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On the Newton–Kantorovich hypothesis for solving equations

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Abstract

The famous Newton–Kantorovich hypothesis has been used for a long time as a sufficient condition for the convergence of Newton's method to a solution of an equation in connection with the Lipschitz continuity of the Fréchet-derivative of the operator involved. Here using Lipschitz and center-Lipschitz conditions we show that the Newton–Kantorovich hypothesis can be weakened. The error bounds obtained under our semilocal convergence result are more precise than the corresponding ones given by the dominating Newton–Kantorovich theorem.

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1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of equation

$$F(x) = 0, \tag{1}$$

where, F is a Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ (for

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some suitable operator Q), where x is the state. Then the equilibrium states are determined by solving Eq. (1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

The famous Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geqslant 0)\ (x_0 \in D) \tag{2}$$

has long played a central role in approximating solutions x^* of non-linear equations and systems. Here $F'(x_n)$ denotes the Fréchet-derivative of operator F evaluated at $x = x_n (n \geqslant 0)$ [4,6,11]. The geometric interpretation of Newton's method is well known, if F is a real function. In such a case x_{n+1} is the point where the tangential line $y - F(x_n) = F'(x_n)(x - x_n)$ of function $F(x)$ at the point $(x_n, F(x_n))$ intersects the x-axis. The geometric interpretation of the complex Newton method ($F: \mathbf{C} \to \mathbf{C}$) is given in [18].

There is much literature concerning the convergence of Newton's method as well as error estimates. Among others, in the real case, Fourier studied the quadratic convergence of Newton's method in 1818, provided that a solution x^* of Eq. (1) exists [9]. In 1829, Cauchy first proved a semilocal convergence theorem which does not require any knowledge of existence of solution and asserted that the iterates (2) converge to a solution x^* if the initial guess x_0 satisfies certain conditions [7]. Ostrowski refined Fourier's and Cauchy's results for the case $X = \mathbf{R}$ or $X = \mathbf{C}$ [14].

For the general case when X, Y are Banach spaces, Kantorovich established a now famous and dominating semilocal convergence theorem for Newton's method which is called Kantorovich's or Newton–Kantorovich's theorem [13] (see Theorem 3 that follows) based on the famous Newton–Kantorovich hypothesis (see (37)). Three years later, he introduced the majorant principle to present a new proof [13]. His technique is so powerful that many authors have applied it to establish convergence theorems for variants of Newton's method, the so-called Newton-like methods [2–6,8,19].

Despite the fact that many decades have passed the Newton–Kantorovich hypothesis has not been challenged or improved. That is all results have been based or can be reduced to this hypothesis. Our new approach is to use center-Lipschitz (see (13)) instead of Lipschitz conditions (see (14)) for the bounds on $\|F'(x_n)^{-1}F'(x_0)\|$ (semilocal) or $\|F'(x_n)^{-1}F'(x^*)\|$ (local case) $(n \geqslant 0)$. This idea arises from the observation that under center-Lipschitz the bounds are more precise and cheaper to compute than in the case of Lipschitz conditions used so far. This idea can be extended for Newton-like methods, and in the case Lipschitz conditions are replaced by "small" functions (see [1,6]).

This way we show that under our weaker hypothesis (3) (for $\delta = 1$, say) ((3) implies (29) but not vice versa unless if $\ell = \ell_0$) Newton's method converges to a locally unique solution x^* of Eq. (1). Our estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are more precise than the ones obtained by the Newton–Kantorovich theorem. Moreover, our uniqueness ball is at least as small as the one given by the Newton–Kantorovich theorem; hence providing better information on the location of the solution.

Finally we provide numerical examples to show our results:

(1) apply to solve nonlinear equations where others fail;
(2) provide more precise error estimates on the distances involved;
(3) provide a better information on the location of the solution.

2. Semilocal analysis of Newton's method

We provide the following result on majorizing sequences for Newton's method (2).

Lemma 1. *Assume there exist parameters* $\ell \geqslant 0$, $\ell_0 \geqslant 0$ *with* $\ell_0 \leqslant \ell$, $\eta \geqslant 0$, *and* $\delta \in [0,1]$ *such that*

$$h_\delta = (\delta\ell_0 + \ell)\eta \leqslant \delta. \tag{3}$$

Then, iteration $\{t_n\}$ $(n \geqslant 0)$ *given by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\ell(t_{n+1} - t_n)^2}{2(1 - \ell_0 t_{n+1})} \ (n \geqslant 0) \tag{4}$$

is non-decreasing, bounded above by $t^{**} = 2\eta/(2-\delta)$ *and converges to some* t^* *such that*

$$0 \leqslant t^* \leqslant t^{**}. \tag{5}$$

Moreover, the following error bounds hold for all $n \geqslant 0$

$$0 \leqslant t_{n+2} - t_{n+1} \leqslant \frac{\delta}{2}(t_{n+1} - t_n) \leqslant \left(\frac{\delta}{2}\right)^{n+1}\eta. \tag{6}$$

Proof. The result clearly holds if $\delta = 0$, or $\ell = 0$ or $\eta = 0$. Let us assume $\delta \neq 0$, $\ell \neq 0$ and $\eta \neq 0$. We must show for all $k \geqslant 0$

$$\ell(t_{k+1} - t_k) + \delta\ell_0 t_{k+1} \leqslant \delta, \quad t_{k+1} - t_k \geqslant 0 \text{ and } 1 - \ell_0 t_{k+1} > 0. \tag{7}$$

Estimate (6) can then follow immediately from (4) and (7). Using induction on the integer k we have for $k = 0$

$$\ell(t_1 - t_0) + \delta\ell_0 t_1 = \ell\eta + \delta\ell_0\eta \leqslant \delta, \quad t_1 \geqslant t_0 \text{ and } 1 - \ell_0\eta > 0 \quad \text{(by (3))}.$$

But then (4) gives

$$0 \leqslant t_2 - t_1 \leqslant \frac{\delta}{2}(t_1 - t_0).$$

Let us assume (6) and (7) holds for all $k \leqslant n+1$.

We can have in turn

$$\begin{aligned}\ell(t_{k+2}-t_{k+1})+\delta\ell_0 t_{k+2} &\leqslant \ell\eta\left(\frac{\delta}{2}\right)^{k+1}+\delta\ell_0\left[t_1+\frac{\delta}{2}(t_1-t_0)+\left(\frac{\delta}{2}\right)^2(t_1-t_0)\right.\\ &\qquad\left.+\cdots+\left(\frac{\delta}{2}\right)^{k+1}(t_1-t_0)\right]\\ &\leqslant \ell_\eta\left(\frac{\delta}{2}\right)^{k+1}+\delta\ell_0\eta\frac{1-(\delta/2)^{k+2}}{1-\delta/2}\\ &= \ell_\eta\left(\frac{\delta}{2}\right)^{k+1}+\frac{2\delta\ell_0\eta}{2-\delta}\left[1-\left(\frac{\delta}{2}\right)^{k+2}\right]\\ &= \left\{\ell\left(\frac{\delta}{2}\right)^{k+1}+\frac{2\ell_0\delta}{2-\delta}\left[1-\left(\frac{\delta}{2}\right)^{k+2}\right]\right\}\eta. \end{aligned} \tag{8}$$

By (3) and (8) it suffices to show

$$\ell\left(\frac{\delta}{2}\right)^{k+1}+\frac{2\ell_0\delta}{2-\delta}\left[1-\left(\frac{\delta}{2}\right)^{k+2}\right]\leqslant \ell+\delta\ell_0$$

or

$$\delta\ell_0\left\{\frac{2}{2-\delta}\left(1-\left(\frac{\delta}{2}\right)^{k+2}\right)-1\right\}\leqslant \ell\left[1-\left(\frac{\delta}{2}\right)^{k+1}\right]$$

or

$$\left[\frac{\ell_0\delta^2}{2-\delta}-\ell\right]\left[1-\left(\frac{\delta}{2}\right)^{k+1}\right]\leqslant 0,$$

or

$$\frac{\ell_0\delta^2}{2-\delta}\leqslant \ell \tag{9}$$

which is true by the choice of δ. Hence, the first estimate in (7) holds for all $n \geqslant 0$. We must also show:

$$t_k \leqslant t^{**}.$$

For $k = 0, 1, 2$ we have

$$t_0 = 0 \leqslant t^{**}, \quad t_1 = \eta \leqslant t^{**} \quad \text{and} \quad t_2 \leqslant \eta + \frac{\delta}{2}\eta = \frac{2+\delta}{2}\eta \leqslant t^{**}.$$

It follows from (6) that for all $k \geqslant 0$,

$$\begin{aligned} t_{k+2} &\leqslant t_{k+1} + \frac{\delta}{2}(t_{k+1} - t_k) \leqslant t_k + \frac{\delta}{2}(t_k - t_{k-1}) + \frac{\delta}{2}(t_{k+1} - t_k) \\ &\leqslant \cdots \leqslant t_1 + \frac{\delta}{2}(t_1 - t_0) + \cdots + \left(\frac{\delta}{2}\right)(t_k - t_{k-1}) + \frac{\delta}{2}(t_{k+1} - t_k) \\ &\leqslant \eta + \frac{\delta}{2}\eta + \left(\frac{\delta}{2}\right)^2 \eta + \cdots + \left(\frac{\delta}{2}\right)^{k+1} \eta \\ &\leqslant \left[1 + \frac{\delta}{2} + \left(\frac{\delta}{2}\right)^2 + \cdots + \left(\frac{\delta}{2}\right)^{k+1}\right]\eta \\ &\leqslant \frac{1-(\delta/2)^{k+2}}{1-\delta/2}\eta < \frac{2}{2-\delta}\eta = t^{**}. \end{aligned}$$

Moreover, we have

$$\ell_0 t_{k+2} < \frac{2\ell_0\eta}{2-\delta} \leqslant 1 \quad \text{(by (3))}. \tag{10}$$

Hence, sequence $\{t_n\}$ $(n \geqslant 0)$ is bounded above by t^{**}. It also follows from (4) that $\{t_n\}$ $(n \geqslant 0)$ is non-decreasing and as such it converges to some t^* satisfying (5).

That completes the proof of Lemma 1. □

Remark 1. The conclusions of Theorem 1 hold in a more general setting as we can easily see if we just follow the above proof (see, e.g. (9) and (10)). Indeed with ℓ_0, ℓ, η as above and $\delta_0 \in [0,2)$ replace (3) by the more difficult to verify conditions:

$$h_{\delta_0} \leqslant \delta_0, \quad \frac{2\ell_0\eta}{2-\delta_0} \leqslant 1, \ \frac{\ell_0\delta_0^2}{2-\delta_0} \leqslant \ell. \tag{3$'$}$$

Note that if $\delta_0 \in [0,1]$ then $(3)'$ reduces to (3), as we can set $\delta_0 = \delta$. As an example, let $\ell = 2\ell_0$. The last condition in $(3)'$ holds if $\delta_0 \in [0, \sqrt{5}-1]$. Choose $\delta_0 = \sqrt{5}-1$. Then the first two conditions in $(3)'$ hold if

$$h = 2\ell\eta \leqslant \frac{4\delta_0}{2+\delta_0} = 1.527864045\ldots \ .$$

However, (3) for $\delta = 1$ is true if the stronger condition

$$h = 2\ell\eta \leqslant \tfrac{4}{3} = 1.\bar{3}$$

holds. Call the result using $(3)'$ instead of (3) Lemma $1'$.

Conditions $(3)'$ can be combined in one to look like Newton–Kantorovich-type hypotheses in some cases. Indeed, if $\delta = \ell_0 = 0$, then $(3)'$ hold if $\ell = 0$ or $\eta = 0$. Assume $\ell_0 \neq 0$, $\ell \neq 0$. Define

$$d = \frac{\ell}{\ell_0}.$$

Then

$$\delta_0 = \frac{-d + \sqrt{d^2 + 8d}}{2}$$

satisfies the last inequality in $(3)'$ as equality. Moreover, define

$$c = \frac{2\ell_0}{2 - \delta_0},$$
$$b = \ell_0 + \delta_0^{-1}\ell,$$
$$a = \max\{c, b\}$$

and

$$L = \begin{cases} \frac{1}{2}a, & \ell_0 \neq 0, \\ 0, & \ell_0 = \ell = 0 \text{ or } \ell_0 = \eta = 0, \\ \frac{1}{2}\frac{\ell}{\delta_0}, & \ell_0 = 0. \end{cases}$$

Then conditions $(3)'$ can be written as

$$h^* = 2L\eta \leqslant 1. \tag{3''}$$

Below is the main semilocal convergence theorem for Newton's method (2) using Lipschitz (14) and center-Lipschitz conditions (13).

Theorem 1. *Let $F: D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume hypotheses of Lemma 1 hold, and there exist a point $x_0 \in D$ and parameters $\eta \geqslant 0$, $\ell_0 \geqslant 0$, $\ell \geqslant 0$, such that*

$$F'(x_0)^{-1} \in L(Y, X), \tag{11}$$

$$\|F'(x_0)^{-1}F(x_0)\| \leqslant \eta, \tag{12}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leqslant \ell_0\|x - x_0\|, \tag{13}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leqslant \ell\|x - y\| \quad \text{for all } x, y \in D, \tag{14}$$

$$\bar{U}(x_0, t^{**}) = \{x \in X \mid \|x - x_0\| \leqslant t^{**}\} \subseteq D, \tag{15}$$

where, t^{**} is given in Theorem 1.

Then, sequence $\{x_n\}$ $(n \geqslant 0)$ generated by Newton's method (2) is well defined, remains in $\bar{U}(x_0, t^*)$ for all $n \geqslant 0$ and converges to a unique solution $x^* \in \bar{U}(x_0, t^*)$ of equation $F(x) = 0$.

Moreover, the following error bounds hold for all $n \geqslant 0$:

$$\|x_{n+2} - x_{n+1}\| \leqslant \frac{\ell\|x_{n+1} - x_n\|^2}{2[1 - \ell_0\|x_{n+1} - x_0\|]} \leqslant t_{n+2} - t_{n+1} \tag{16}$$

and

$$\|x_n - x^*\| \leqslant t^* - t_n, \tag{17}$$

where, iteration $\{t_n\}$ $(n \geqslant 0)$ is given by (4).

Furthermore, if there exists $R > t^*$ such that

$$U(x_0, R) \subseteq D \tag{18}$$

and

$$\ell_0(t^* + R) \leqslant 2, \tag{19}$$

the solution x^* is unique in $U(x_0, R)$.

Proof. Let us prove

$$\|x_{k+1} - x_k\| \leqslant t_{k+1} - t_k \tag{20}$$

and

$$\bar{U}(x_{k+1}, t^* - t_{k+1}) \subseteq \bar{U}(x_k, t^* - t_k) \tag{21}$$

hold for all $k \geqslant 0$.

For every $z \in \bar{U}(x_1, t^* - t_1)$,

$$\|z - x_0\| \leqslant \|z - x_1\| + \|x_1 - x_0\| \leqslant t^* - t_1 + t_1 = t^* - t_0, \tag{22}$$

implies $z \in \bar{U}(x_0, t^* - t_0)$. Since also

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leqslant \eta = t_1 - t_0,$$

(20) and (21) hold for $k = 0$. Given they hold for $n = 0, 1, \ldots, k$, then

$$\|x_{k+1} - x_0\| \leqslant \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leqslant \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \leqslant t_k + \theta(t_{k+1} - t_k) < t^* \quad \theta \in [0, 1].$$

Using (2) we obtain the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) \\ &= \int_0^1 [F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)](x_{k+1} - x_k)\,\mathrm{d}\theta \end{aligned} \tag{23}$$

and by (14)

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{k+1})\| &\leqslant \int_0^1 \|F'(x_0)^{-1}[F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)]\|\,\mathrm{d}\theta \|x_{k+1} - x_k\| \\ &\leqslant \frac{\ell}{2}\|x_{k+1} - x_k\|^2 \leqslant \frac{\ell}{2}(t_{k+1} - t_k)^2. \end{aligned} \tag{24}$$

It follows from (13) and (3)

$$\|F'(x_0)^{-1}[F'(x_{k+1}) - F'(x_0)]\| \leqslant \ell_0\|x_{k+1} - x_0\| \leqslant \ell_0 t_{k+1} < 1,$$

and the Banach Lemma on invertible operators [11] that the inverse $F'(x_{k+1})^{-1}$ exists and

$$\|F'(x_{k+1})^{-1}F'(x_0)\| \leqslant \frac{1}{1 - \ell_0\|x_{k+1} - x_0\|} \leqslant \frac{1}{1 - \ell_0 t_{k+1}}. \tag{25}$$

Therefore, by (2), (4), (29) and (25) we obtain in turn

$$\begin{aligned}\|x_{k+2}-x_{k+1}\| &= \|F'(x_{k+1})^{-1}F(x_{k+1})\|\\ &\leqslant \|F'(x_{k+1})^{-1}F'(x_0)\|\cdot\|F'(x_0)^{-1}F(x_{k+1})\|\\ &\leqslant \frac{\ell\|x_{k+1}-x_k\|^2}{2(1-\ell_0\|x_{k+1}-x_0\|)} \leqslant \frac{\ell(t_{k+1}-t_k)^2}{2(1-\ell_0 t_{k+1})} = t_{k+2}-t_{k+1}.\end{aligned} \tag{26}$$

Thus for every $z\in\bar U(x_{k+2},t^*-t_{k+2})$ we have

$$\|z-x_{k+1}\| \leqslant \|z-x_{k+2}\|+\|x_{k+2}-x_{k+1}\| \leqslant t^*-t_{k+2}+t_{k+2}-t_{k+1} = t^*-t_{k+1}.$$

That is,

$$z\in\bar U(x_{k+1},t^*-t_{k+1}). \tag{27}$$

Estimates (26) and (27) imply that (20) and (21) hold for $n=k+1$. By induction the proof of (20) and (21) is completed.

Lemma 1 implies that $\{t_n\}(n\geqslant 0)$ is a Cauchy sequence. From (20) and (21) $\{x_n\}$ $(n\geqslant 0)$ becomes a Cauchy sequence too, and as such it converges to some $x^*\in\bar U(x_0,t^*)$ (since $\bar U(x_0,t^*)$ is a closed set) such that

$$\|x^*-x_k\| \leqslant t^*-t_k. \tag{28}$$

The combination of (24) and (28) yields $F(x^*)=0$. Finally to show uniqueness let y^* be a solution of equation $F(x)=0$ in $U(x_0,R)$. It follows from (13), the estimate

$$\begin{aligned}&\left\|F'(x_0)^{-1}\int_0^1 [F'(y^*+\theta(x^*-y^*))-F'(x_0)]\right\| \mathrm{d}\theta\\ &\quad\leqslant \ell_0\int_0^1 \|y^*+\theta(x^*-y^*)-x_0\|\,\mathrm{d}\theta\\ &\quad\leqslant \ell_0\int_0^1 [\theta\|x^*-x_0\|+(1-\theta)\|y^*-x_0\|]\,\mathrm{d}\theta < \frac{\ell_0}{2}(t^*+R)\leqslant 1\end{aligned}$$

and the Banach Lemma on invertible operators that linear operator

$$L=\int_0^1 F'(y^*+\theta(x^*-y^*))\,\mathrm{d}\theta$$

is invertible.

Using the identity

$$0=F(x^*)-F(y^*)=L(x^*-y^*)$$

we deduce

$$x^*=y^*.$$

The uniqueness in $\bar U(x_0,t^*)$ follows as above by setting $t^*=R$.

That completes the proof of Theorem 1. □

Remark 2. Note that the conclusions of Theorem 1 hold if conditions (3) are replaced by (3)′.

In order for us to compare Theorem 1 with the famous Newton–Kantorovich theorem which is based on the Newton–Kantorovich hypothesis (29) we recall it below (see also [1,4,6,15–17]):

Theorem 2 (Moret [13], Miel [12], Kantorovich and Akilov [11]). *Let $F: D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume: there exist a point $x_0 \in D$ and parameters $\eta \geqslant 0$, $\ell > 0$ such that* (11), (12), (14),

$$h = 2\ell\eta \leqslant 1 \tag{29}$$

and

$$\bar{U}(x_0, s^*) \subseteq D,$$

where

$$s^* = \frac{1 - \sqrt{1-h}}{\ell}, \tag{30}$$

hold below s^*.

Then, sequence $\{x_n\}$ $(n \geqslant 0)$ *generated by Newton's method* (2) *is well defined, remains in* $\bar{U}(x_0, s^*)$ *for all* $n \geqslant 0$ *and converges to a unique solution* $x^* \in \bar{U}(x_0, s^*)$ *of equation* $F(x) = 0$.

Moreover, the following error bounds hold:

$$\|x_{n+1} - x_n\| \leqslant \frac{\ell \|x_n - x_{n-1}\|^2}{2[1 - \ell_0 \|x_n - x_0\|]} \leqslant s_n - s_{n-1} \quad (n \geqslant 1), \tag{31}$$

$$\|x_n - x^*\| \leqslant s^* - s_n \quad (n \geqslant 0), \tag{32}$$

$$0 \leqslant s_{n+1} - s_n = \frac{\frac{1}{2}\ell s_n^2 - s_n + \eta}{1 - \ell s_n} = \frac{\frac{1}{2}\ell (s_n - s_{n-1})^2}{1 - \ell s_n} \quad (n \geqslant 1), \tag{33}$$

and

$$s^* - s_{n+1} = \frac{\frac{1}{2}\ell (s^* - s_n)^2}{1 - \ell s_n} \leqslant \frac{1}{\ell 2^{n+1}} h^{2^{n+1}} \quad (n \geqslant 0) \ (\textit{for } h < 1). \tag{34}$$

We can now compare Theorems 1 and 2.

Theorem 3. *Under hypotheses of Theorems* 1 (*for* $\ell_0 < \ell$) *and* 2 *the following error bounds hold*:

$$t_{n+1} < s_{n+1} \quad (n \geqslant 1), \tag{35}$$

$$t_{n+1} - t_n < s_{n+1} - s_n \quad (n \geqslant 1), \tag{36}$$

$$t^* - t_n \leqslant s^* - s_n \quad (n \geqslant 0), \tag{37}$$

$$t^* \leqslant s^*, \tag{38}$$

$$0 \leqslant t_{n+1} - t_n \leqslant \alpha^{2^{n-1}} (s_{n+1} - s_n) \quad (n \geqslant 1), \ \alpha = \frac{1 - \ell\eta}{1 - \ell_0 \eta} \in [0, 1) \tag{39}$$

and

$$0 \leqslant t^* - t_n \leqslant \alpha^{2^{n-1}}(s^* - s_n) \quad (n \geqslant 1). \tag{40}$$

Moreover, we have: $t_n = s_n$ $(n \geqslant 0)$ *if* $\ell = \ell_0$.

Proof. We use induction on the integer k to show (35) and (36) first. For $n = 0$ in (4) we obtain

$$t_2 - \eta = \frac{\ell\eta^2}{2(1-\ell_0\eta)} \leqslant \frac{\ell\eta^2}{2(1-\ell\eta)} = s_2 - s_1$$

and

$$t_2 \leqslant s_2.$$

Assume:

$$t_{k+1} < s_{k+1}, \quad t_{k+1} - t_k < s_{k+1} - s_k \quad (k \leqslant n+1).$$

Using (4) and (33) we get

$$t_{k+2} - t_{k+1} = \frac{\ell/2(t_{k+1}-t_k)^2}{1-\ell_0 t_{k+1}} < \frac{\ell/2(s_{k+1}-s_k)^2}{1-\ell s_{k+1}} = s_{k+2} - s_{k+1}$$

and

$$t_{k+2} - t_{k+1} < s_{k+2} - s_{k+1}.$$

Let $m \geqslant 0$, we can obtain

$$\begin{aligned} t_{k+m} - t_k &< (t_{k+m} - t_{k+m-1}) + (t_{k+m-1} - t_{k+m-2}) + \cdots + (t_{k+1} - t_k) \\ &< (s_{k+m} - s_{k+m-1}) + (s_{k+m-1} - s_{k+m-2}) + \cdots + (s_{k+1} - s_k) \\ &< s_{k+m} - s_k. \end{aligned} \tag{41}$$

By letting $m \to \infty$ in (41) we obtain (37). For $n = 1$ in (37) we get (38).

Finally, (39) and (40) follow easily from (4) and (33). Note also that (39) holds as a strict inequality if $n \geqslant 2$.

That completes the proof of Theorem 3. □

Remark 3. It follows from Theorems 1–3 that whenever the Newton–Kantorovich hypothesis (29) holds so does (3) (for $\delta = 1$) (but not vice versa unless if $\ell = \ell_0$). Moreover, our error bounds are more precise, since

$$\ell_0 \leqslant \ell \tag{42}$$

in general. Note also that $t^* \in [\eta, 2\eta]$ and under the hypotheses of Theorem 2 $t^* \in [\eta, s^*]$. Moreover, ℓ/ℓ_0 can be arbitrarily large. Indeed:

Example 1. Define the scalar function F by $F(x) = c_0 x + c_1 + c_2 \sin e^{c_3 x}$, $x_0 = 0$, where c_i, $i = 0, 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small ℓ/ℓ_0 can be arbitrarily large. That is (3) may be satisfied but not (29).

We provide the additional error bounds without proof since these are similar to the ones given by others in the special case $\ell_0 = \ell$.

Proposition 1 (Yamamoto [17]). *Under the hypotheses of Theorem* 1 *for* $\ell_0 < \ell$ *the following error bounds hold for all* $n \geqslant 0$:

$$\frac{2\|x_{n+1}-x_n\|}{1+\sqrt{1+4a_n\|x_{n+1}-x_n\|}} \leqslant \|x_n - x^*\| \leqslant b_n \leqslant c_n, \tag{43}$$

$$\|x_n - x^*\| \leqslant d_n, \tag{44}$$

$$a_n\|x_n - x^*\|^2 + \|x_n - x^*\| - \|x_{n+1}-x_n\| \geqslant 0, \tag{45}$$

where

$$a_n = \frac{\ell}{2[1-\ell_0\|x_n - x_0\|]}, \quad b_n = \frac{\ell\|x_n - x_{n-1}\|^2}{2[1-\ell_0/2(\|x_n - x_0\| + \|x^* - x_0\|)]}, \tag{46}$$

$$c_n = \frac{\ell\|x_n - x_{n-1}\|^2}{1-\ell_0\|x_n - x_0\| + \sqrt{(1-\ell_0\|x_n - x_0\|)^2 - \ell_0\ell\|x_n - x_{n-1}\|^2}} \tag{47}$$

and

$$d_n = \frac{\ell}{2[1-\ell_0\|x_{n-1} - x_0\|]}\|x_{n-1} - x^*\|^2. \tag{48}$$

Proposition 2 (Yamamoto [17]). *Under the hypotheses of Proposition* 1 *the following additional error bounds hold for all* $n \geqslant 1$:

$$\|x_n - x^*\| \leqslant \frac{2\|x_{n+1}-x_n\|}{1+\sqrt{1-2M_n\|x_{n+1}-x_n\|}}$$

$$\leqslant \frac{2\|x_{n+1}-x_n\|}{1+\sqrt{1-2N_n\|x_{n+1}-x_n\|}} \leqslant t^* - t_n, \tag{49}$$

where

$$M_n = \frac{\ell}{1-\ell_0\|x_n - x_0\|}, \quad N_n = \frac{\ell}{1-\ell_0 t_n}. \tag{50}$$

Remark 4. Yamamoto in [17, p. 210] showed estimates of the form (49) with

$$\bar{M}_n = \frac{\ell}{1-\ell\|x_n - x_0\|} \tag{51}$$

and

$$\bar{N}_n = \frac{\ell}{1-\ell s_n}. \tag{52}$$

Note for all $n \geqslant 1$

$$M_n \leqslant \bar{M}_n \tag{53}$$

and

$$N_n \leqslant \bar{N}_n. \tag{54}$$

Hence our estimates (49) are finer than Yamamoto's if strict inequality holds in (42).

Remark 5. Gragg and Tapia [10] showed

$$\frac{2\|x_{n+1}-x_n\|}{1+\sqrt{1+4\theta^{2^n}/(1+\theta^{2^n})^2}} \leqslant \|x^*-x_n\| \leqslant \theta^{2^{n-1}}\|x_n-x_{n-1}\|, \quad (n \geqslant 1), \tag{55}$$

where

$$\theta = \frac{1-\sqrt{1-h}}{1+\sqrt{1-h}}. \tag{56}$$

Our error bounds compare favorably with (55). In the case of the lower bounds (similarly for the upper bounds) our error bounds (43) are larger than (55). This is true since the coefficient of the quadratic term in (45) is larger than ours (see [10, p. 12]).

Let us compare (43) with (55) using a simple example.

Example 2. Let $X = Y = \mathbf{R}$, $D = [-0.5, 0.5]$, $x_0 = 0.25$ and define F on $\mathbf{R}$ by

$$F(x) = \frac{x^3}{3} + x. \tag{57}$$

Using (12)–(14), (56) and (57) we obtain

$$\eta = 0.240196078, \quad \ell = \frac{16}{17}, \quad \ell_0 = \frac{12}{17}, \quad h = 0.452133794,$$

$$h_1 = (\ell + \ell_0)\eta = 0.339100346, \quad \theta = 0.149306495,$$

$$x_1 = 0.009803922, \quad x_2 = 0.000000628, \quad x_3 = x^* = 0.$$

We obtain by (55)

$$0.217860463 \leqslant \|x^* - x_0\|,$$

$$0.000960265 \leqslant \|x^* - x_1\|,$$

$$\|x^* - x_1\| \leqslant 0.035862835,$$

$$\|x^* - x_2\| \leqslant 0.000218539,$$

and by (43)

$$0.217860463 \leqslant \|x^* - x_0\|,$$

$$0.009766021 \leqslant \|x^* - x_1\|,$$

$$\|x^* - x_1\| \leqslant 0.032830147,$$

$$\|x^* - x_2\| \leqslant 0.000054917.$$

Hence our estimates (43) are more precise than (55).

Note that lower bounds were given also by Miel [12] and Yamamoto [17] using different techniques but the same type of quadratic inequality. Since their coefficient of the quadratic term is larger than ours (see (45)) we deduce that our lower bound is also more precise than the corresponding ones given by Miel and Yamamoto.

Remark 6. Under the Newton–Kantorovich hypothesis (29) Miel [12] showed

$$\|x_{n+1}-x_n\| \leqslant \frac{\ell}{2[1-\ell\|x_n-x_0\|]}\|x_n-x_{n+1}\|^2 \leqslant \frac{\ell}{2(1-\ell s_n)}\|x_n-x_{n-1}\|^2$$
$$= \frac{s_{n+1}-s_n}{(s_n-s_{n-1})^2}\|x_n-x_{n-1}\|^2 \tag{58}$$

and

$$\|x^*-x_n\| \leqslant A_n\|x_n-x_{n-1}\|^2 \leqslant B_n\|x_n-x_{n-1}\| \leqslant C_n\|x_1-x_0\| \tag{59}$$

are valid and best possible, where,

$$A_n = \frac{s^*-s_n}{(s_n-s_{n-1})^2}, \quad B_n = \frac{s^*-s_n}{s_n-s_{n-1}}, \quad C_n = \frac{s^*-s_n}{s_1}. \tag{60}$$

But for $\ell_0 < \ell$

$$\frac{\ell}{2[1-\ell_0\|x_n-x_0\|]} < \frac{\ell}{2[1-\ell\|x_n-x_0\|]} \tag{61}$$

and

$$\frac{\ell}{2[1-\ell_0 t_n]} < \frac{\ell}{2[1-\ell_0 s_n]}. \tag{62}$$

Hence under the hypotheses of Theorem 1

$$\|x_{n+1}-x_n\| \leqslant \frac{\ell}{2[1-\ell_0\|x_n-x_0\|]}\|x_n-x_{n-1}\|^2 \leqslant \frac{\ell}{2[1-\ell_0 t_n]}\|x_n-x_{n-1}\|^2$$
$$= \frac{t_{n+1}-t_n}{(t_n-t_{n-1})^2}\|x_n-x_{n-1}\|^2 \tag{63}$$

and

$$\|x^*-x_n\| \leqslant \bar{A}_n\|x_n-x_{n-1}\|^2 \leqslant \bar{B}_n\|x_n-x_{n-1}\| \leqslant \bar{C}_n\|x_1-x_0\| \tag{64}$$

are valid, best possible and finer than (58) because of (61) and (62), where

$$\bar{A}_n = \frac{t^*-t_n}{(t_n-t_{n-1})^2}, \quad \bar{B}_n = \frac{t^*-t_n}{t_n-t_{n-1}}, \quad \bar{C}_n = \frac{t^*-t_n}{t_1}. \tag{65}$$

Miel also gave the following lower bounds:

$$\frac{2\|x_{n+1}-x_n\|}{1+\sqrt{1+4[s_{n+1}-s_n/s_n-s_{n-1}^2]^2}} \leqslant \frac{2\|x_{n+1}-x_n\|}{1+\sqrt{1+4(s^*-s_{n+1})/(s^*-s_n)^2\|x_{n+1}-x_n\|}}$$
$$\leqslant \|x^*-x_n\|. \tag{66}$$

Simply replace the s^*, s_n by t^*, t_n to obtain our lower bounds which from the discussion above are also finer (closer to $\|x^*-x_n\|$) than the corresponding ones in (66).

Remark 7. Moret [13] showed

$$\|x_n-x^*\| \leqslant u(\|x_{n+1}-x_n\|)-u(0)=\varepsilon_n^1 \quad (n \geqslant 0) \tag{67}$$

and

$$\|x_n - x^*\| \leqslant u(\|x_n - x_{n-1}\|) - \|x_n - x_{n-1}\| - u(0) = \varepsilon_n^2 \quad (n \geqslant 1), \tag{68}$$

under condition (29) using the "r" functions which Potra and Pták call "rates of convergence" [16]. The functions u, r are given by

$$u(t) = u(t; g_n, p_n) = t + \sqrt{t^2 + \frac{1 - 2g_n p_n}{g_n^2}}, \quad t \in [0, \eta] \tag{69}$$

$$r(t) = \frac{t^2}{2(u(t) - t)}, \quad t \in [0, \eta] \tag{70}$$

$$g_0 = \ell, \quad q_n = 1 - \ell\|x_n - x_0\|, \quad p_n = \|x_{n+1} - x_n\|, \quad g_n = \frac{\ell}{q_n}.$$

Define functions $\bar{u}$, $\bar{r}$ by

$$\bar{u}(t) = \bar{u}(t; \bar{g}_n, z_n, p_n) = t + \frac{1 - z_n p_n}{\bar{g}_n}, \quad t \in [0, \eta] \tag{71}$$

$$\bar{r}(t) = \frac{t^2}{2(\bar{u}(t) - t)}, \quad t \in [0, \eta], \tag{72}$$

$$z_0 = \ell_0, \quad \bar{g}_0 = \ell, \quad z_n = \bar{g}_n \quad (n \geqslant 1), \quad \bar{g}_n = \frac{\ell}{1 - \ell_0\|x_{n+1} - x_0\|}.$$

Moret's results (67) and (68) are deduced from

$$\|x_{n+1} - x_n\| \leqslant r(\|x_n - x_{n-1}\|) \quad (n \geqslant 1). \tag{73}$$

But under (3) we have already showed

$$\|x_{n+1} - x_n\| \leqslant \bar{r}(\|x_n - x_{n-1}\|) < r(\|x_n - x_{n-1}\|) \quad (\ell \neq \ell_0)\ (n \geqslant 1). \tag{74}$$

Exactly as in Moret [11, pp. 67–70] we get under the hypotheses of Theorem 1

$$\|x_n - x^*\| \leqslant \sum_{k=0}^{\infty} \bar{r}^{(k)}(\|x_{n+1} - x_n\|) \leqslant \varepsilon_n^1 \tag{75}$$

and

$$\|x_n - x^*\| \leqslant \sum_{k=0}^{\infty} \bar{r}^{(k)}(\|x_{n+1} - x_n\|) \leqslant \varepsilon_n^2 \quad (n \geqslant 1). \tag{76}$$

That is, our upper bounds are at least as fine as (67) and (68). The results obtained by Potra and Pták in [16] can also be improved along the same lines since our function "$\bar{r}$" is smaller than "r". However we leave the details to the motivated reader.

Remark 8. The error bounds obtained in the Newton–Kantorovich Theorem 2 have been improved by several authors and under different techniques. The interested reader can find a list of error bounds and the relationship between them in [4,15,17]. It is clear from the error estimates in Theorem 3 that a parallel and more favorable list to the one in [17] can be obtained for error bounds using the same information (see Propositions 1 and 2 and Remarks 3–6) under our weaker hypotheses and for error bounds using the same information.

Let us give a brief sketch of what else we mean. Under the assumptions of Theorem 1 set:

$$\bar{U} = \bar{U}_0 = \bar{U}(x_0, t^*), \quad \bar{U}_n = \bar{U}(x_n, t^* - t_n) \quad (n \geqslant 1), \quad K_0 = \ell, \quad \bar{K}_0 = \ell_0,$$

$$K_n = \sup_{\substack{x,y \in \bar{U}_n \\ x \neq y}} \frac{\|F'(x_n)^{-1}[F'(x) - F'(y)]\|}{\|x - y\|}, \tag{77}$$

$$\bar{K}_n = \sup_{\substack{x \neq x_n \\ x \in \bar{U}_n}} \frac{\|F'(x_n)^{-1}[F'(x) - F'(x_n)]\|}{\|x - x_n\|}. \tag{78}$$

It can easily be seen that

$$K_n \leqslant K_0 \frac{1}{1 - \bar{K}_0 \|x_n - x_0\|} \tag{79}$$

and

$$\bar{K}_n \leqslant K_0 \frac{1}{1 - \bar{K}_0 \|x_n - x_0\|}. \tag{80}$$

We show

$$(K_n + \delta \bar{K}_n) \|x_{n+1} - x_n\| \leqslant \delta, \tag{81}$$

or

$$\frac{(1 + \delta) K_0}{1 - \bar{K}_0 \|x_n - x_0\|} \|x_{n+1} - x_n\| \leqslant \delta,$$

or

$$\frac{(1 + \delta) K_0}{1 - 2\bar{K}_0 [1 - \left(\frac{1}{2}\right)^n] \eta} \left(\frac{1}{2}\right)^n \quad \eta \leqslant \delta,$$

or

$$(1 + \delta) \left(\tfrac{1}{2}\right)^n K_0 \eta \leqslant \delta - 2\delta \bar{K}_0 \left[1 - \left(\tfrac{1}{2}\right)^n\right] \eta$$

or

$$\left\{(1 + \delta) \left(\tfrac{1}{2}\right)^n K_0 + 2\delta \left[1 - \left(\tfrac{1}{2}\right)^n\right] \bar{K}_0\right\} \eta \leqslant \delta. \tag{82}$$

It suffices to show

$$(1 + \delta) \left(\tfrac{1}{2}\right)^n K_0 + 2\delta \left[1 - \left(\tfrac{1}{2}\right)^n\right] \bar{K}_0 \leqslant K_0 + \bar{K}_0 \delta$$

or

$$\bar{K}_0 \delta \left[2 \left(1 - \left(\tfrac{1}{2}\right)^n\right) - 1\right] \leqslant K_0 \left[1 - (1 + \delta) \left(\tfrac{1}{2}\right)^n\right]$$

or

$$\delta \left[1 - \left(\tfrac{1}{2}\right)^{n-1}\right] \leqslant 1 - (1 + \delta) \left(\tfrac{1}{2}\right)^n$$

or

$$\delta \left[1 - \left(\tfrac{1}{2}\right)^n\right] \leqslant 1 - \left(\tfrac{1}{2}\right)^n$$

or

$$\delta \leqslant 1$$

which is true by hypothesis.

That is, the conclusions of Theorem 1 can pass from $(0, \delta, F, x_0, \bar{U}_0, K_0, \bar{K}_0)$ to the class $(\eta, \delta, F, x_n, \bar{U}_n, K_n, \bar{K}_n)$ $(n \geqslant 1)$ in order to obtain finer error bounds than the ones given by Yamamoto in [17] (see, e.g., Theorem 3.2 or the chart on page 213). Similar remarks can be made if we take the sup on $\bar{U}$ instead of $\bar{U}_n$ $(n \geqslant 1)$.

We now complete this section with three numerical examples. In the first as well as in the third one we show that hypothesis (29) fails whereas (3) holds. In the second example used also in [16] we compare estimates (4), (16) and (33), (32), respectively.

Example 3. Let $X = Y = \mathbf{R}$, $D = [\sqrt{2} - 1, \sqrt{2} + 1]$, $x_0 = \sqrt{2}$ and define function F on D by

$$F(x) = \frac{1}{6}x^3 - \left(\frac{2^{3/2}}{6} + 0.23\right). \tag{83}$$

Using (11)–(13) and (83) we obtain

$$\eta = 0.23, \quad \ell = 2.4142136, \quad \ell_0 = 1.914213562,$$

$$h = 2\ell\eta = 1.1105383 > 1 \tag{84}$$

and (3) for $\delta = 1$

$$h_1 = (\ell + \ell_0)\eta = 0.995538247 < 1. \tag{85}$$

That is, there is no guarantee that Newton's method $\{x_n\}$ $(n \geqslant 0)$ starting at x_0 converges to a solution x^* of equation $F(x) = 0$, since (29) is violated. However since (85) holds, Theorem 1 guarantees the convergence of Newton's method to $x^* = 1.614507018$.

Example 4. Let $X = Y = \mathbf{R}$, $x_0 = 1.3$, $D = [x_0 - 2\eta,\ x_0 + 2\eta]$ and define function F on D by

$$F(x) = \tfrac{1}{3}(x^3 - 1). \tag{86}$$

As in Example 2 we obtain

$$\eta = 0.236094674, \quad \ell = 2.097265501, \quad \ell_0 = 1.817863519$$

$$h = 2\ell\eta = 0.990306428 < 1, \quad h_1 = (\ell + \ell_0)\eta = 0.92434111 < 1, \quad (\text{for } \delta = 1)$$

$$t^* = 0.369677842 \quad \text{and} \quad s^* = 0.429866445.$$

That is, we provide a better information on the location of the solution x^* since

$$\bar{U}(x_0, t^*) \subset \bar{U}(x_0, s^*). \tag{87}$$

Moreover, using (2), (86), (4), (33) and (32) we can tabulate the following results (Table 1):

A more interesting example is given by the following:

Table 1
Comparison table

x_n	Estimates (16)	Estimates (17)	Estimates (33)	Estimates (32)
$x_1 = 1.0639053254$	0.236094674	0.133583172	0.236094674	0.193771771
$x_2 = 1.0037617275$	0.102400629	0.031182539	0.115780708	0.0779910691
$x_3 = 1.0000140800$	0.028585756	0.002596783	0.053649732	0.024342893
$x_4 = 1.0000000002$	0.002575575	0.000021208	0.020186667	0.004156226
$n = 5$	0.000021207	0.000000001	0.003987206	0.00016902
$n = 6$	0.000000001	0	0.000166761	0.000002259

Example 5. Let $X = Y = \mathbf{R}$, $x_0 = 1$ and define function F by

$$F(x) = x^3 - a \quad \text{for all } x \in [a, 2-a], \; a \in [0, \tfrac{1}{2}).$$

Using (12)–(14) we find

$$\eta = \tfrac{1}{3}(1-a), \quad \ell_0 = 3 - a \text{ and } \ell = 2(2-a).$$

The Newton–Kantorovich hypothesis (29) does not hold since

$$h = \tfrac{4}{3}(1-a)(2-a) > 1 \quad \text{for all } a \in [0, \tfrac{1}{2}).$$

That is there is no guarantee that Newton's method (2) converges to the solution $x^* = \sqrt[3]{a}$ of equation $F(x) = 0$. However, (3) holds for all $a \in [\frac{5-\sqrt{13}}{3}, \frac{1}{2})$ if $\delta = 1$, since

$$h_1 = \tfrac{1}{3}(1-a)[3 - a + 2(2-a)] \leqslant 1.$$

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