine the functions $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_0$ in terms of $\boldsymbol{E}_0$ and $\boldsymbol{J}^*$.

By solving Eqs. (4.30) and (4.31) for $\tilde{\boldsymbol{\phi}}_0$ we find

$$\tilde{\boldsymbol{\phi}} = \frac{-\mu s}{2\sigma(C_3^{1/2} - p_\perp)} \int_{-\infty}^{\infty} (1 - \mathrm{i}p_3)\tilde{H}\tilde{\boldsymbol{J}}^* \, \mathrm{d}p_3. \tag{4.33}$$

By substituting (4.33) into (4.25) and (4.24)–(4.28) into (4.10) and maintaining only the dominant contribution, we have

$$\boldsymbol{E}^- \cong \frac{1}{2}(\boldsymbol{E}_s^{*-} + \boldsymbol{E}_s^{**-}) = \frac{\mu}{2\sigma}\int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} s \int \left[\frac{\sigma \tilde{I}}{p_3^2 + C_3} - \tilde{H}\right] \tilde{\boldsymbol{J}}(\boldsymbol{p}, s) \times \mathrm{e}^{\mathrm{i}\boldsymbol{p}.\boldsymbol{x}+st} \, \mathrm{d}\tilde{\boldsymbol{p}} \, \mathrm{d}\tilde{s}, \tag{4.34}$$

where $\tilde{H}(p,s)$ is given by (4.32), $\mathrm{d}\tilde{\boldsymbol{p}} = \mathrm{d}\boldsymbol{p}/(2\pi)^3$ and $\mathrm{d}\tilde{s} = \mathrm{d}s/2\pi\mathrm{i}$.

By substituting for $\tilde{\boldsymbol{J}}(\boldsymbol{p}, s)$ and carrying out the integrals in the $\boldsymbol{p}$ space and the s-plane, we obtain

$$\begin{aligned} E_1 &= \int_0^t \int^* G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) J_1^*(\boldsymbol{x}_0, t_0) \, \mathrm{d}\boldsymbol{x}_0 \, \mathrm{d}t_0, \\ E_2 &= \int_0^t \int^* G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) J_2^*(\boldsymbol{x}_0, t_0) \, \mathrm{d}\boldsymbol{x}_0 \, \mathrm{d}t_0, \\ E_3 &= \int_0^t \int^* G^*(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) J_3^*(\boldsymbol{x}_0, t_0) \, \mathrm{d}\boldsymbol{x}_0 \, \mathrm{d}t_0, \end{aligned} \tag{4.35}$$

where $\int^* \mathrm{d}\boldsymbol{x}_0 = \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_{-\infty}^{\infty} \mathrm{d}y_0 \int_{-\infty}^{0} \mathrm{d}z_0$ and

$$G = \frac{3\mu\sigma^{3/4}\,\mathrm{e}^{-Q}}{16\pi^{3/2}R^{3/2}(t - t_0)^{7/4}} M_{7/4,1/4}(2Q) - G^*, \tag{4.36}$$

$$G^* = \frac{3\mu\sigma^{3/2}\,\mathrm{e}^{-2Q}}{16\pi^{3/2}(t - t_0)^{5/2}},$$

$$Q = (R^2\sigma/8(t - t_0)), \quad R = |\boldsymbol{x} - \boldsymbol{x}_0| \tag{4.37}$$

and $M_{r,s}(z)$ is the Whittaker function.

In (4.35), the function G is the Green function which corresponds to the solution of the problem in the presence of a current source located at the point (x_0, y_0, z_0), and at time t_0. If $\boldsymbol{x}_0 = \boldsymbol{0}$, we find that the solution (4.35–37) is quasi-Gaussian and symmetric in x and y. The results (4.35–37) are displayed in Figs. 1 and 2. Calculations have been carried out for $\mu = 1$, $\sigma = 0.1$, $\boldsymbol{E}_0 = 0$ and $\boldsymbol{J}(\boldsymbol{x}, t) = \boldsymbol{J}_0\delta(x)\delta(y)\delta(z)$. In Fig. 1, the values of $E_2^* = 32\pi^{3/2}\sigma E_2/3J_0$ are displayed versus y, $z = 0$ and for $t = 1, 3$. It is assumed that $\boldsymbol{J}_0 = (0, J_0, 0)$ and all variables are dimensionless. The results of this figure verify the condition (4.8). As the electric field depends on the variable σ/t and from the results of Fig. 1, we may conclude that it depends weakly on the time and the conductivity. But, it depends significantly on σ/t. In Fig. 2, a 3D plot with contour lines is done for E_2^*. The results of Figs. 1 and 2 agree qualitatively with the numerical results found in [6] (cf. Figs. 5 and 6).

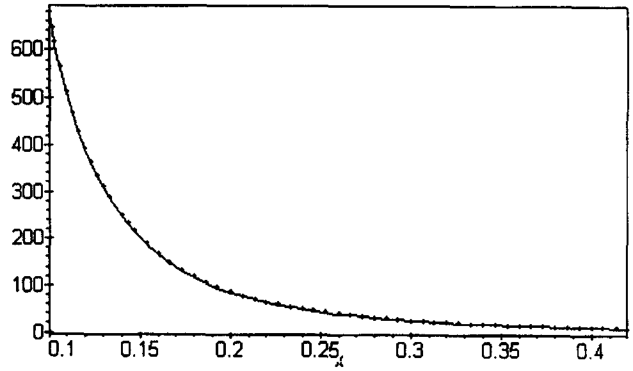

Fig. 1. The electric field (normalized) E_2^* is displayed against y when $x=z=0$ and for $\sigma=0.1$. We use the symmetry of the results for the electric field in x and y and the parity in y. The solid curve corresponds to the value $t=1$ and the doted one corresponds to the value $t=3$.

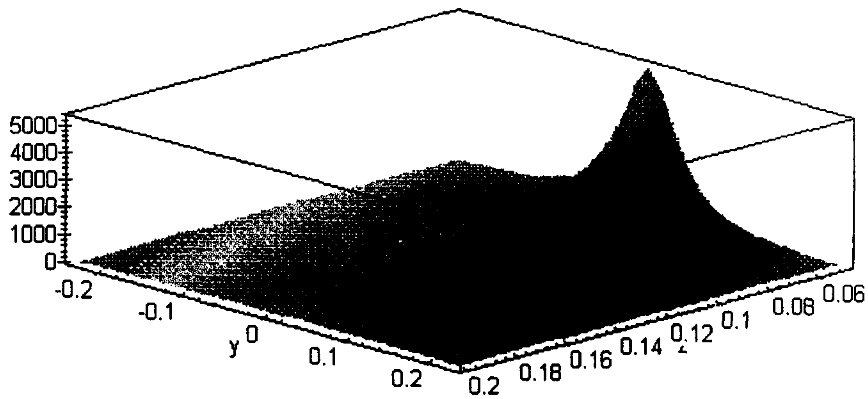

Fig. 2. The electric field E_2^* is displayed against y and $-z$ for $\sigma=0.1$ and $t=1$.

We assume in what follows that the electric conductivity σ is space dependent and it depends only weakly on the depth z. So that, we can consider $(\sigma^{(n)}/6)\ll 1$; $n\geqslant 3$, and $(\sigma'/\sigma)^2\ll 1$, where $\sigma'=\mathrm{d}\sigma/\mathrm{d}z$. The electric field in the substratum is given by

$$\boldsymbol{E}=\boldsymbol{E}_{\mathrm{IV}}^{-}+\boldsymbol{E}_{\mathrm{BV}}^{-}+\tfrac{1}{2}(\boldsymbol{E}_s^{*}+\boldsymbol{E}_s^{**-}). \tag{4.38}$$

In (4.38), we have used the same notations as in case (a). Now, we have the same formal results for $\boldsymbol{E}_{\mathrm{IV}}^{*}$ and $\boldsymbol{E}_s^{*-}$ as given by (4.13) and (4.17), respectively, but $\sigma=\sigma(z)$. To find $\boldsymbol{E}_{\mathrm{BV}}^{-}$, we combined (4.20) and (4.21) bearing in mind that σ depends on z, we have

$$(\hat{\boldsymbol{A}}\partial_z^2+\hat{\boldsymbol{B}}\partial_z+\hat{\boldsymbol{C}})\boldsymbol{E}_{\mathrm{BV}}^{*-}=\hat{0} \tag{4.39}$$

$$\hat{\boldsymbol{A}}=\begin{pmatrix} -1-\partial_x^2 & -\partial_{xy}^2+\hat{C}_z^{-1} \\ -\partial_{yx}^2\hat{C}_z^1 & -1-\partial_y^2\hat{C}_z^{-1} \end{pmatrix}, \tag{4.40}$$

$$\hat{\boldsymbol{B}} = \hat{C}_z^{-1}\sigma^1\partial_t\begin{pmatrix} \partial_x^2 & \partial_{xy}^2 \\ \partial_{yx}^2 & \partial_y^2 \end{pmatrix}, \tag{4.41}$$

$$\hat{\boldsymbol{C}} = \begin{pmatrix} \sigma\partial_t - \partial_y^2 & \partial_{xy}^2 \\ \partial_{yx}^2 & \sigma\partial_t - \partial_x^2 \end{pmatrix}. \tag{4.42}$$

where $\boldsymbol{E}_{\rm BV}^{*-} = (E_1, E_2)$. If $\sigma' = 0$, then (4.39) reduces to (4.22). We multiply Eq. (4.39) from the left by $\hat{A}^{-1}$ to obtain

$$(\partial_z^2 + \hat{\boldsymbol{A}}^{-1}\hat{\boldsymbol{B}}\partial_z + \hat{\boldsymbol{A}}^{-1}\hat{\boldsymbol{C}})\boldsymbol{E}_{\rm BV}^{*-} = \boldsymbol{0}, \tag{4.43}$$

where $\hat{A}^{-1}$ is evaluated according to the definition (2.5) and is given by

$$\hat{\boldsymbol{A}}^{-1} = (\sigma\partial_t)^{-1}\begin{pmatrix} \hat{C}_z + \partial_y^2 & -\partial_{xy}^2 \\ -\partial_{yx}^2 & \hat{C}_z + \partial_x^2 \end{pmatrix}. \tag{4.44}$$

After a set of manipulation, we find that

$$\hat{\boldsymbol{A}}^{-1}\hat{\boldsymbol{C}} = {}^-\hat{C}_z I, \qquad \hat{\boldsymbol{A}}^{-1}\hat{\boldsymbol{B}} = {}^-\hat{C}_z^{-1}\frac{\sigma^{-1}}{\sigma}\hat{C}_z\hat{\boldsymbol{B}}. \tag{4.45}$$

For obtaining the above equations, we have used that $\partial_x\hat{C}_z^{-1} = \hat{C}_z^{-1}\partial_x$ and

$$\partial_z\hat{C}_z^{-1} = -\hat{C}_z^{-2}\sigma^1\partial_t + \hat{C}_z^{-1}\partial_z. \tag{4.46}$$

We claim that the solution of (4.43) can be written in the form

$$\boldsymbol{E}_{\rm BV}^{*-} = \exp\left(\int_0^z \hat{y}(z_1, \partial_t, \partial_x, \partial_y)\,{\rm d}z_1\right)\psi(x, y, t). \tag{4.47}$$

It is worth noticing that $\hat{y}$ depends on z as a parameter through the electric conductivity σ which does not depend either on x or on y. Consequently, the commutator $[\hat{y}(z_1), \hat{y}(z_2)]$ vanishes. So that, we have

$$\partial_z \exp\left(\int_0^z \hat{y}\,{\rm d}z_1\right) = \hat{y}\exp\left(\int_0^z \hat{y}\,{\rm d}z_1\right). \tag{4.48}$$

By substituting (4.47) into (4.43), we find that the operator $\hat{y}$ satisfies the equation

$$\partial_z\hat{y}I + \hat{y}^2 I - \hat{C}_z^{-1}\frac{\sigma'}{\sigma}\hat{B}\hat{C}_z\hat{y} = \hat{C}_z I. \tag{4.49}$$

Since the dependence of σ on z is not explicitly known, we solve (4.49) approximately in terms of the small parameter (σ'/σ).

To this end, we make use of the expansion

$$\hat{y} = \hat{y}_0 + \frac{\sigma'}{\sigma}\hat{y}_1 + \left(\frac{\sigma'}{\sigma}\right)^2\hat{y}_2 + \cdots, \tag{4.50}$$

where we look for $\hat{y}_0$ as the solution of (4.49) as if σ is independent of z. Thus, we have $\hat{y}_0^2 = \hat{C}_z$, or $\hat{y}_0 = \hat{C}_z^{1/2}$ where the minus sign is discarded for the reason of finiteness. By substituting (4.50) into (4.49) and maintaining only terms up to first order in (σ'/σ), we find

$$\partial_z \hat{y}_1 + \frac{\sigma''}{\sigma'} \hat{y}_1 + 2\hat{y}_1 \hat{y}_0 = \hat{0}. \tag{4.51}$$

It is worth noticing that the third term in (4.47) is of order $(\sigma'/\sigma)^2$. Now, equation (4.51) solves to

$$\hat{y}_1 = \sigma' \exp\left(2 \int_0^z \hat{y}_0 \, dz_1\right). \tag{4.52}$$

Finally, the solution of (4.43) is given by

$$\boldsymbol{E}_{\mathrm{BV}}^{*-} \simeq \exp\left[\int_0^z \left(\hat{C}_z^{1/2} + \frac{\sigma'^2}{\sigma} e^2 \int_0^{z_1} \hat{y}_0 \, dz_2\right) dz_1\right] \boldsymbol{\psi}^*. \tag{4.53}$$

When bearing in mind Eq. (4.20), the solution $\boldsymbol{E}_{\mathrm{BV}}^- = (E_1, E_2, E_3)$ is determined by

$$\boldsymbol{E}_{\mathrm{BV}}^- \simeq \exp\left(\int_0^z \hat{C}_z^{1/2} \, dz_1\right) \boldsymbol{\psi} \left[1 + 0\left(\frac{\sigma'}{\sigma}\right)^2\right]. \tag{4.54}$$

In the same way, we have

$$\boldsymbol{E}_s^{**-} \simeq -\int_0^z \int_0^{z_1} \exp\left[\int_{z_1}^z \hat{C}_z^{1/2} \, ds - \int_{z_2}^{z_1} \hat{C}_z^{1/2} \, ds\right] \times \sigma(z_2) \partial_t \boldsymbol{J}^*(x, y, z_2, t) \, dz_2 \, dz_1 \left[1 + 0\left(\frac{\sigma'}{\sigma}\right)^2\right], \tag{4.55}$$

where $\hat{C}_z = \sigma(z)\partial_t - \partial_x^2 - \partial_y^2$.

To continue with (4.55) and (4.17) bearing in mind that σ depends on z, we use the Fourier Laplace transform of the vector function $\boldsymbol{J}^*(\boldsymbol{x}, t)$. Then, we have to evaluate the following $\exp \hat{H} e^{i\boldsymbol{p}.\boldsymbol{x}}$, where the operator $\hat{H} = \hat{H}(\sigma(z), \partial_x, \partial_y)$ is equal to either $(t/\sigma(z))\nabla^2$ or to that in the exponential in (4.55). By using the assumptions on $\sigma(z)$ and the results of Ref. [1], we obtain

$$\exp\left(\frac{t}{\sigma(z)} \nabla^2\right) e^{i\boldsymbol{p}.\boldsymbol{x}} \simeq e^{i\boldsymbol{p}.\boldsymbol{x} - \frac{tp^2}{\sigma(z)} + 0(\frac{\sigma'}{\sigma})^2}. \tag{4.56}$$

We remark that, when substituting (4.56) into (4.17), we obtain mainly the same result for $\boldsymbol{E}_s^{*-}$ as before.

Returning to (4.55), as the operator in the exponential is free of ∂_z, we have directly the result of

$$\exp\left(\int_0^z \hat{C}_{z_1}^{1/2} \, dz_1\right) e^{i(p_1 x + p_2 y) + st} = \exp\left(\int_0^z C_{z_1}^{1/2} \, dz_1\right). \; e^{i(p_1 x + p_2 y) + st}, \tag{4.57}$$

where $C_z = \sigma(z)s + p_1^2 + p_2^2$.

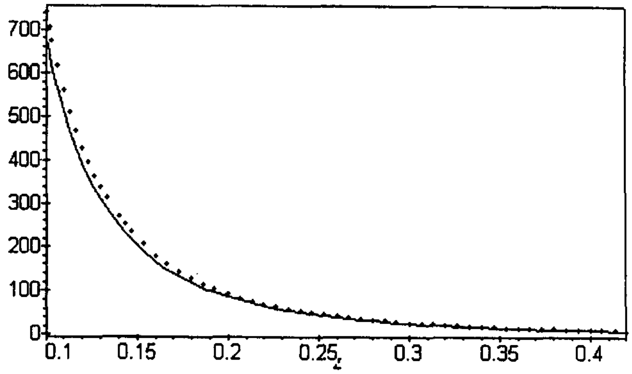

Fig. 3. The electric field E_2^* is displayed against $-z$ for $x = y = 0$ and $t = 1$. The solid curve corresponds to $\sigma = 0.1$ and the dotted one corresponds to $\sigma = 0.1\ (1 + 0.1 \exp z)$.

In fact as the dependence of σ on z is implicit, we cannot progress in the calculation. However, the results (4.55–57) suggest that we may adopt the formal results (4.35–37) for the electric field to hold also when σ depends on z. Estimated corrections to these results are of the order $(\sigma'/\sigma)^2$. Accordingly, we have evaluated the electric field under the same conditions as in Figs. 1 and 2 by taking $\sigma = \text{const.} = 0.1$ and σ varies with z as $0.1(1 + 0.1 \exp z)$, $z<0$. The results are shown in Fig. 3. They confirm our theoretical predictions that weak dependence on the depth leads to a small correction.

5. Conclusions

We have developed an approach for solving systems of linear PDE. This approach is more adequate for finding the (anharmonic) solution of a given system of PDE. This situation is produced in problems with anharmonic source term. It occurs also in problems where the source term is harmonic but is set at a specific value of time. The present approach can be applied as well for solving the problems with complicated initial and boundary conditions. In what concerns the problem of evaluating the electric field in the air and substratum as a model in geophysical prospecting, our results are in a qualitative agreement with the numerical ones. An important result is that if the conductivity depends weakly on the depth then the values of the electric field deviate from these found when $\sigma = \text{const}$ by a small correction. This can be shown after Fig. 3.

References

[1] H.I. Abdel-Gawad, Solution of a class of partial differential equations: Some diffusion equations, Math. Meth. Appl. Sci. 12 (1990) 221–228.

[2] H.I. Abdel-Gawad, Solution of the equation $\partial_t^n f(x,t) = \hat{L}(x,t) f(x,t) + S(x,t)$. Exact solution of some partial differential equations in mechanics and physics, Math. Meth. Appl. Sci. 14 (1991) 615–633.

[3] H.I. Abdel-Gawad, Anharmonic solution of the problem of diffraction of acoustic waves, Arab Gulf J. Sci. Res. 13(1) (1995) 75–91.

[4] H.I. Abdel-Gawad, On the resonance of acoustic waves in slightly deformed and slowly moving slabs, J. Eng. Math. 29 (1995) 241–253.

[5] Y. Goldman, C. Hubans, Nicolctis, S. Spitz, A finite element solution for the transient electromagnetic response of an arbitrary two dimensional resistivity distribution, Geophysics 51(7) (1986) 1450–1461.

[6] Y. Goldman, P. Joly, M. Kern, The electric field in the conductive half-space as a model in mining and petroleum prospecting, Math. Meth. Appl. Sci 11 (1989) 373–401.

[7] K.S. Miller, B. Ross, An introduction to fractional calculus and fractional differential equations, Wiley, New York, 1993, p. 463.

[8] M. Oristaglio, G. Hohmann, Diffusion of an electromagnetic field into a two dimensional earth: A finite element Approach, Geophysics 39 (1984) 870–894.

[9] M.C. Pease, Methods of matrix Algebra, Academic Press, New York, 1965, p. 406.