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Canonical products of small order and related Pick functions

Henrik L. Pedersen[1]

Department of Natural Sciences, Royal Veterinary and Agricultural University, Thorvaldsensvej 40, DK-1870 Copenhagen, Denmark

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Abstract

We show that various functions related to the logarithms of the canonical products $P_\rho(z)=\prod_{n=1}^{\infty}(1+z/n^\rho)$, $\rho>1$ and $Q(z)=\prod_{n=0}^{\infty}(1+zq^n)$, $q\in(0,1)$ are Pick functions. As a consequence we find an integral expansion of a function involving the logarithm of Jacksons q-gamma function.

MSC: Primary 30E20; Secondary 33D05

Keywords: Pick function; Completely monotone function; q-gamma function

1. Introduction

We shall investigate certain entire functions of genus 0 having only negative zeros. These functions are canonical products and have the representation

$$f(z)=\prod_{n=1}^{\infty}\left(1+\frac{z}{a_n}\right), \tag{1}$$

where the positive numbers $\{a_n\}$ are arranged in increasing order of magnitude and where the series $\sum_{n=1}^{\infty}1/a_n$ converges. The corresponding zero counting function n is defined as

$$n(r)=\#\{n\mid |a_n|\leqslant r\}. \tag{2}$$

E-mail address: henrikp@dina.kvl.dk (H.L. Pedersen).
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We describe the motivation for the investigations carried out in this paper. The maximal growth of a function of the form (1) occurs in the direction of the positive axis;

$$m(f,r)\equiv \max\{|f(z)|\,|z|=r\}=f(r),$$

for $r>0$. The order κ of f is defined as the infimum of the numbers s such that there exists a constant C with the property that $m(f,r)\leqslant C\ \exp(r^s)$, for all $r>0$. In our situation $\kappa\leqslant 1$ and in the case where κ is positive we see that the related function

$$\varphi(z)=\frac{\log f(z)}{z^\kappa},$$

varies between 0 and a constant when z runs through the positive axis. The constant could be equal to infinity, although it is finite if we impose further restrictions on the zero distribution (which we shall do in this paper). The natural question about the detailed behaviour of φ on the positive axis is addressed here for some classes of functions of the form (1). This behaviour is read off from the properties of the holomorphic function φ in the cut plane $\mathscr{A}=\mathbb{C}\setminus(-\infty,0)]$ and our investigation focuses on its behaviour in $\mathscr{A}$.

A Pick function is a holomorphic function in the upper half plane having nonnegative imaginary part. For the general theory about these functions, see [6]. It is known that any Pick function admits an integral representation in terms of a positive measure. Furthermore, if this measure is supported on the negative axis then it is easily seen that the derivative of the function is a completely monotone function. A completely monotone function g is a C^∞ function such that $(-1)^n g^{(n)}(x)\geqslant 0$ for $x>0$. For an introduction to these functions see the monograph [14]. Many references on results about completely monotone functions can be found in e.g. [7]. A positive function $f(x)$ whose derivative is completely monotone is sometimes called a Bernstein function.

The general goal of this paper is to explore what can be called a Pick property for some classes of entire functions having negative zeros. The investigations were initiated by the study of Eulers gamma function (see [4]) and have subsequently been carried out for a more general class of functions of genus one (see [12]) as well as for the double gamma function introduced by Barnes (see [13]).

In our results below the expression $\log w(z)$ (where w is some analytic function) denotes the branch of the logarithm obtained by analytic continuation of $\log w(x)$ for positive real x. The principal logarithm is denoted by Log.

Theorem 1.1. *Let P be a canonical product of the form* (1) *where* $|n(r)-r^{1/\rho}|\leqslant \text{Const}$. *Then*

$$\frac{\log P(z)}{z^{1/\rho}}=\Re\left(\frac{\log P(i)}{i^{1/\rho}}\right)+\frac{1}{\pi}\int_{-\infty}^{0}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)\mathrm{d}(t)\mathrm{d}t,$$

where the function d is given by

$$\mathrm{d}(t)=\frac{\sin(\pi/\rho)}{|t|^{1/\rho}}\left(-\log|P(t)|+\pi\cot(\pi/\rho)n(-t)\right).$$

This is the Stieltjes representation of $(\log P(z))/z^{1/\rho}$.

We put for $\rho > 1$

$$P_\rho(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^\rho}\right).$$

Theorem 1.2. *The function* $(\log P_\rho(z))/z^{1/\rho}$ *is a Pick function and*

$$\frac{\log P_\rho(z)}{z^{1/\rho}} = \frac{\pi}{\sin(\pi/\rho)} + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{d}_\rho(t)}{t - z}\,\mathrm{d}t,$$

where the positive *function* d_ρ *is given by*

$$\mathrm{d}_\rho(t) = \frac{\sin(\pi/\rho)}{|t|^{1/\rho}}\,(-\log|P_\rho(t)| + \pi\,\cot(\pi/\rho)n(-t)).$$

As a corollary of this result we notice

Corollary 1.3. *The function* $(\log P_\rho(x))/x^{1/\rho}$ *is a Bernstein function*.

We shall also investigate canonical products of order 0. Here we shall limit ourselves by considering functions of the form

$$Q(z) = \prod_{n=0}^{\infty} (1 + zq^n), \quad q \in (0, 1).$$

This function appears in the theory of q-special functions. The infinite product $Q(-q^x)$ is in fact the limit of the q-shifted factorials (or q-Pochhammer symbols) $(q^x; q)_n$.

One well-known function defined in terms of these factorials is Jackson's q-gamma function. It is given as

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}\,(1-q)^{-(z-1)}.$$

Historical remarks and several references on this function can be found in [1].

We shall explain the motivation for our choice of the function related to Q that we consider in Theorem 1.6. It has recently been proved that

$$z \mapsto \frac{\log \Gamma(z+1)}{z\,\log z}$$

is a Pick function (see [4]). The question arises if there is a q-analogue of this result involving Jacksons q-gamma function Γ_q. This function has poles at all points of the form $\{-n + 2\pi i m/\log q\}$, where $n \geqslant 0$ and m are integers. In particular there are poles in the upper half plane so the function $\log \Gamma_q(z)$ is not even holomorphic there.

If we restrict our viewpoint to the positive real axis, then the function

$$x \mapsto \frac{\log \Gamma_q(x+1)}{x\,\log\left(\frac{1-q^x}{1-q}\right)}$$

varies between zero and some positive constant (which easily is seen to be 1). As a corollary to Theorem 1.6 we show that this function is increasing. This may be considered as a q-analogue of the corresponding result about the gamma function (see [2]), since $\Gamma_q(x+1) \to \Gamma(x+1)$ and $(1-q^x)/(1-q) \to x$ as $q \to 1_-$.

Corollary 1.4. *The function*

$$x \mapsto \frac{\log \Gamma_q(x+1)}{x \, \log\left(\frac{1-q^x}{1-q}\right)}$$

increases *on the positive line from* 0 *to* 1. *It has the following integral representation:*

$$\frac{\log \Gamma_q(x+1)}{x \, \log\left(\frac{1-q^x}{1-q}\right)} = \Re\left\{\frac{\log \Gamma_q(i\pi/(2\log q)+1)}{i\pi \, \log\left(\frac{1+i}{1-q}\right)/(2\log q)}\right\} + \int_{-\infty}^{-1}\left(\frac{1}{t+q^x} - \frac{t}{t^2+1}\right) h_-(t)\, \mathrm{d}t$$
$$+ \int_0^{\infty}\left(\frac{1}{t+q^x} - \frac{t}{t^2+1}\right) h_+(t)\, \mathrm{d}t,$$

where h_- and h_+ are the functions defined in (11) *and* (12).

(We remark that the function in the corollary above does *not* have a completely monotone derivatives.)

Concerning the function Q we notice:

Theorem 1.5. *The function*

$$\psi(z) = \frac{\log Q(z) - \log Q(-q)}{\mathrm{Log}\left(\frac{1+z}{1-q}\right)}$$

is a Pick function.

Theorem 1.6. *The function*

$$\psi(z) = \frac{\log Q(qz) - \log Q(-q) + \frac{\log(1-q)}{\log q}\,\mathrm{Log}(-z)}{\mathrm{Log}\left(\frac{1+z}{1-q}\right)\,\mathrm{Log}(-z)}$$

is a Pick function and it has the following integral representation

$$\psi(z) = \Re\psi(i) + \int_{-\infty}^{-1}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) h_-(t)\, \mathrm{d}t$$
$$+ \int_0^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) h_+(t)\, \mathrm{d}t,$$

where h_- and h_+ are certain positive functions (*defined in* (11) *and* (12)).

The development to follow is based on a maximum principle in the upper half plane. However, it is burdened with many technical estimates.

2. Positive order

Here we investigate the situation where the zero counting function n associated with the canonical product P of the form (1) satisfies

$$|n(r) - r^{1/\rho}| \leqslant \mathrm{Const}, \tag{3}$$

for all $r > 0$. Here $\rho > 1$. We define

$$\log P(z) = \sum_{n=1}^{\infty} \mathrm{Log}(1 + z/a_n),$$

where Log denotes the principal logarithm. The function P is of order $1/\rho$. Indeed we have

$$\lim_{x\to\infty} \frac{\log P(x)}{x^{1/\rho}} = \frac{\pi}{\sin(\pi/\rho)}.$$

This may be verified using the relation

$$\log P(x) = \int_0^\infty \frac{n(t)}{t} \frac{x}{t+x} \, \mathrm{d}t, \tag{4}$$

see e.g. [5, Chapter 4] or [11, Lecture 12]. In the proof of Theorem 1.1 we need an estimate of $\log |P(z)|$. That is the contents of Lemma 2.1.

Lemma 2.1. *If $n(r) \leqslant Cr^{1/\rho}$ there exists a sequence $\{r_n\}$ tending to infinity and a constant such that we have*

$$|\log P(z)| \leqslant \mathrm{Const}|z|^{1/\rho}, \quad |z| = r_n,$$

for all n.

Proof. Since the maximum growth of P occurs along the positive real line we obtain

$$\log m(P, r) \leqslant \mathrm{Const}\, \frac{\pi}{\sin(\pi/\rho)}\, r^{1/\rho},$$

for $r > 0$. By a result of Littlewood (see [8, Chapter 6]) this is also a lower bound on $\log |P|$ on circles of certain radii $\{r_n\}$ tending to infinity. The estimate

$$|\arg\ P(z)| \leqslant \mathrm{Const}|z|^{1/\rho}, \quad z \in \mathbb{C},$$

may be proved by as follows. We suppose that $A > 0$ and define $K = \min\{k \mid a_k \geqslant A + 1\}$. Then, if $z = x + iy$ where $x \in [-A, -A + 1]$ and $y \geqslant 0$, we get

$$\begin{aligned}\arg\ P(z) &= \sum_{n=1}^{\infty} \mathrm{Arg}(1 + z/a_n)\\ &\leqslant \pi(K-1) + \sum_{n=K}^{\infty} \arctan\left(\frac{y}{a_n - A}\right).\end{aligned}$$

One may rewrite the infinite sum as an integral in terms of the counting function $n(t) - n(a_{K-1})$ and perform integration by parts. In this way one obtains

$$\sum_{n=K}^{\infty} \arctan\left(\frac{y}{a_n - A}\right) \leqslant \int_1^{\infty} n(s+A)\,\frac{y}{s^2+y^2}\,\mathrm{d}s,$$

and this last integral is seen to be less than $\mathrm{Const}(A^{1/\rho} + y^{1/\rho})$. Therefore

$$\arg\, P(z) \leqslant \mathrm{Const}(K - 1 + A^{1/\rho} + y^{1/\rho}) \leqslant \mathrm{Const}|z|^{1/\rho}. \qquad \Box$$

Proof of Theorem 1.1. We let $B > 0$ and consider the auxiliary function

$$\varphi_B(z) = \frac{\log P(z)}{z^{1/\rho}} + B\,\mathrm{Log}\, z, \quad z \in \mathscr{A}.$$

We set out to prove that φ_B is a Pick function if $B > 0$ is chosen sufficiently large. This amounts to showing that the harmonic function $\Im\varphi_B$ is nonnegative in the upper half plane. We shall verify this by appealing to a Phragmén–Lindelöf principle. To this end we need to investigate the boundary behaviour of $\Im\varphi_B$ on the real line. A routine computation shows that,

$$\Im\varphi_B(z) \to \frac{-\sin(\pi/\rho)\log|P(x)| + \pi\cos(\pi/\rho)n(-x)}{|x|^{1/\rho}} + B\pi$$

as $z \to x \in (-\infty, 0)\backslash\{-a_n\}$ within the upper half plane. We see that $\Im\varphi_B$ has nonnegative boundary values on the real line provided

$$\log|P(x)| \leqslant \pi\cot(\pi/\rho)n(-x) + \frac{B\pi|x|^{1/\rho}}{\sin(\pi/\rho)}, \quad x < 0. \tag{5}$$

We now show that we can choose B so large that this holds. We see from [5, Theorem 4.1] that there exists $r_0 > 0$ such that

$$\log|P(x)| \leqslant (\pi\cot(\pi/\rho) + 1)|x|^{1/\rho}, \quad x < -r_0.$$

Suppose that $\rho \in (1, 2]$. From (3) and the line above we get (noting that $\cot(\pi/\rho) \leqslant 0$)

$$\begin{aligned}\log|P(x)| &\leqslant \pi\cot(\pi/\rho)(n(-x) - C) + |x|^{1/\rho}\\ &= \pi\cot(\pi/\rho)n(-x) + |x|^{1/\rho} + \mathrm{Const}, \quad x < -r_0.\end{aligned}$$

For $x \in [-r_0, 0]$, $\log|P(x)| - \cot(\pi/\rho)n(-x)$ is bounded from above and it is less than or equal to $\mathrm{Const}|x|$ as $x \to 0$. Hence we may choose $B > 0$ such that (5) is satisfied.

If $\rho > 2$, we use $\log|P(x)| \leqslant \pi\cot(\pi/\rho)(n(-x) + C) + |x|^{1/\rho}$ to obtain (5) in this case.

We may rephrase (5) as: $\liminf_{z\to x} \Im\varphi_B(z) \geqslant 0$ for all real x. Furthermore, from Lemma 2.1,

$$|\Im\varphi_B(z)| \leqslant \mathrm{Const}\,\log|z|, \quad |z| = r_n,$$

for some sequence $\{r_n\}$ tending to infinity. We obtain from these facts that $\Im\varphi_B \geqslant 0$ throughout the upper half plane by a Phragmén–Lindelöf principle or extended maximum principle [10, Chapter III]. (See also [12].) Therefore φ_B is a Pick function.

It is not hard to show that (using the general integral representation of Pick functions, see e.g. [4, Section 1])

$$\varphi_B(z) = \Re\left(\frac{\log P(i)}{i^{1/\rho}}\right) + \frac{1}{\pi}\int_{-\infty}^{0}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) v_B(t)\mathrm{d}t,$$

where

$$v_B(t) = \frac{\sin(\pi/\rho)}{|t|^{1/\rho}}\left(-\log|P(t)| + \pi\cot(\pi/\rho)n(-t)\right) + B\pi.$$

Furthermore, since Log z is itself a Pick function and in fact

$$\operatorname{Log} z = \int_{-\infty}^{0}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\mathrm{d}t,$$

we get the desired Stieltjes representation

$$\frac{\log P(z)}{z^{1/\rho}} = \Re\left(\frac{\log P(i)}{i^{1/\rho}}\right) + \frac{1}{\pi}\int_{-\infty}^{0}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\mathrm{d}(t)\mathrm{d}t,$$

where

$$\mathrm{d}(t) = \frac{\sin(\pi/\rho)}{|t|^{1/\rho}}\left(-\log|P(t)| + \pi\cot(\pi/\rho)n(-t)\right).$$

We have proved Theorem 1.1 □

Proof of Theorem 1.2. From the integral representation in Theorem 1.1 and the positivity of the measure in Proposition 2.2 below we see that $(\log P(z))/z^{1/\rho}$ is a Pick function with integral representation

$$\frac{\log P_\rho(z)}{z^{1/\rho}} = \Re\left(\frac{\log P_\rho(i)}{i^{1/\rho}}\right) + \frac{1}{\pi}\int_{-\infty}^{0}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\mathrm{d}_\rho(t)\mathrm{d}t,$$

where

$$\mathrm{d}_\rho(t) = \frac{\sin(\pi/\rho)}{|t|^{1/\rho}}\left(-\log|P_\rho(t)| + \pi\cot(\pi/\rho)n(-t)\right), \quad t<0.$$

We find (by differentiating under the integral sign) that $(\log P(x))/x^{1/\rho}$ increases from 0 to $\pi/\sin(\pi/\rho)$ as x increases from zero to infinity. This means however that the function $F(z) = \pi/\sin(\pi/\rho) - (\log P(z))/z^{1/\rho}$ is positive on the positive axis and has negative imaginary part in the upper half plane. The function is thus a Stieltjes transform and hence it has an integral representation of the form

$$\frac{\pi}{\sin(\pi/\rho)} - \frac{\log P(z)}{z^{1/\rho}} = \alpha + \int_0^\infty \frac{\mathrm{d}\sigma(t)}{z+t},$$

where $\alpha = \lim_{x\to\infty} F(x) = 0$ and $\int_0^\infty \mathrm{d}\sigma(t)/(1+t) < \infty$ (see e.g. [3]). It follows that $\mathrm{d}\sigma(t) = \mathrm{d}(-t)\mathrm{d}t/\pi$ and hence that

$$\frac{\log P(z)}{z^{1/\rho}} = \frac{\pi}{\sin(\pi/\rho)} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}_\rho(-t)}{z+t}\mathrm{d}t.$$

This proves Theorem 1.2 (once the proposition below is verified). □

Proposition 2.2. *For any $\rho > 1$ we have*

$$\log|P_\rho(x)| \leqslant \pi\cot(\pi/\rho)n(-x), \quad x<0.$$

This proposition furnishes an upper bound on $|P_\rho|$ on the negative axis. Theorems as [5, Theorem 4.1] give an asymptotic upper bound where the right-hand side is replaced by $(\pi\cot(\pi/\rho)+\varepsilon)|x|^{1/\rho}$. This is however not precise enough for our needs and in fact the proof of our proposition becomes an involved and technical argument.

The precise asymptotic behaviour of $\log P_\rho$ has been described by Hardy, see [8]. We shall use some of the formulae in that paper and also extend them. We formulate the following lemma.

Lemma 2.3. *For $\rho \in (1,4)$ and $a>0$ we have*

$$\begin{aligned}\log|P_\rho(-a)| =& \pi\cot(\pi/\rho)a^{1/\rho} + \log|\sin \pi a^{1/\rho}| - (\log a)/2 \\ &- \log\pi + (1-\rho/2)\log(2\pi) - (\rho/\pi)\sin(\pi\rho/2)aI(a,\rho),\end{aligned}$$

where

$$I(a,\rho) = \int_0^\infty (\log(2\sinh(\pi\eta)) - \pi\eta)\,\frac{\eta^{\rho-1}}{\eta^{2\rho}+a^2-2a\eta^\rho\cos(\pi\rho/2)}\,\mathrm{d}\eta.$$

In the situation where $\rho<2$ this lemma is obtained in [8] and it is easy to see that it holds also for $\rho=2$. We shall verify the relation in the lemma for $\rho\in(2,4)$ in the Appendix.

About the integral $I(a,\rho)$ we notice

Lemma 2.4. *Let $I(a,\rho)$ be as in Lemma 2.3. Then*

$$-aI(a,\rho) \leqslant \frac{\pi}{12a}, \quad \text{for } 1<\rho\leqslant 2.$$

Proof. We find

$$\begin{aligned}-aI(a,\rho) =& \int_0^\infty (-\log(1-\mathrm{e}^{-2\pi\eta}))\,\frac{a\eta^{\rho-1}}{\eta^{2\rho}+a^2-2a\eta^\rho\cos(\pi\rho/2)}\,\mathrm{d}\eta \\ \leqslant& \int_0^\infty (-\log(1-\mathrm{e}^{-2\pi\eta}))\,\frac{a\eta^{\rho-1}}{\eta^{2\rho}+a^2}\,\mathrm{d}\eta \\ =& \frac{1}{a}\int_0^\infty (-\log(1-\mathrm{e}^{-2\pi\eta}))\,\frac{a^2\eta^{\rho-1}}{\eta^{2\rho}+a^2}\,\mathrm{d}\eta \\ \leqslant& \frac{1}{a}\int_0^\infty (-\log(1-\mathrm{e}^{-2\pi\eta}))\eta^{\rho-1}\,\mathrm{d}\eta \\ =& \frac{1}{a(2\pi)^\rho}\int_0^\infty (-\log(1-\mathrm{e}^{-t}))t^{\rho-1}\,\mathrm{d}t \\ =& \frac{1}{a(2\pi)^\rho}\,\Gamma(\rho)\zeta(\rho+1) \leqslant \frac{1}{a(2\pi)^\rho}\,\Gamma(1)\zeta(2) = \frac{\pi}{12a}.\end{aligned}$$

The relation involving Riemanns zeta function and the gamma function can be verified by performing partial integration on the integral and by using the well-known formula

$$\Gamma(\rho+1)\zeta(\rho+1)=\int_0^\infty \frac{t^\rho}{\mathrm{e}^t-1}\,\mathrm{d}t.$$

This completes the proof of the lemma. □

The assertion of Proposition 2.2 may be written as follows:

$$\log|P_\rho(-t)|\leqslant k\pi\cot(\pi/\rho),\quad t\in(k^\rho,(k+1)^\rho), \tag{6}$$

for $k\geqslant 0$. If $k=0$, (6) is easily verified regardless of the value of ρ. ($|P_\rho(-t)|$ varies between 0 and 1 for t in the interval (0,1).)

Proof of Proposition 2.2 when $\rho\in(1,4)$**.** We suppose that $t\in(k^\rho,(k+1)^\rho)$ for some $k\geqslant 1$. From Lemma 2.3 we see that (6) holds provided

$$\begin{aligned}&\pi\cot(\pi/\rho)(t^{1/\rho}-k)+\log|\sin\pi t^{1/\rho}|-(\log t)/2\\&\quad-\log\pi+(1-\rho/2)\log(2\pi)-(\rho/\pi)t\sin(\pi\rho/2)I(t,\rho)\leqslant 0,\end{aligned}$$

for $t\in(k^\rho,(k+1)^\rho)$. Here it is elementary to see that (introducing the variable $s=t^{1/\rho}-k\in(0,1)$)

$$\pi\cot(\pi/\rho)(t^{1/\rho}-k)+\log|\sin\pi t^{1/\rho}|\leqslant\pi\cot(\pi/\rho)(1-1/\rho)+\log|\sin(\pi/\rho)|,$$

and therefore it is enough to verify

$$\begin{aligned}&\pi\cot(\pi/\rho)(1-1/\rho)+\log|\sin(\pi/\rho)|-(\log t)/2-\log\pi\\&\quad+(1-\rho/2)\log(2\pi)-(\rho/\pi)t\sin(\pi\rho/2)I(t,\rho)\leqslant 0.\end{aligned}\tag{7}$$

Here we consider two situations, namely $\rho\in(1,2]$ and $\rho\in(2,4)$. If $\rho\in(1,2]$ we have from Lemma 2.4 that $-tI(t,\rho)\leqslant\pi/(12t)$. Hence, the left-hand side of (7) is less than or equal to (remembering that $\cot(\pi/\rho)\leqslant 0$ and $t\geqslant 1$)

$$-\log\pi+(\log(2\pi))/2+1/6,$$

which in fact is negative. Therefore (7) is verified in this situation.

If $\rho\in(2,4)$,

$$-(\rho/\pi)t\sin(\pi\rho/2)I(t,\rho)\leqslant 0,$$

since both the integral $I(t,\rho)$ and $\sin(\pi/\rho)$ are negative. Thus (7) follows if we can verify that

$$\pi\cot(\pi/\rho)(1-1/\rho)+\log|\sin(\pi/\rho)|-\log\pi+(1-\rho/2)\log(2\pi)\leqslant 0,$$

for $\rho\in(2,4)$. Here we put $r=\pi/\rho\in(\pi/4,\pi/2)$ and have to verify

$$\cot(r)(\pi-r)+\log|\sin r|+\log 2-(\pi\log(2\pi))/(2r)\leqslant 0.$$

This relation is indeed true as one can prove by standard calculus methods. Proposition 2.2 is thus proved when $\rho\in(1,4)$. □

To prove the proposition for $\rho \geqslant 4$ we need some technical lemmas. We prove them below.

Lemma 2.5. *We have*

$$\log |P_\rho(-t)| \leqslant \pi \cot(\pi/\rho), \quad t \in [1, 2^\rho], \quad \rho \geqslant 4.$$

Lemma 2.6. *We have*

$$\log P_\rho((k+1)^\rho) \leqslant k\pi/\sin(\pi/\rho) + 1, \quad \rho \in [2,4], \quad k \geqslant 2, \tag{8}$$

$$\log P_\rho((k+1)^\rho) \leqslant k\pi/\sin(\pi/\rho), \quad \rho \in [4,\infty), \quad k \geqslant 2. \tag{9}$$

Proof of Proposition 2.2 when $\rho \geqslant 4$. We begin by noticing that the desired inequality does hold for $1 \leqslant t \leqslant 2^\rho$ and for any $\rho \geqslant 4$ by Lemma 2.5.

Suppose next that $t \in (k^\rho, (k+1)^\rho)$ for some $k \geqslant 2$ and that $\rho \in [4, 8)$. We put $\sigma = \rho/2$ and $s = \sqrt{t}$ and we notice that $k^\rho \leqslant t \leqslant (k+1)^\rho$ if and only if $k^\sigma \leqslant s \leqslant (k+1)^\sigma$. The idea is now to use that

$$\log |P_\rho(-t)| = \log |P_{2\sigma}(-s^2)| = \log |P_\sigma(-s)| + \log P_\sigma(s).$$

From Lemma 2.6,

$$\log P_\sigma(s) \leqslant k\pi/\sin(\pi/\sigma) + 1,$$

for $k \geqslant 2$ and from Lemma 2.3 (since $s \geqslant 2^\sigma$)

$$\begin{aligned}\log |P_\sigma(-s)| \leqslant{}& k\pi \cot(\pi/\sigma) + \pi(1 - 1/\sigma) \cot(\pi/\sigma) \\ &+ \log|\sin(\pi/\sigma)| + \log 2 - (\sigma/2)\log(2\pi) - (\sigma/2) \log 2.\end{aligned}$$

Since furthermore $\cot(\pi/\rho) = \cot(\pi/\sigma) + 1/\sin(\pi/\sigma)$ we get

$$\begin{aligned}\log |P_\rho(-t)| - k\pi \cot(\pi/\rho) \leqslant{}& 1 + (\pi - \pi/\sigma) \cot(\pi/\sigma) + \log|\sin(\pi/\sigma)| \\ &+ \log 2 - (\sigma/2) \log(2\pi) - (\sigma/2) \log 2.\end{aligned}$$

It is seen by standard calculus arguments that this quantity is negative for $\sigma \in [\pi/4, \pi/2]$. This proves the proposition for $\rho < 8$.

For $\rho \in [2^m, 2^{m+1}]$ for $m = 3, 4, \ldots$ we use Lemma 2.5 and relation (9) repeatedly. In this way the proposition is finally proved. $\square$

Remark 2.7. In the proof we used the relation

$$\log |P_{2\sigma}(-t^2)| = \log |P_\sigma(-t)| + \log P_\sigma(t).$$

It is easy to show that $\log P_\sigma(t) \leqslant (\pi/\sin(\pi/\sigma))t^{1/\sigma}$ (see (4)). When t is small (e.g. for $t \leqslant 2^\sigma$), it may happen that $\log P_\sigma(t) > (\pi/\sin(\pi/\sigma))n(t)$. This accounts for many of the technicalities in the situation where $\rho \geqslant 4$.

Proof of Lemma 2.5. Since $\log(1-s) \leqslant -s$ for $s < 1$ we have for $t \in (1, 2^\rho)$,

$$\log |P_\rho(-t)| \leqslant \log(t-1) + \sum_{n=2}^{\infty} \frac{-t}{n^\rho} = \log(t-1) - t(\zeta(\rho) - 1).$$

The asserted inequality will follow from the inequality

$$-t(\zeta(\rho)-1)+\log(t-1)\leqslant \pi\cot(\pi/\rho),\quad t\in(1,2^{\rho}). \tag{10}$$

It is easy to see that the left-hand side of this relation has a unique maximum point in $[1,2^{\rho}]$, namely at $t=1+1/(\zeta(\rho)-1)$ (which actually is less that 2^{ρ}). If we evaluate (10) at this point we obtain the relation.

$$-\zeta(\rho)-\log(\zeta(\rho)-1)\leqslant \pi\cot(\pi/\rho),\quad \rho>4,$$

which we set out to verify. We let

$$g(\rho)=-\zeta(\rho)-\log(\zeta(\rho)-1)-\pi\cot(\pi/\rho)$$

and find

$$g'(\rho)=-\frac{\zeta(\rho)\zeta'(\rho)}{\zeta(\rho)-1}-\left(\frac{\pi/\rho}{\sin\pi/\rho}\right)^2.$$

Here,

$$-\frac{\zeta(\rho)\zeta'(\rho)}{\zeta(\rho)-1}=\zeta(\rho)\frac{\sum_{n=2}^{\infty}(\log n)/n^{\rho}}{\sum_{n=2}^{\infty}1/n^{\rho}}\geqslant\log 2,$$

so

$$-\zeta(\rho)-\log(\zeta(\rho)-1)-\rho\log 2$$

is increasing as a function of ρ. Its limit as ρ tends to infinity is easily seen to equal -1. Therefore the function itself must be less than or equal to -1 for all $\rho>4$. To show that $g(\rho)\leqslant 0$ for all $\rho>4$ it is therefore enough to show that

$$\rho\log 2-\pi\cot\pi/\rho\leqslant 1$$

for all $\rho>4$. This is indeed the case. We get

$$\begin{aligned}\rho\log 2-\pi\cot\pi/\rho&=\rho\left(\log 2-\frac{\pi/\rho}{\sin\pi/\rho}\cos\pi/\rho\right)\\&\leqslant\rho(\log 2-\cos\pi/\rho)\\&\leqslant\rho\left(\log 2-\frac{1}{\sqrt{2}}\right)<0.\qquad\square\end{aligned}$$

Proof of Lemma 2.6. We shall first show that the function

$$g(x)\equiv\sum_{n=1}^{\infty}\log(1+((x+1)/n)^{\rho})-\frac{x\pi}{\sin(\pi/\rho)}$$

decreases in x for $x>1$ and $\rho>1$. It is easily seen that

$$g'(x)=\frac{\rho}{x+1}\sum_{n=1}^{\infty}\frac{(x+1)^{\rho}}{n^{\rho}+(x+1)^{\rho}}-\frac{\pi}{\sin(\pi/\rho)},$$

and therefore, replacing the sum by an integral, that

$$\begin{aligned} g'(x) &\leqslant \frac{\rho}{x+1}\int_0^\infty \frac{(x+1)^\rho}{t^\rho+(x+1)^\rho}\,\mathrm{d}t - \frac{\pi}{\sin(\pi/\rho)} \\ &= \rho\int_0^\infty \frac{\mathrm{d}s}{s^\rho+1} - \frac{\pi}{\sin(\pi/\rho)} = 0. \end{aligned}$$

(See e.g. [9, 3.222 (2)].) The assertions of the lemma will then follow from the inequalities

$$\begin{aligned} &\log P_\rho(3^\rho) \leqslant 2\pi/\sin(\pi/\rho)+1, \quad \rho\in[2,4], \\ &\log P_\rho(3^\rho) \leqslant 2\pi/\sin(\pi/\rho), \quad \rho\in[4,\infty). \end{aligned}$$

We begin by verifying the first inequality. We have

$$\begin{aligned} \log P_\rho(3^\rho) &= \log(1+3^\rho)+\log(1+(3/2)^\rho)+\log 2+\sum_{n=4}^\infty \log(1+(3/n)^\rho) \\ &\leqslant \log(1+3^\rho)+\log(1+(3/2)^\rho)+\log 2+\int_3^\infty \log(1+(3/t)^\rho)\mathrm{d}t. \end{aligned}$$

Here

$$\int_3^\infty \log(1+(3/t)^\rho)\mathrm{d}t \leqslant 3\int_1^\infty \log(1+s^{-2})\mathrm{d}s = 3(\pi/2-\log 2).$$

(See e.g. [9, 4.293 (2), 8.375 (1)].) Therefore the first assertion holds provided that

$$\log(1+3^\rho)+\log(1+(3/2)^\rho)+3\pi/2-2\,\log 2 \leqslant 2\pi/\sin(\pi/\rho)+1, \quad \rho\in[2,4].$$

Now this inequality can be verified using convexity of the logarithmic terms, e.g.

$$\log(1+3^\rho) \leqslant (\log(1+3^4)-\log(1+3^2))(\rho-2)/2+\log(1+3^2).$$

We shall not give the details.

The second inequality follows in the same way, using that

$$\int_1^\infty \log(1+s^{-4})\mathrm{d}s = \frac{\pi}{\sqrt{2}} - \log 2 + \frac{1}{\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right).$$

(See the same references as above.) Here one should verify that

$$\log(1+3^\rho)+\log\left(1+\left(\frac{3}{2}\right)^\rho\right) - 2\,\log 2 + \frac{3\pi}{\sqrt{2}} + \frac{3}{\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) \leqslant \frac{2\pi}{\sin(\pi/\rho)},$$

for $\rho > 4$ and this can be done by showing that the derivative is negative. This completes the sketch of the proof. □

3. Zero order

We shall investigate functions related to the function Q defined by

$$Q(z)=\prod_{n=0}^{\infty}(1+zq^n),\quad q\in(0,1).$$

Proof of Theorem 1.5. We have

$$\frac{\log Q(z)-\log Q(-q)}{\operatorname{Log}\left(\frac{1+z}{1-q}\right)}=\sum_{n=0}^{\infty}\frac{\operatorname{Log}\left(\frac{1+zq^n}{1-qq^n}\right)}{\operatorname{Log}\left(\frac{1+z}{1-q}\right)},$$

and here it is easy to see that

$$\frac{\operatorname{Log}\left(\frac{1+za}{1-qa}\right)}{\operatorname{Log}\left(\frac{1+z}{1-q}\right)}$$

is a Pick function for $a\in(0,1)$. Thus the infinite sum is also a Pick function. □

Concerning the growth of Q we notice.

Lemma 3.1. *There exists a sequence $\{r_n\}$ tending to infinity and a constant such that we have*

$$|\log Q(z)|\leqslant \text{Const}\,(\log|z|)^2,\quad |z|=r_n,$$

for all n.

We shall not give the proof of this lemma, since the method is the same as we have used before: we obtain an upper bound on Q on the positive real line and use it as an upper bound on Q in the entire plane. A lower bound comes from the result of Littlewood mentioned above. Estimation of arg $Q(z)$ is also straight forward.

Proposition 3.2 below expresses positivity of the boundary values of the imaginary part of the function

$$\psi(z)=\frac{\log Q(qz)-\log Q(-q)+\frac{\log(1-q)}{\log q}\operatorname{Log}(-z)}{\operatorname{Log}\left(\frac{1+z}{1-q}\right)\operatorname{Log}(-z)}$$

in the upper half plane. Theorem 1.6 follows now by the same general arguments as we have used before. So the function ψ is a Pick function. It is not difficult to find the integral representation of ψ and we shall not write down the derivation. We get

$$\psi(z)=\Re\psi(i)+\int_{-\infty}^{-1}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)h_-(t)\mathrm{d}t+\int_{0}^{\infty}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)h_+(t)\mathrm{d}t,$$

where n is the zero counting function associated with Q,

$$h_-(x) = \frac{n(-qx)\log\left|\frac{1+x}{1-q}\right| - \left(\log|Q(qx)| - \log Q(-q) + \frac{\log(1-q)}{\log q}\log(-x)\right)}{\left(\log\left|\frac{1+x}{1-q}\right|\right)^2 + \pi^2}, \tag{11}$$

for $x < -1$ and

$$h_+(x) = \frac{\log Q(qx) - \log Q(-q)}{\log\left(\frac{1+x}{1-q}\right)((\log x)^2 + \pi^2)}, \tag{12}$$

for $x > 0$. In the integral representation of ψ, the positive measure is now supported by set $(-\infty, -1] \cup [0, \infty)$, and not on any half line. It is however still possible to differentiate under the integral sign when $z \in (-1, 0)$. In this way we obtain that $\psi'(t) > 0$ for $t \in (-1, 0)$. We therefore conclude that

$$x \mapsto \psi(-q^x), \quad x \in (0, \infty)$$

is increasing. A simple substitution finally shows that

$$\psi(-q^x) = \frac{\log \Gamma_q(x+1)}{x \log\left(\frac{1-q^x}{1-q}\right)},$$

and this proves Corollary 1.4.

Proposition 3.2. *We have*

$$\log|Q(qx)| - \log|Q(-q)| + \frac{\log(1-q)}{\log q}\log(-x) \leqslant n(-qx)\log\left|\frac{1+x}{1-q}\right|, \quad x < -1,$$

and

$$\log Q(qx) \geqslant \log Q(-q), \quad x > 0.$$

Proof. The second assertion of the proposition is easily verified so we go directly to the first. We split the line $x < -1$ into the intervals $[-q^{-k-1}, -q^{-k})$, $k \geqslant 0$, and use the relation $Q(qz) = (1+z)Q(z)$ in an inductive argument. We shall begin by considering the case $k = 0$ and this amounts to showing that

$$\log Q(qx) - \log Q(-q) + \frac{\log(1-q)}{\log q}\log(-x) \leqslant 0, \quad x \in (-1/q, -1).$$

Here $\log Q(qx) - \log Q(-q) < 0$ but the remaining term is in fact positive so we need to make a more detailed analysis. When $x = -1$ the relation holds and we shall differentiate the expression on the left-hand side and show that it is positive. We get (for $x < -1$)

$$\begin{aligned}\left(\log Q(qx) - \log Q(-q) + \frac{\log(1-q)}{\log q}\log(-x)\right)' &= \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{q^{n+1}x}{1+q^{n+1}x} + \frac{\log(1-q)}{\log q}\right)\\ &> \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{-q^{n+1}}{1-q^{n+1}} + \frac{\log(1-q)}{\log q}\right).\end{aligned}$$

Here we claim that

$$\sum_{n=0}^{\infty} \frac{q^{n+1}}{1-q^{n+1}} - \frac{\log(1-q)}{\log q} > 0.$$

Indeed, we have $-\log(1-q) = \sum_{n=0}^{\infty} q^{n+1}/(n+1)$ so that

$$\sum_{n=0}^{\infty} \frac{q^{n+1}}{1-q^{n+1}} - \frac{\log(1-q)}{\log q} = \sum_{n=0}^{\infty} q^{n+1}\left(\frac{1}{1-q^{n+1}} + \frac{1}{(n+1)\log q}\right).$$

Now, each summand in this sum is positive due to the fact that $\log t < t-1$ for $t \in (0,1)$ (put $t = q^{n+1}$). This completes the basis for our inductive argument. We shall not write that down since it is straight forward. □

Appendix A. Appendix. Hardy's investigations

In [8] Hardy found precise asymptotic relations involving the function P_ρ, when $1<\rho<2$. In this appendix we shall extend these relations to also cover $2 \leqslant \rho < 4$. We adapt the ideas of Hardy to our situation. We shall show that for $a>0$ and $\rho \in (2,4)$,

$$\begin{aligned}\log|P_\rho(-a)| = {}& \pi\cot(\pi/\rho)a^{1/\rho} + \log|\sin \pi a^{1/\rho}| - (\log a)/2 \\ & - \log\pi + (1-\rho/2)\log(2\pi) - (\rho/\pi)\sin(\pi\rho/2)aI(a,\rho),\end{aligned}$$

where

$$I(a,\rho) = \int_0^\infty (\log(2\sinh(\pi\eta)) - \pi\eta)\frac{\eta^{\rho-1}}{\eta^{2\rho}+a^2-2a\eta^\rho\cos(\pi\rho/2)}\,\mathrm{d}\eta.$$

We denote by $\Omega_{\varepsilon,R}$ (for $R,\varepsilon>0$) the region

$$\Omega_{\varepsilon,R} = \{u \in \mathbb{C} \mid \varepsilon < |u| < R, \Re u > 0\}.$$

For $z = r\mathrm{e}^{\mathrm{i}\phi}$ and $\phi \in (-\delta,\delta)$ the function

$$u \mapsto \frac{1}{u^\rho - z}$$

is meromorphic in $\Omega_{\varepsilon,R}$ with a single and simple pole at $u = r^{1/\rho}\mathrm{e}^{\mathrm{i}\phi/\rho}$. Indeed if $u = |u|\mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in (-\pi/2, \pi/2)$, the solutions to $u^\rho = z$ are given by $u = r^{1/\rho}\mathrm{e}^{\mathrm{i}(\phi/\rho+2\pi k/\rho)}$, $k \in \mathbb{Z}$. This forces $k=0$ if we assume that δ is chosen so that $2\pi/\rho - \delta/\rho > \pi/2$, which we henceforth do.

The residue at this pole is

$$\operatorname{Res}\left(\frac{1}{u^\rho - z}, z^{1/\rho}\right) = \frac{z^{1/\rho-1}}{\rho}$$

and the Residue theorem now yields (when $z^{1/\rho}$ is not a pole of $\cot(\pi u)$)

$$\frac{1}{2\pi i}\int_{\partial\Omega_{\varepsilon,R}} \frac{\pi\cot(\pi u)}{u^\rho - z}\,\mathrm{d}u = \frac{\pi}{\rho}\cot(\pi z^{1/\rho})z^{1/\rho-1} + \sum_{\varepsilon<n\pi<R}\frac{1}{n^\rho - z}.$$

We let ∂C_r denote the right half of the circle $|z| = r$ traversed in the usual counter clockwise direction. We use that

$$\left|\int_{\partial C_R} \frac{\pi\,\cot(\pi u)}{u^\rho - z}\,\mathrm{d}u\right| \to 0 \quad \text{as } R \to \infty$$

(at least through a sequence $\{R_n\}$) and

$$\left|\int_{\partial C_\varepsilon} \frac{\pi\,\cot(\pi u)}{u^\rho - z}\,\mathrm{d}u\right| \to -\frac{\pi i}{z} \quad \text{as } \varepsilon \to 0.$$

The integral along the vertical parts of $\partial\Omega_{\varepsilon,R}$ is seen (after some computation) to equal

$$2\pi i\,\sin(\pi\rho/2)\int_\varepsilon^R \frac{\eta^\rho\,\coth\,\pi\eta}{\eta^{2\rho}+z^2-2\eta^\rho z\,\cos(\pi\rho/2)}\,\mathrm{d}\eta.$$

Therefore, letting R tend to infinity and ε tend to zero, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^\rho - z} = &-\frac{\pi}{\rho}\cot(\pi z^{1/\rho})z^{1/\rho-1}+\frac{1}{2z}\\ &+\sin(\pi\rho/2)\int_0^\infty \frac{\eta^\rho\,\coth\,\pi\eta}{\eta^{2\rho}+z^2-2\eta^\rho z\,\cos(\pi\rho/2)}\,\mathrm{d}\eta.\end{aligned} \tag{A.1}$$

We shall consider the integral in this relation more closely. We write it as a sum $A_1(z)+A_2(z)$, where

$$A_1(z)=\frac{1}{\pi}\int_0^\infty \frac{\eta^{\rho-1}}{\eta^{2\rho}+z^2-2\eta^\rho z\,\cos(\pi\rho/2)}\,\mathrm{d}\eta=\frac{1/\rho-1/2}{z\,\sin(\pi\rho/2)},$$

(this identity can be verified by making the substitution $t=\eta^\rho$ followed by [9, 3.252(12)]) and

$$A_2(z)=\int_0^\infty\left(\coth\,\pi\eta-\frac{1}{\pi\eta}\right)\frac{\eta^\rho}{\eta^{2\rho}+z^2-2\eta^\rho z\,\cos(\pi\rho/2)}\,\mathrm{d}\eta.$$

The expression on the left-hand side of (A.1) we recognize as $(\log P_\rho)'(-z)$ and we integrate both sides of the relation to obtain information about $\log P_\rho(-z)$. We integrate along a straight line from a small positive ε to $a \in \{|\mathrm{Arg}\,w| < \delta\}\backslash\mathbb{R}$. (At this moment we wish to avoid bringing in principal values of integrals and therefore we take a to be nonreal. Later we shall allow a to be real again.) We find

$$\begin{aligned}-(\log P_\rho(-a)-\log P_\rho(-\varepsilon)) &= \int_\varepsilon^a\left(-\frac{\pi}{\rho}\cot(\pi z^{1/\rho})z^{1/\rho-1}+\frac{1}{\rho z}\right)\mathrm{d}z\\ &\quad+\sin(\pi\rho/2)\int_\varepsilon^a A_2(z)\mathrm{d}z\\ &= -\log\sin(\pi a^{1/\rho})+\frac{1}{\rho}\,\mathrm{Log}\,a+\log\sin(\pi\varepsilon^{1/\rho})\\ &\quad-\frac{1}{\rho}\,\mathrm{Log}\,\varepsilon+\sin(\pi\rho/2)\int_\varepsilon^a A_2(z)\mathrm{d}z.\end{aligned}$$

Now,

$$\int_\varepsilon^a A_2(z)\mathrm{d}z=\int_\varepsilon^a\int_0^\infty\left(\coth\,\pi\eta-\frac{1}{\pi\eta}\right)\frac{\eta^\rho}{\eta^{2\rho}+z^2-2\eta^\rho z\,\cos(\pi\rho/2)}\,\mathrm{d}\eta\,\mathrm{d}z$$
$$=\int_0^\infty\int_\varepsilon^a\frac{\eta^\rho\mathrm{d}z}{\eta^{2\rho}+z^2-2\eta^\rho z\,\cos(\pi\rho/2)}\left(\coth\,\pi\eta-\frac{1}{\pi\eta}\right)\mathrm{d}\eta.$$

A computation shows that

$$\int_\varepsilon^a\frac{\sin(\pi\rho/2)\eta^\rho}{\eta^{2\rho}+z^2-2\eta^\rho z\,\cos(\pi\rho/2)}\,\mathrm{d}z=\left[\arctan\left(\frac{z-\eta^\rho\,\cos(\pi\rho/2)}{\sin(\pi\rho/2)\eta^\rho}\right)\right]_\varepsilon^a,$$

where arctan as usual denotes the branch that is zero at the origin. Therefore,

$$-\sin(\pi\rho/2)\int_\varepsilon^a A_2(z)\mathrm{d}z$$
$$=-\int_0^\infty\left[\arctan\left(\frac{z-\eta^\rho\,\cos(\pi\rho/2)}{\sin(\pi\rho/2)\eta^\rho}\right)\right]_\varepsilon^a\left(\coth\,\pi\eta-\frac{1}{\pi\eta}\right)\mathrm{d}\eta$$
$$=-\int_0^\infty\left(\arctan\left(\frac{a-\eta^\rho\,\cos(\pi\rho/2)}{\sin(\pi\rho/2)\eta^\rho}\right)-L_\rho\right)\left(\coth\,\pi\eta-\frac{1}{\pi\eta}\right)\mathrm{d}\eta$$
$$+\int_0^\infty\left(\arctan\left(\frac{\varepsilon-\eta^\rho\,\cos(\pi\rho/2)}{\sin(\pi\rho/2)\eta^\rho}\right)-L_\rho\right)\left(\coth\,\pi\eta-\frac{1}{\pi\eta}\right)\mathrm{d}\eta,$$

where

$$L_\rho=\lim_{\eta\to\infty}\arctan\left(\frac{a-\eta^\rho\,\cos(\pi\rho/2)}{\sin(\pi\rho/2)\eta^\rho}\right)=\arctan(-\cot(\pi\rho/2)).$$

We get by integration by parts,

$$\int_0^\infty\left(\arctan\left(\frac{a-\eta^\rho\,\cos(\pi\rho/2)}{\sin(\pi\rho/2)\eta^\rho}\right)-L_\rho\right)\left(\coth\,\pi\eta-\frac{1}{\pi\eta}\right)\mathrm{d}\eta$$
$$=\left[\frac{1}{\pi}\log\left(\frac{\sinh\,\pi\eta}{\pi\eta}\right)\left(\arctan\left(\frac{a-\eta^\rho\,\cos(\pi\rho/2)}{\sin(\pi\rho/2)\eta^\rho}\right)-L_\rho\right)\right]_0^\infty$$
$$-\frac{1}{\pi}\int_0^\infty\log\left(\frac{\sinh\,\pi\eta}{\pi\eta}\right)\frac{a\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+a^2-2a\eta^\rho\,\cos(\pi\rho/2)}\,\mathrm{d}\eta$$
$$=-\frac{1}{\pi}\int_0^\infty\log\left(\frac{\sinh\,\pi\eta}{\pi\eta}\right)\frac{a\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+a^2-2a\eta^\rho\,\cos(\pi\rho/2)}\,\mathrm{d}\eta.$$

We therefore get

$$\log P_\rho(-a)=\log\sin(\pi a^{1/\rho})-\frac{1}{\rho}\,\mathrm{Log}\,a-\log\sin(\pi\varepsilon^{1/\rho})+\frac{1}{\rho}\,\mathrm{Log}\,\varepsilon+\log P_\rho(-\varepsilon)$$
$$-\frac{1}{\pi}\int_0^\infty\log\left(\frac{\sinh\,\pi\eta}{\pi\eta}\right)\frac{a\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+a^2-2a\eta^\rho\,\cos(\pi\rho/2)}\,\mathrm{d}\eta$$
$$+\frac{1}{\pi}\int_0^\infty\log\left(\frac{\sinh\,\pi\eta}{\pi\eta}\right)\frac{\varepsilon\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+\varepsilon^2-2\varepsilon\eta^\rho\,\cos(\pi\rho/2)}\,\mathrm{d}\eta.$$

In this relation we let ε tend to zero. Clearly,

$$-\operatorname{logsin}(\pi\varepsilon^{1/\rho}) + \frac{1}{\rho}\operatorname{Log}\varepsilon + \log P_\rho(-\varepsilon) \to -\log\pi,$$

as $\varepsilon \to 0$. Furthermore

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \log\left(\frac{\sinh \pi\eta}{\pi\eta}\right) \frac{\varepsilon\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+\varepsilon^2-2\varepsilon\eta^\rho\cos(\pi\rho/2)}\,\mathrm{d}\eta \\
&\quad = \frac{1}{\pi}\int_0^\infty \log\left(\frac{\sinh \pi t^{1/\rho}}{\pi t^{1/\rho}}\right) \frac{\varepsilon\sin(\pi\rho/2)}{(t-\varepsilon\cos(\pi\rho/2))^2+(\varepsilon\sin(\pi\rho/2))^2}\,\mathrm{d}t \\
&\quad = \frac{1}{\pi}\int_{-\infty}^\infty G(t)\,\frac{Y}{(t-X)^2+Y^2}\,\mathrm{d}t,
\end{aligned}$$

where $X = \varepsilon\cos(\pi\rho/2)$, $Y = \varepsilon\sin(\pi\rho/2)$ and

$$G(t) = \log\left(\frac{\sinh \pi t^{1/\rho}}{\pi t^{1/\rho}}\right)$$

for $t > 0$ and $G(t) = 0$ for $t \leqslant 0$. Since G is continuous and $G(t)/(t^2+1)$ is integrable on the real line we conclude that

$$\frac{1}{\pi}\int_0^\infty \log\left(\frac{\sinh \pi\eta}{\pi\eta}\right) \frac{\varepsilon\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+\varepsilon^2-2\varepsilon\eta^\rho\cos(\pi\rho/2)}\,\mathrm{d}\eta \to 0,$$

as $\varepsilon \to 0$. This yields

$$\begin{aligned}
\log P_\rho(-a) = &\operatorname{logsin}(\pi a^{1/\rho}) - \frac{1}{\rho}\operatorname{Log}a - \log\pi \\
&- \frac{1}{\pi}\int_0^\infty \log\left(\frac{\sinh \pi\eta}{\pi\eta}\right) \frac{a\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+a^2-2a\eta^\rho\cos(\pi\rho/2)}\,\mathrm{d}\eta.
\end{aligned}$$

We write

$$\log\left(\frac{\sinh \pi\eta}{\pi\eta}\right) = (\log(2\sinh \pi\eta) - \pi\eta) + (\pi\eta - \log(2\pi\eta))$$

and use the facts (these follow by some computation from the relation [9, 3.252 (12)])

$$\begin{aligned}
&\int_0^\infty \frac{\eta^\rho}{\eta^{2\rho}+a^2-2\eta^\rho a\cos(\pi\rho/2)}\,\mathrm{d}\eta = -\frac{\pi}{\rho}\,\frac{a^{1/\rho-1}\cot \pi/\rho}{\sin(\pi\rho/2)}, \\
&\int_0^\infty \frac{\eta^{\rho-1}\log\eta}{\eta^{2\rho}+a^2-2\eta^\rho a\cos(\pi\rho/2)}\,\mathrm{d}\eta = \frac{\pi\operatorname{Log}a}{\rho^2 a}\,\frac{1-\rho/2}{\sin(\pi\rho/2)},
\end{aligned}$$

to obtain

$$
\begin{aligned}
\log P_\rho(-a) = & \log\sin(\pi a^{1/\rho}) - \frac{1}{\rho}\operatorname{Log} a - \log\pi - \frac{\rho}{\pi}\sin(\pi\rho/2) a I(a,\rho) \\
& - \frac{1}{\pi}\int_0^\infty (\pi\eta - \log(2\pi\eta)) \frac{a\rho\eta^{\rho-1}\sin(\pi\rho/2)}{\eta^{2\rho}+a^2-2a\eta^\rho\cos(\pi\rho/2)}\,d\eta \\
= & \log\sin(\pi a^{1/\rho}) - \frac{1}{2}\operatorname{Log} a - \log\pi + \pi a^{1/\rho}\cot(\pi/\rho) \\
& + (1-\rho/2)\log(2\pi) - \frac{\rho}{\pi}\sin(\pi\rho/2) a I(a,\rho).
\end{aligned}
$$

Now extend this relation to $a > 0$ and take its real part. That gives the desired relation for $a > 0$.

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