Available online at www.sciencedirect.com
ScienceDirect
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
ELSEVIER
Journal of Computational and Applied Mathematics 217 (2008) 252–258
www.elsevier.com/locate/cam

Convergence of SSOR multisplitting method for an H-matrix

Jae Heon Yun*

Department of Mathematics, Institute for Basic Sciences & College of Natural Sciences, Chungbuk National University, Cheongju, Chungbuk 361-763, Korea

Received 1 December 2006; received in revised form 22 May 2007

Abstract

In this paper, we study the convergence of both the multisplitting method and the relaxed multisplitting method associated with SSOR multisplitting for solving a linear system whose coefficient matrix is an H-matrix. We also introduce an application of the SSOR multisplitting method.

MSC: 65F10; 65F15

Keywords: SSOR multisplitting; Multisplitting method; Relaxed multisplitting method; H-matrix; M-matrix

1. Introduction

In this paper, we consider both the multisplitting method and the relaxed multisplitting method for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse H-matrix. Multisplitting method was introduced by [6] and was further studied by many authors [3–5,7,9]. The multisplitting method can be thought of as an extension and parallel generalization of the classical block Jacobi method [2].

A matrix $A=(a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if $a_{ij} \leqslant 0$ for $i \neq j$ and $A^{-1} \geqslant 0$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geqslant 0$ and $N \geqslant 0$, and *weak regular* if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$.

* Tel.: +82 43 272 9658; fax: +82 43 274 9619.
E-mail address: gmjae@chungbuk.ac.kr.

0377-0427/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2007.06.030

A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's, called weighting matrices, are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$. The *multisplitting method* associated with this multisplitting for solving the linear system (1) is as follows.

Algorithm 1. Multisplitting method

Given an initial vector x_0
For $i = 1, 2, \ldots$, until convergence
    For $k = 1$ to ℓ
        $M_k y_k = N_k x_{i-1} + b$,
    $x_i = \sum_{k=1}^{\ell} E_k y_k$.

The *relaxed multisplitting method* with a positive relaxation parameter β associated with a multisplitting of A, (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, for solving the linear system (1) is as follows.

Algorithm 2. Relaxed multisplitting method

Given an initial vector x_0
For $i = 1, 2, \ldots$, until convergence
    For $k = 1$ to ℓ
        $M_k y_k = N_k x_{i-1} + b$,
    $x_i = \beta \sum_{k=1}^{\ell} E_k y_k + (1 - \beta) x_{i-1}$.

Notice that the loop k in Algorithms 1 and 2 can be executed in parallel by different processors.

In 1991, Deren [3] studied the convergence of both the multisplitting method and the relaxed multisplitting method associated with AOR multisplitting for solving the linear system (1). In this paper, we study the convergence of both the multisplitting method and the relaxed multisplitting method associated with SSOR multisplitting for solving the linear system (1). This paper is organized as follows. In Section 2, we present some notation and well-known results. In Section 3, we provide convergence results of both the multisplitting method and the relaxed multisplitting method associated with SSOR multisplitting. In Section 4, we introduce an application of the SSOR multisplitting method.

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geqslant 0$ $(x > 0)$ denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geqslant y$ $(x > y)$ means that $x - y \geqslant 0$ $(x - y > 0)$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices. It follows that $|A| \geqslant 0$ for any matrix A and $|AB| \leqslant |A||B|$ for any two matrices A and B of compatible size. Let $\mathrm{diag}(A)$ denote a diagonal matrix whose diagonal part coincides with the diagonal part of A, and let $\rho(A)$ denote the *spectral radius* of a square matrix A. Varga [8] showed that for any square matrices A and B, $|A| \leqslant B$ implies $\rho(A) \leqslant \rho(B)$. It is well known that if $A \geqslant 0$ and there exists a vector $x > 0$ such that $Ax < \alpha x$, then $\rho(A) < \alpha$ (see [1,8]).

The SSOR multisplitting to be used in this paper is defined as follows.

Definition 2.1. Let $0 < \omega < 2$ and $A = D - L_k - U_k$ for $k = 1, 2, \ldots, \ell$, where $D = \mathrm{diag}(A)$, L_k's are strictly lower triangular matrices, and U_k's are general matrices. $(M_k(\omega), N_k(\omega), E_k)$, $k=1, 2, \ldots, \ell$, is called the *SSOR multisplitting* of A if $(M_k(\omega), N_k(\omega), E_k)$, $k=1, 2, \ldots, \ell$, is a multisplitting of A, $M_k(\omega) = 1/(\omega(2-\omega))(D - \omega L_k) D^{-1} (D - \omega U_k)$, and $N_k(\omega) = 1/(\omega(2 - \omega))\,((1 - \omega)D + \omega L_k)\, D^{-1}((1 - \omega)D + \omega U_k)$.

3. Convergence results

We first consider convergence of the multisplitting method (Algorithm 1) associated with SSOR multisplitting for solving the linear system (1). Algorithm 1 can be written as $x_i = Hx_{i-1} + Pb$, $i = 1, 2, \ldots$, where

$$H = \sum_{k=1}^{\ell} E_k M_k^{-1} N_k \quad \text{and} \quad P = \sum_{k=1}^{\ell} E_k M_k^{-1}.$$

The H is called an iteration matrix for Algorithm 1, and it is well known that Algorithm 1 converges to the exact solution of $Ax=b$ for any initial vector x_0 if and only if $\rho(H)<1$. O'Leary and White [6] showed that $\rho(H)<1$ when $A^{-1}\geqslant 0$ and the splittings $A=M_k-N_k$ are weak regular.

Theorem 3.1. *Let $A\in\mathbb{R}^{n\times n}$ be an H-matrix. Let $A=D-B=D-L_k-U_k$ $(1\leqslant k\leqslant \ell)$, where $D=\operatorname{diag}(A)$, L_k is a strictly lower triangular matrix, and U_k is a general matrix, and let $(M_k(\omega), N_k(\omega), E_k)$, $k=1,2,\ldots,\ell$, be the SSOR multisplitting of A. Assume that $\langle A\rangle=|D|-|L_k|-|U_k|$ for $k=1,2,\ldots,\ell$. Then, the multisplitting method associated with the SSOR multisplitting converges to the exact solution of $Ax=b$ for any initial vector x_0 if $0<\omega<2/(1+\alpha)$, where $\alpha=\rho(|D|^{-1}|B|)$.*

Proof. Let $H_\omega=\sum_{k=1}^{\ell}E_kM_k(\omega)^{-1}N_k(\omega)$. Then, it suffices to show that $\rho(H_\omega)<1$ for $0<\omega<2/(1+\alpha)$. Clearly, $D-\omega L_k$ is an H-matrix for $\omega>0$. Let $C=D-\omega U_k$. Then $\langle C\rangle=|D|-\omega|U_k|$, which is a regular splitting of $\langle C\rangle$. Since $\alpha<1$ and $\omega<2/(1+\alpha)$, $\omega<1/\alpha$. It follows that $\rho(\omega|D|^{-1}|U_k|)\leqslant\rho(\omega|D|^{-1}|B|)=\omega\alpha<1$ and thus $\langle C\rangle^{-1}\geqslant 0$. That is, $D-\omega U_k$ is an H-matrix for $0<\omega<2/(1+\alpha)$. Let

$$\tilde{M}_k(\omega)=(|D|-\omega|L_k|)|D|^{-1}(|D|-\omega|U_k|),$$

$$\tilde{N}_k^1(\omega)=((1-\omega)|D|+\omega|L_k|)|D|^{-1}((1-\omega)|D|+\omega|U_k|),$$

$$\tilde{N}_k^2(\omega)=((\omega-1)|D|+\omega|L_k|)|D|^{-1}((\omega-1)|D|+\omega|U_k|).$$

Since $D-\omega L_k$ and $D-\omega U_k$ are H-matrices, one obtains

$$\begin{aligned}|((D-\omega L_k)D^{-1}(D-\omega U_k))^{-1}|&\leqslant|(D-\omega U_k)^{-1}|\,|D|\,|(D-\omega L_k)^{-1}|\\&\leqslant\langle D-\omega U_k\rangle^{-1}|D|\,\langle D-\omega L_k\rangle^{-1}\\&=(|D|-\omega|U_k|)^{-1}\,|D|\,(|D|-\omega|L_k|)^{-1}\\&=\tilde{M}_k(\omega)^{-1}.\end{aligned}\tag{2}$$

First, we consider the case of $0<\omega\leqslant 1$. Using (2),

$$|H_\omega|\leqslant\sum_{k=1}^{\ell}E_k|M_k(\omega)^{-1}N_k(\omega)|\leqslant\sum_{k=1}^{\ell}E_k\tilde{M}_k(\omega)^{-1}\tilde{N}_k^1(\omega).\tag{3}$$

Note that $\tilde{M}_k(\omega)-\tilde{N}_k^1(\omega)=\omega(2-\omega)\langle A\rangle$ for every k. Since $\omega(2-\omega)\langle A\rangle=\tilde{M}_k(\omega)-\tilde{N}_k^1(\omega)$ is a regular splitting of $\omega(2-\omega)\langle A\rangle$ for each k and $\langle A\rangle^{-1}\geqslant 0$, $\rho(\sum_{k=1}^{\ell}E_k\tilde{M}_k(\omega)^{-1}\tilde{N}_k^1(\omega))<1$. From (3), $\rho(H_\omega)<1$. Next, we consider the case of $1<\omega<2/(1+\alpha)$. Using (2),

$$|H_\omega|\leqslant\sum_{k=1}^{\ell}E_k|M_k(\omega)^{-1}N_k(\omega)|\leqslant\sum_{k=1}^{\ell}E_k\tilde{M}_k(\omega)^{-1}\tilde{N}_k^2(\omega).\tag{4}$$

Let $\tilde{A}=\tilde{M}_k(\omega)-\tilde{N}_k^2(\omega)$. Then $\tilde{A}=\omega(2-\omega)|D|-\omega^2|B|$, which is a regular splitting of $\tilde{A}$. Since $\omega<2/(1+\alpha)$, $\rho(\omega/(2-\omega)|D|^{-1}|B|)=\omega/(2-\omega)\alpha<1$. It follows that $\tilde{A}^{-1}\geqslant 0$. Since $\tilde{A}=\tilde{M}_k(\omega)-\tilde{N}_k^2(\omega)$ is a regular splitting of $\tilde{A}$ for each k, $\rho(\sum_{k=1}^{\ell}E_k\tilde{M}_k(\omega)^{-1}\tilde{N}_k^2(\omega))<1$. From (4), $\rho(H_\omega)<1$. $\square$

Corollary 3.2. *Let $A\in\mathbb{R}^{n\times n}$ be an M-matrix. Let $A=D-B=D-L_k-U_k$ $(1\leqslant k\leqslant\ell)$, where $D=\operatorname{diag}(A)$, $L_k\geqslant 0$ is a strictly lower triangular matrix, and $U_k\geqslant 0$ is a general matrix, and let $(M_k(\omega), N_k(\omega), E_k)$, $k=1,2,\ldots,\ell$, be the SSOR multisplitting of A. Then, the multisplitting method associated with the SSOR multisplitting converges to the exact solution of $Ax=b$ for any initial vector x_0 if $0<\omega<2/(1+\alpha)$, where $\alpha=\rho(D^{-1}B)$.*

Proof. Since A is an M-matrix, A is an H-matrix and $\langle A\rangle=A=D-L_k-U_k=|D|-|L_k|-|U_k|$ for $k=1,2,\ldots,\ell$. By Theorem 3.1, the corollary follows.

We next consider convergence of the relaxed multisplitting method (Algorithm 2) associated with SSOR multisplitting for solving the linear system (1). Algorithm 2 can be written as $x_i = H_\beta x_{i-1} + P_\beta b$, $i = 1, 2, \ldots,$ where

$$H_\beta = \beta \sum_{k=1}^{\ell} E_k M_k^{-1} N_k + (1-\beta)I \quad \text{and} \quad P_\beta = \beta \sum_{k=1}^{\ell} E_k M_k^{-1}.$$

The H_β is called an iteration matrix for Algorithm 2, and it is well known that Algorithm 2 converges to the exact solution of $Ax = b$ for any initial vector x_0 if and only if $\rho(H_\beta) < 1$. □

Theorem 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix. Let $A = D - B = D - L_k - U_k$ $(1 \leqslant k \leqslant \ell)$, where $D = \operatorname{diag}(A)$, L_k is a strictly lower triangular matrix, and U_k is a general matrix, and let $(M_k(\omega), N_k(\omega), E_k)$, $k = 1, 2, \ldots, \ell$, be the SSOR multisplitting of A. Assume that $\langle A\rangle = |D| - |L_k| - |U_k|$ for $k = 1, 2, \ldots, \ell$. Let $H_{\omega,\beta} = \beta\sum_{k=1}^{\ell} E_k M_k(\omega)^{-1} N_k(\omega) + (1-\beta)I$ be an iteration matrix of the relaxed multisplitting method associated with the SSOR multisplitting, $H(\omega) = |1-\omega|I + \omega|D|^{-1}|B|$ and $\alpha = \rho(|D|^{-1}|B|)$. Then the following hold*:

(a) *if $0 < \omega \leqslant 1$ and $0 < \beta < \frac{2}{1+\rho(H(\omega))}$, then $\rho(H_{\omega,\beta}) < 1$,*
(b) *if $1 < \omega < \frac{2}{1+\alpha}$ and $0 < \beta \leqslant 1$, then $\rho(H_{\omega,\beta}) < 1$,*
(c) *if $1 < \omega < \sqrt{\frac{2}{1+\alpha}}$ and $0 < \beta < \frac{2}{\omega(1+\rho(H(\omega)))}$, then $\rho(H_{\omega,\beta}) < 1$.*

Proof. Notice that $|D| - \omega|L_k|$ and $|D| - \omega|U_k|$ are M-matrices for $0 < \omega < 2/(1+\alpha)$ since $D - \omega L_k$ and $D - \omega U_k$ are H-matrices for $0 < \omega < 2/(1+\alpha)$. Let

$$\tilde{M}_k(\omega) = (|D| - \omega|L_k|)|D|^{-1}(|D| - \omega|U_k|),$$
$$\tilde{N}_k^1(\omega) = ((1-\omega)|D| + \omega|L_k|)|D|^{-1}((1-\omega)|D| + \omega|U_k|),$$
$$\tilde{N}_k^2(\omega) = ((\omega-1)|D| + \omega|L_k|)|D|^{-1}((\omega-1)|D| + \omega|U_k|).$$

Since $\rho(\omega|D|^{-1}|L_k|) < 1$ and $\rho(\omega|D|^{-1}|U_k|) < 1$ for $0 < \omega < 2/(1+\alpha)$,

$$\begin{aligned}\tilde{M}_k(\omega)^{-1}|D| &= (|D| - \omega|U_k|)^{-1}|D|(|D| - \omega|L_k|)^{-1}|D| \\ &= (I - \omega|D|^{-1}|U_k|)^{-1}(I - \omega|D|^{-1}|L_k|)^{-1} \geqslant I \end{aligned} \tag{5}$$

for $0 < \omega < 2/(1+\alpha)$. We first prove the part (a). Since $\tilde{M}_k(\omega) - \tilde{N}_k^1(\omega) = \omega(2-\omega)\langle A\rangle$ for every k, one obtains

$$\begin{aligned}|H_{\omega,\beta}| &\leqslant \beta\sum_{k=1}^{\ell} E_k\tilde{M}_k(\omega)^{-1}\tilde{N}_k^1(\omega) + |1-\beta|I \\ &= \beta\sum_{k=1}^{\ell} E_k\tilde{M}_k(\omega)^{-1}(\tilde{M}_k(\omega) - \omega(2-\omega)\langle A\rangle) + |1-\beta|I \\ &= (\beta + |1-\beta|)I - \omega(2-\omega)\beta\sum_{k=1}^{\ell} E_k\tilde{M}_k(\omega)^{-1}\langle A\rangle \\ &= (\beta + |1-\beta|)I - \omega(2-\omega)\beta\sum_{k=1}^{\ell} E_k\tilde{M}_k(\omega)^{-1}|D|(I - |D|^{-1}|B|) \\ &\leqslant (\beta + |1-\beta|)I - \omega(2-\omega)\beta\sum_{k=1}^{\ell} E_k\tilde{M}_k(\omega)^{-1}|D|(I - |D|^{-1}|B| - \varepsilon ee^{\mathrm{T}}),\end{aligned} \tag{6}$$

where $\varepsilon > 0$ and $e = (1, 1, \ldots, 1)^{\mathrm{T}}$. By the Perron–Frobenius theorem, for any $\varepsilon > 0$ there exists a $x_\varepsilon > 0$ such that $(|D|^{-1}|B| + \varepsilon ee^{\mathrm{T}})x_\varepsilon = \alpha_\varepsilon x_\varepsilon$, where $\alpha_\varepsilon = \rho(|D|^{-1}|B| + \varepsilon ee^{\mathrm{T}})$. Since $\alpha < 1$ and $0 < \omega \leqslant 1$, $\rho(H(\omega)) = 1 - \omega + \omega\alpha < 1$.

By continuity of the spectral radius, $\alpha_\varepsilon < 1$ and $1-\omega+\omega\alpha_\varepsilon < 1$ if $\varepsilon > 0$ is sufficiently small. Let $\varepsilon > 0$ be such a sufficiently small number. Using (5) and (6) and the fact that $\omega(2-\omega) \geqslant \omega$ for $0 < \omega \leqslant 1$,

$$\begin{aligned}|H_{\omega,\beta}|x_\varepsilon &\leqslant (\beta + |1-\beta|)x_\varepsilon - \omega(2-\omega)\beta \sum_{k=1}^{\ell} E_k \tilde{M}_k(\omega)^{-1}|D|(1-\alpha_\varepsilon)x_\varepsilon \\ &\leqslant (\beta + |1-\beta|)x_\varepsilon - \omega(2-\omega)\beta(1-\alpha_\varepsilon)x_\varepsilon \\ &\leqslant (\beta + |1-\beta|)x_\varepsilon - \omega\beta(1-\alpha_\varepsilon)x_\varepsilon \\ &= (|1-\beta| + \beta(1-\omega+\omega\alpha_\varepsilon))x_\varepsilon. \end{aligned} \tag{7}$$

If $0 < \beta \leqslant 1$, from (7) and $1-\omega+\omega\alpha_\varepsilon < 1$ one obtains $|H_{\omega,\beta}|x_\varepsilon < x_\varepsilon$. It follows that $\rho(H_{\omega,\beta}) < 1$ for $0 < \beta \leqslant 1$. If $1 < \beta < 2/(1+\rho(H(\omega)))$, $\beta(1+\rho(H(\omega))) = \beta(2-\omega+\omega\alpha) < 2$. By continuity of the spectral radius, $\beta(2-\omega+\omega\alpha_\varepsilon) < 2$ for sufficiently small $\varepsilon > 0$. Using this fact and (7),

$$\begin{aligned}|H_{\omega,\beta}|x_\varepsilon &\leqslant (\beta - 1 + \beta(1-\omega+\omega\alpha_\varepsilon))x_\varepsilon \\ &= (\beta(2-\omega+\omega\alpha_\varepsilon) - 1)x_\varepsilon < x_\varepsilon. \end{aligned} \tag{8}$$

From (8), $\rho(H_{\omega,\beta}) < 1$ for $1 < \beta < 2/(1+\rho(H(\omega)))$. Hence, part (a) is proved.

We next prove part (b). Let $\tilde{A} = \tilde{M}_k(\omega) - \tilde{N}_k^2(\omega)$. Then $\tilde{A} = \omega(2-\omega)|D| - \omega^2|B|$ for every k, and thus one obtains

$$\begin{aligned}|H_{\omega,\beta}| &\leqslant \beta \sum_{k=1}^{\ell} E_k \tilde{M}_k(\omega)^{-1}\tilde{N}_k^2(\omega) + |1-\beta| I \\ &= \beta \sum_{k=1}^{\ell} E_k \tilde{M}_k(\omega)^{-1}(\tilde{M}_k(\omega) - \tilde{A}) + |1-\beta| I \\ &= (\beta + |1-\beta|)I - \omega\beta \sum_{k=1}^{\ell} E_k \tilde{M}_k(\omega)^{-1}|D|((2-\omega)I - \omega|D|^{-1}|B|) \\ &\leqslant (\beta + |1-\beta|)I - \omega\beta \sum_{k=1}^{\ell} E_k \tilde{M}_k(\omega)^{-1}|D|((2-\omega)I - \omega|D|^{-1}|B| - \omega\varepsilon e e^{\mathrm{T}}). \end{aligned} \tag{9}$$

Since $1 < \omega < 2/(1+\alpha)$, $\omega(1+\alpha) < 2$. By continuity of the spectral radius, $\omega(1+\alpha_\varepsilon) < 2$ if $\varepsilon > 0$ is sufficiently small. Using (5) and (9),

$$\begin{aligned}|H_{\omega,\beta}|x_\varepsilon &\leqslant (\beta + |1-\beta|)x_\varepsilon - \omega\beta \sum_{k=1}^{\ell} E_k \tilde{M}_k(\omega)^{-1}|D|(2-\omega-\omega\alpha_\varepsilon)x_\varepsilon \\ &\leqslant (\beta + |1-\beta|)x_\varepsilon - \omega\beta(2-\omega-\omega\alpha_\varepsilon)x_\varepsilon. \end{aligned} \tag{10}$$

Since $0 < \beta \leqslant 1$, from (10) $|H_{\omega,\beta}|x_\varepsilon \leqslant (1 - \omega\beta(2-\omega-\omega\alpha_\varepsilon))x_\varepsilon < x_\varepsilon$, which proves part (b).

Lastly, we prove part (c). Since $2/(1+\alpha) > 1$, $\sqrt{2/1+\alpha} < 2/(1+\alpha)$. Clearly, $\omega(1+\rho(H(\omega))) < 2$ when $1 < \omega < \sqrt{2/(1+\alpha)}$. Thus, by part (b) it suffices to prove part (c) for $1 < \beta < 2/(\omega(1+\rho(H(\omega))))$. Since $\beta\omega(1+\rho(H(\omega))) = \beta\omega^2(1+\alpha) < 2$, by continuity of the spectral radius $\beta\omega^2(1+\alpha_\varepsilon) < 2$ if $\varepsilon > 0$ is sufficiently small. Since $\omega^2 - 2\omega + 2 < \omega^2$ for $\omega > 1$, from (10) one obtains

$$\begin{aligned}|H_{\omega,\beta}|x_\varepsilon &\leqslant (2\beta - 1 - \omega\beta(2-\omega-\omega\alpha_\varepsilon))x_\varepsilon \\ &= (\beta(\omega^2 - 2\omega + 2 + \omega^2\alpha_\varepsilon) - 1)x_\varepsilon \\ &< (\beta(\omega^2 + \omega^2\alpha_\varepsilon) - 1)x_\varepsilon \\ &= (\beta\omega^2(1+\alpha_\varepsilon) - 1)x_\varepsilon < x_\varepsilon. \end{aligned} \tag{11}$$

From (11), part (c) is proved. □

From Theorem 3.3, the following corollary can be easily obtained.

Corollary 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Let $A=D-B=D-L_k-U_k$ $(1\leqslant k\leqslant \ell)$, where $D=\mathrm{diag}(A)$, $L_k\geqslant 0$ is a strictly lower triangular matrix, and $U_k\geqslant 0$ is a general matrix, and let $(M_k(\omega), N_k(\omega), E_k)$, $k=1,2,\ldots,\ell$, be the SSOR multisplitting of A. Let $H_{\omega,\beta}=\beta\sum_{k=1}^{\ell}E_kM_k(\omega)^{-1}N_k(\omega)+(1-\beta)I$ be an iteration matrix of the relaxed multisplitting method associated with the SSOR multisplitting, $H(\omega)=|1-\omega|I+\omega D^{-1}B$ and $\alpha=\rho(D^{-1}B)$. Then the following hold*:

(a) *if* $0<\omega\leqslant 1$ *and* $0<\beta<\frac{2}{1+\rho(H(\omega))}$, *then* $\rho(H_{\omega,\beta})<1$.
(b) *if* $1<\omega<\frac{2}{1+\alpha}$ *and* $0<\beta\leqslant 1$, *then* $\rho(H_{\omega,\beta})<1$.
(c) *if* $1<\omega<\sqrt{\frac{2}{1+\alpha}}$ *and* $0<\beta<\frac{2}{\omega(1+\rho(H(\omega)))}$, *then* $\rho(H_{\omega,\beta})<1$.

4. Application of the SSOR multisplitting method

In this section, we consider an application of multisplitting method associated with the SSOR multisplitting for solving a linear system $Ax=b$, where A is an M-matrix. As a typical application, we introduce how the SSOR multisplitting can be used as inner splittings of two-stage multisplitting method which was proposed by Szyld and Jones [7]. Let ℓ denote the number of processors to be used. For simplicity of exposition, suppose that $\ell=3$. Then, the M-matrix A is partitioned into a 3×3 block matrix of the form

$$A=\begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix},$$

where the diagonal blocks A_{ii} of A are square matrices. Let $A=Q-R$, where

$$Q=\begin{pmatrix} A_{11} & 0 & 0\\ 0 & A_{22} & 0\\ 0 & 0 & A_{33}\end{pmatrix},\quad R=\begin{pmatrix} 0 & -A_{12} & -A_{13}\\ -A_{21} & 0 & -A_{23}\\ -A_{31} & -A_{32} & 0\end{pmatrix}. \tag{12}$$

Since A is an M-matrix, it is clear that $A=Q-R$ is a regular splitting of A and Q is an M-matrix. Let $Q=D-L_k-U_k$ for $k=1,2,3$, where $D=\mathrm{diag}(Q)$, $L_k\geqslant 0$ is a strictly lower triangular matrix, and $U_k\geqslant 0$ is a general matrix. Then, it is easy to see that D, L_k and U_k are of the form

$$D=\begin{pmatrix} D_1 & 0 & 0\\ 0 & D_2 & 0\\ 0 & 0 & D_3\end{pmatrix},\quad L_k=\begin{pmatrix} L_{k1} & 0 & 0\\ 0 & L_{k2} & 0\\ 0 & 0 & L_{k3}\end{pmatrix},\quad U_k=\begin{pmatrix} U_{k1} & 0 & 0\\ 0 & U_{k2} & 0\\ 0 & 0 & U_{k3}\end{pmatrix}, \tag{13}$$

where $D_i=\mathrm{diag}(A_{ii})$, L_{ki} is a nonnegative strictly lower triangular matrix, and U_{ki} is a nonnegative general matrix such that $A_{ii}=D_i-L_{ki}-U_{ki}$ for $k=1,2,3$. For each k, let

$$M_k(\omega)=\frac{1}{\omega(2-\omega)}(D-\omega L_k)D^{-1}(D-\omega U_k),$$

$$N_k(\omega)=\frac{1}{\omega(2-\omega)}((1-\omega)D+\omega L_k)D^{-1}((1-\omega)D+\omega U_k).$$

Let

$$E_1=\begin{pmatrix} I & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad E_2=\begin{pmatrix} 0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 0\end{pmatrix},\quad E_3=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I\end{pmatrix}. \tag{14}$$

Then, $(M_k(\omega), N_k(\omega), E_k)$, $k=1,2,3$, is the SSOR multisplitting of Q. When $0<\omega\leqslant 1$, it is easy to show that $Q=M_k(\omega)-N_k(\omega)$ is a regular splitting of Q for each k. Since $A=Q-R$ is a regular splitting of A and $Q=M_k(\omega)-N_k(\omega)$ is a regular splitting of Q when $0<\omega\leqslant 1$, from [7] the two-stage multisplitting method using $A=Q-R$ as an outer splitting and the SSOR multisplitting of Q (i.e., $Q=M_k(\omega)-N_k(\omega)$) as inner splittings converges to the exact solution of $Ax=b$ when $0<\omega\leqslant 1$. Notice that the two-stage multisplitting method can be fully parallelized without increasing computational amount because of the special structure of D, L_k, U_k and E_k.

For various ranges of ω, convergence of two-stage multisplitting method using an outer splitting and the SSOR multisplitting as inner splittings for solving a linear system whose coefficient matrix is an M-matrix or an H-matrix will be discussed in future work.

Acknowledgements

The author would like to thank the anonymous referee for useful comments and constructive suggestions which substantially improved the quality of this paper.

References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.
[2] R. Bru, V. Migallón, J. Penadés, D.B. Szyld, Parallel, synchronous and asynchronous two-stage multisplitting methods, Electron. Trans. Numer. Anal. 3 (1995) 24–38.
[3] W. Deren, On the convergence of the parallel multisplitting AOR algorithm, Linear Algebra Appl. 154–156 (1991) 473–486.
[4] A. Frommer, G. Mayer, Convergence of relaxed parallel multisplitting methods, Linear Algebra Appl. 119 (1989) 141–152.
[5] M. Neumann, R.J. Plemmons, Convergence of parallel multisplitting methods for M-matrices, Linear Algebra Appl. 88–89 (1987) 559–574.
[6] D.P. O'Leary, R.E. White, Multisplittings of matrices and parallel solution of linear systems, SIAM J. Algebraic Discrete Methods 6 (1985) 630–640.
[7] D.B. Szyld, M.T. Jones, Two-stage and multisplitting methods for the parallel solution of linear systems, SIAM J. Matrix Anal. Appl. 13 (1992) 671–679.
[8] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, 1962.
[9] J.H. Yun, Performance of ILU factorization preconditioners based on multisplittings, Numer. Math. 95 (2003) 761–779.