Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 222 (2008) 477–486
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Some higher-order modifications of Newton's method for solving nonlinear equations

YoonMee Ham[a], Changbum Chun[b,*], Sang-Gu Lee[b]

[a] Department of Mathematics, Kyonggi University, Suwon 443-760, Republic of Korea
[b] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea

Received 15 April 2007; received in revised form 16 July 2007

Abstract

In this paper we consider constructing some higher-order modifications of Newton's method for solving nonlinear equations which increase the order of convergence of existing iterative methods by one or two or three units. This construction can be applied to any iteration formula, and per iteration the resulting methods add only one additional function evaluation to increase the order. Some illustrative examples are provided and several numerical results are given to show the performance of the presented methods.

MSC: 65H05; 65B99

Keywords: Newton's method; Modified Newton's method; Iterative methods; Nonlinear equations; Order of convergence; Root-finding

1. Introduction

Solving a nonlinear equation is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root α, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$.

Newton's method is the most frequently used iterative method to solve the nonlinear equation; it is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \tag{1}$$

which, as is well known, converges quadratically in a sufficiently small neighbourhood of the root α, the desired one to find to solve the nonlinear equation [17].

In recent years, numerous higher-order iterative methods have been developed and analysed for solving nonlinear equations that improve some classical methods such as Newton's, Euler's, Chebyshev–Halley's, etc., methods, in order of convergence, most frequently composed of more than two existing formulas and derived in various manners, see [1,2,4–16] and the references therein. The increased order of convergence is usually achieved at the expense of

* Corresponding author.
E-mail addresses: ymham@kyonggi.ac.kr (Y. Ham), cbchun64@yahoo.com (C. Chun), sglee@skku.edu (S.-G. Lee).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.11.018

additional function or derivative evaluations to carry out iterations, which may affect the efficiency of the method. It has been shown that these methods are efficient in their performance, can compete with Newton's method.

Motivated by the recent results in this area, in this paper we consider constructing higher-order modifications of Newton's method. Any existing method can be used in the construction and its order of convergence is increased by one, two or three units by the expense of one additional function evaluation per iteration, this makes the computational efficiency better. This will be explored in the following sections, together with numerical examples showing the performance of some of the obtained methods.

2. Development of methods and convergence analysis

Throughout the paper, we assume that ϕ_p represents any iteration function whose order of convergence is at least p, which means that the corresponding iterative method defined by

$$x_{n+1} = \phi_p(x_n), \quad n = 0, 1, 2, \ldots \tag{2}$$

is of order p, that is, the error $|\alpha - x_{n+1}|$ is proportional to $|\alpha - x_n|^p$ as $n \to \infty$. A general study of iteration functions can be found in [17].

To derive the new methods, we consider a two-step iteration scheme with any existing method of order p as the first step and a corrected form of the first-step method depending on an unknown function as the second step, namely

$$z_n = \phi_p(x_n), \tag{3}$$

$$x_{n+1} = z_n - H(x_n, y_n)\frac{f(z_n)}{f'(x_n)}, \tag{4}$$

where $H(x, y)$ represents a two-variable function to be determined later and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{5}$$

Our concern here is to find $H(x, y)$ for which the method defined by (3) and (4) has a higher order of convergence. This can be answered in the following theorem, which gives a detailed analysis of convergence.

Theorem 2.1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. Let ϕ_p be any iteration function whose order of convergence is at least p. If $H(x, y)$ satisfies condition*

$$H(\alpha, \alpha) = 1, \tag{6}$$

then the method defined by (3) and (4) is of order at least $p + 1$. If $H(x, y)$ satisfies condition

$$H(\alpha, \alpha) = 1, \quad H_x(\alpha, \alpha) = \frac{f''(\alpha)}{f'(\alpha)}, \tag{7}$$

then the method defined by (3) and (4) is of order at least $p + 2$; furthermore, if $H(x, y)$ satisfies condition (7) and also the following condition,

$$\frac{f''(\alpha)}{f'(\alpha)}H_y(\alpha, \alpha) + H_{xx}(\alpha, \alpha) = \frac{f^{(3)}(\alpha)}{f'(\alpha)}, \tag{8}$$

then it is of order at least $p + 3$.

Proof. Let α be a simple zero of f and let $z_n = \phi_3(x_n)$. Throughout the proof, $H(\alpha, \alpha)$ will be denoted by H, $H_x(\alpha, \alpha)$ by H_x, $H_y(\alpha, \alpha)$ by H_y, $H_{xx}(\alpha, \alpha)$ by H_{xx}, $H_{xy}(\alpha, \alpha)$ by H_{xy}, $H_{yy}(\alpha, \alpha)$ by H_{yy}, and $H_{xxx}(\alpha, \alpha)$ by H_{xxx}.

Using Taylor's expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)], \tag{9}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)], \tag{10}$$

where $e_n = x_n - \alpha$ and $c_k = \frac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}$, $k = 2, 3, 4 \ldots$.

Dividing (9) by (10) gives

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2\, e_n^2 - 2(c_2^2 - c_3)\, e_n^3 + O(e_n^4). \tag{11}$$

By the Taylor expansion of $f(z_n)$ about α, we get

$$f(z_n) = f'(\alpha)[z_n - \alpha + c_2\,(z_n - \alpha)^2 + O((z_n - \alpha)^3)], \tag{12}$$

then dividing it by (10) gives

$$\frac{f(z_n)}{f'(x_n)} = (z_n - \alpha)[1 - 2c_2\, e_n + (4c_2^2 - 3c_3)\, e_n^2 + (c_2(z_n - \alpha) + (-8c_2^3 + 12c_2c_3 - 4c_4))\, e_n^3] + O(e_n^7). \tag{13}$$

Since ϕ_p is of order at least p, there exists a constant A such that

$$z_n - \alpha = Ae_n^p + O(e_n^{p+1}). \tag{14}$$

If we apply the 2-variable form of Taylor's theorem around the point (α, α), then from (11), we obtain

$$\begin{aligned} H(x_n, y_n) &= H + H_x(x_n - \alpha) + H_y(y_n - \alpha) + H_{xx}\frac{(x_n - \alpha)^2}{2} + H_{xy}(x_n - \alpha)(y_n - \alpha) \\ &\quad + H_{yy}\frac{(y_n - \alpha)^2}{2} + H_{xxx}\frac{(x_n - \alpha)^3}{6} + \cdots \\ &= H + H_x e_n + \left[H_y c_2 + \frac{H_{xx}}{2}\right] e_n^2 + \left[(2c_3 - 2c_2^2)H_y + c_2 H_{xy} + \frac{H_{xxx}}{6}\right] e_n^3 + O(e_n^4). \end{aligned} \tag{15}$$

Thus, from (13) and (15), we obtain

$$\begin{aligned} e_{n+1} &= (z_n - \alpha) - H(x_n, y_n)\frac{f(z_n)}{f'(x_n)} \\ &= (z_n - \alpha) - (z_n - \alpha)\Bigg[H + (-2c_2 H + H_x)e_n \\ &\quad + \left((4c_2^2 - 3c_3)H - 2c_2 H_x + c_2 H_y + \frac{H_{xx}}{2}\right) e_n^2 + K_6\Bigg] + O(e_n^7), \end{aligned} \tag{16}$$

where

$$\begin{aligned} K_6 &= H[c_2(z_n - \alpha) + (-8c_2^3 + 12c_2c_3 - 4c_4)e_n^3] + (4c_2^2 - 3c_3)H_x\, e_n^3 \\ &\quad - [2H_y c_2^2 + H_{xx} c_2]e_n^3 + \left[(2c_3 - 2c_2^2)H_y + c_2 H_{xy} + \frac{H_{xxx}}{6}\right] e_n^3. \end{aligned} \tag{17}$$

Thus, if $H = 1$ then using (14), (16) reduces to

$$e_{n+1} = A[2c_2 - H_x]e_n^{p+1} + O(e_n^{p+2}), \tag{18}$$

which implies that the method defined by (3) and (4) is of order at least $p + 1$. Thus, if $H = 1$ and $H_x = 2c_2$ then using (14), (16) reduces to

$$e_{n+1} = A\left[3c_3 - c_2 H_y - \frac{H_{xx}}{2}\right] e_n^{p+2} + O(e_n^{p+3}), \tag{19}$$

so that the method defined by (3) and (4) is of order at least $p + 2$. Furthermore, if $H = 1, H_x = 2c_2$ and $c_2 H_y + \frac{H_{xx}}{2} = 3c_3$, then using (14), (16) reduces to

$$e_{n+1} = A\left[-(A + H_{xy})c_2 + 2H_y c_2^2 - 2H_y c_3 + 4c_4 - \frac{H_{xxx}}{6}\right] e_n^{p+3} + O(e_n^{p+4}), \tag{20}$$

therefore the method defined by (3) and (4) is of order at least $p + 3$, this completes the proof. $\square$

As a result of Theorem 2.1 the order of any existing iterative method may be improved to three more higher at the expense of but one additional function evaluation, this also improving the computational efficiency of the method much better.

3. Some iterative methods

Many new third-order, fifth-order, and sixth-order methods are special cases of Theorem 2.1. In this section, we present some examples of the methods obtained as the special cases. It is not that difficult to find $H(x, y)$ satisfying conditions (6) or (7) or conditions (7) and (8), this can be possibly done with the help of mathematical packages such as Maple, Mathematica; for example, in condition (6) case,

$$H(x, y) = 1, \quad H(x, y) = \frac{\beta f(x) + \gamma f'(x)}{\mu f(x) + \gamma f'(x)}, \quad H(x, y) = 1 + x - \varphi(x), \tag{21}$$

where φ is any iteration function with $\varphi(\alpha) = \alpha$, in condition (7) case,

$$H(x, y) = \frac{f'(y)}{2f'(y) - f'(x)}, \quad H(x, y) = \frac{f'(x)}{f'(y)}, \quad H(x, y) = \frac{3f'(x) - f'(y)}{f'(x) + f'(y)}, \tag{22}$$

or in (7) and (8) case,

$$H(x, y) = \frac{2f'^2(y)}{f'^2(x) - 4f'(x)f'(y) + 5f'^2(y)}, \quad H(x, y) = \frac{f'(x) + f'(y)}{3f'(y) - f'(x)}, \tag{23}$$

$$H(x, y) = -\frac{2f'^2(x)}{f'^2(x) - 4f'(x)f'(y) + f'^2(y)}, \tag{24}$$

etc. Therefore, many higher-order methods are easily constructible by the approach proposed in this contribution. Some of the above examples of $H(x, y)$ will be used in the following examples. Throughout the rest of this section, y_n is defined by (5).

3.1. Some third-order methods

Example 3.1. For the function H defined by

$$H(x, y) = \frac{\beta f(x) + \gamma f'(x)}{\mu f(x) + \gamma f'(x)}, \tag{25}$$

where $\beta, \mu, \gamma (\neq 0) \in \mathbf{R}$, we obtain the new third-order family of methods

$$z_n = \phi_2(x_n), \tag{26}$$

$$x_{n+1} = z_n - \frac{\beta f(x) + \gamma f'(x)}{\mu f(x) + \gamma f'(x)} \frac{f(z_n)}{f'(x_n)}. \tag{27}$$

If we take Newton's iteration function, for example,

$$\phi_2(x) = x - \frac{f(x)}{f'(x)}, \tag{28}$$

then we obtain the new three-parameter third-order family of methods

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{29}$$

$$x_{n+1} = z_n - \frac{\beta f(x_n) + \gamma f'(x_n)}{\mu f(x_n) + \gamma f'(x_n)} \frac{f(z_n)}{f'(x_n)}. \tag{30}$$

We note that if $\beta = \mu = 0$, then we obtain the well-known third-order iteration result [17],

$$z_n = \phi_2(x_n), \tag{31}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}. \tag{32}$$

Example 3.2. For the function H defined by

$$H(x, y) = 1 + x - \varphi(x), \tag{33}$$

where φ is any function with $\varphi(\alpha) = \alpha$, we obtain the new third-order class of methods

$$z_n = \phi_2(x_n), \tag{34}$$

$$x_{n+1} = z_n - [1 + x - \varphi(x)]\,\frac{f(z_n)}{f'(x_n)}. \tag{35}$$

If we take Newton's iteration function, for example,

$$\phi_2(x) = x - \frac{f(x)}{f'(x)}, \tag{36}$$

and

$$\varphi(x) = x - \frac{f(x)f'(x)}{f(x)^2 + f'(x)^2}, \tag{37}$$

which is the known iteration function of order two [8], then we obtain the new third-order method

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{38}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n)^2 + f'(x_n)^2 + f(x_n)f'(x_n)}{f(x_n)^2 + f'(x_n)^2}\right]\frac{f(z_n)}{f'(x_n)}. \tag{39}$$

If we consider the definition of efficiency index [3] as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functional evaluations per iteration required by the method, we have that the methods obtained in the above examples have the efficiency index equal to $3^{\frac{1}{3}} \approx 1.441$, which is better than that of Newton's method $\sqrt{2} \approx 1.414$.

3.2. Some fifth-order methods

Example 3.3. For the function H defined by

$$H(x, y) = \frac{f'(y)}{2f'(y) - f'(x)}, \tag{40}$$

we obtain the new fifth-order scheme

$$z_n = \phi_3(x_n), \tag{41}$$

$$x_{n+1} = z_n - \frac{f'(y_n)}{2f'(y_n) - f'(x_n)}\,\frac{f(z_n)}{f'(x_n)}. \tag{42}$$

If we take [19], for example,

$$\phi_3(x) = x - \frac{2f(x)}{f'(x) + f'(y(x))}, \tag{43}$$

where $y(x) = x - f(x)/f'(x)$, then we obtain the new fifth-order method

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \tag{44}$$

$$x_{n+1} = z_n - \frac{f'(y_n)}{2f'(y_n) - f'(x_n)} \frac{f(z_n)}{f'(x_n)}. \tag{45}$$

Example 3.4. For function H defined by

$$H(x, y) = \frac{3f'(x) - f'(y)}{f'(x) + f'(y)}, \tag{46}$$

we obtain the new fifth-order scheme

$$z_n = \phi_3(x_n), \tag{47}$$

$$x_{n+1} = z_n - \frac{3f'(x_n) - f'(y_n)}{f'(x_n) + f'(y_n)} \frac{f(z_n)}{f'(x_n)}. \tag{48}$$

If we take [7], for example,

$$\phi_3(x) = x - \frac{f(x)}{2} \left[\frac{1}{f'(x)} + \frac{1}{f'(y(x))} \right], \tag{49}$$

where $y(x) = x - f(x)/f'(x)$, then we obtain the new fifth-order method

$$z_n = x_n - \frac{f(x_n)}{2} \left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right), \tag{50}$$

$$x_{n+1} = z_n - \frac{3f'(x_n) - f'(y_n)}{f'(x_n) + f'(y_n)} \frac{f(z_n)}{f'(x_n)}. \tag{51}$$

The methods obtained in the fifth-order examples have an efficiency index equal to $5^{\frac{1}{4}} \approx 1.4953$, which is better than that of Newton's method $\sqrt{2} \approx 1.414$.

3.3. Some sixth-order methods

Example 3.5. For the function H defined by

$$H(x, y) = \frac{2f'^2(y)}{f'^2(x) - 4f'(x)f'(y) + 5f'^2(y)}, \tag{52}$$

we obtain the new sixth-order class of methods

$$z_n = \phi_3(x_n), \tag{53}$$

$$x_{n+1} = z_n - \frac{2f'^2(y_n)}{f'^2(x_n) - 4f'(x_n)f'(y_n) + 5f'^2(y_n)} \frac{f(z_n)}{f'(x_n)}. \tag{54}$$

If we take [19], for example,

$$\phi_3(x) = x - \frac{2f(x)}{f'(x) + f'(y(x))}, \tag{55}$$

where $y(x) = x - f(x)/f'(x)$, then we obtain the new sixth-order method

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \tag{56}$$

$$x_{n+1} = z_n - \frac{2f'^2(y_n)}{f'^2(x_n) - 4f'(x_n)f'(y_n) + 5f'^2(y_n)} \frac{f(z_n)}{f'(x_n)}. \tag{57}$$

Example 3.6. For the function H defined by

$$H(x, y) = -\frac{2f'^2(x)}{f'^2(x) - 4f'(x)f'(y) + f'^2(y)}, \tag{58}$$

we obtain the new sixth-order class of methods

$$z_n = \phi_3(x_n), \tag{59}$$

$$x_{n+1} = z_n + \frac{2f'(x_n)f(z_n)}{f'^2(x_n) - 4f'(x_n)f'(y_n) + f'^2(y_n)}. \tag{60}$$

If we take [7], for example,

$$\phi_3(x) = x - \frac{f(x)}{2}\left(\frac{1}{f'(x)} + \frac{1}{f'(y(x))}\right), \tag{61}$$

where $y(x) = x - f(x)/f'(x)$, then we obtain the new sixth-order method

$$z_n = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right) \tag{62}$$

$$x_{n+1} = z_n + \frac{2f'(x_n)f(z_n)}{f'^2(x_n) - 4f'(x_n)f'(y_n) + f'^2(y_n)}. \tag{63}$$

The presented sixth-order methods add only one evaluation of the function at the point iterated by the third-order methods to obtain the sixth-order, so that they have the efficiency index equal to $6^{\frac{1}{4}} \approx 1.565$, which is much better than that of Newton's method $\sqrt{2} \approx 1.414$. It should be pointed out that by considering many other possible combinations of the third-order formulas for ϕ_3 and the functions H satisfying condition (7) and (8), we can continuously derive many new sixth-order methods.

We would like to mention that some existing methods can be obtained as special cases of Theorem 2.1. In the sixth-order case, if we take, for example,

$$H(x, y) = \frac{f'(x) + f'(y)}{3f'(y) - f'(x)}, \tag{64}$$

or

$$H(x, y) = \frac{2f'(x)f'(y)}{f'^2(y) + 2f'(x)f'(y) - f'^2(x)}, \tag{65}$$

then the sixth-order methods proposed by Kou [12] are obtained, so our results can also be viewed as an important advance on the previously known results.

4. Numerical examples

All computations were done with the MAPLE using 256 digit floating point arithmetics (Digits := 256). We set $\epsilon = 10^{-225}$ as an iteration tolerance number. We used the following test functions and display the approximate zeros x_* found up to the 28th decimal place.

$$f_1(x) = x^3 + 4x^2 - 10, \quad x_* = 1.3652300134140968457608068290,$$
$$f_2(x) = \sin^2 x - x^2 + 1, \quad x_* = 1.4044916482153412260350868178,$$
$$f_3(x) = x^2 - e^x - 3x + 2, \quad x_* = 0.2575302854398607604553673049 4,$$
$$f_4(x) = \cos x - x, \quad x_* = 0.7390851332151606416553120876 7,$$
$$f_5(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad x_* = -1.2076478271309189270094167584,$$
$$f_6(x) = x^2\sin^2(x) + e^{x^2\cos x \sin x} - 28, \quad x_* = 4.6221041635528383439278532516,$$

Table 1
Comparison of various iterative methods and Newton's method

$f(x)$	NFE(COC)							
	NM	WF3	IW5	IW6	HM3	IH5	IH6	KM3
$f_1, x_0 = 0.8$	18(2.0)	18(3.0)	16(4.999)	16(6.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)
$f_1, x_0 = 1$	18(2.0)	18(3.0)	16(5.0)	16(6.0)	15(3.0)	16(5.0)	16(6.0)	15(3.0)
$f_2, x_0 = 2.3$	20(2.0)	18(3.0)	20(5.0)	16(6.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)
$f_2, x_0 = 1$	20(2.0)	18(3.0)	20(5.0)	16(6.0)	18(3.0)	16(5.004)	16(5.99)	18(3.0)
$f_3, x_0 = 0$	16(2.0)	15(3.0)	16(5.0)	12(6.0)	15(3.0)	16(5.0)	12(5.983)	15(3.0)
$f_3, x_0 = 1$	16(2.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)
$f_4, x_0 = 1.7$	16(2.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)
$f_4, x_0 = 0$	18(2.0)	18(3.0)	16(4.999)	16(6.0)	18(3.0)	16(4.999)	16(6.0)	18(3.0)
$f_5, x_0 = -1$	18(2.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)	16(5.0)	16(6.0)	21(3.0)
$f_5, x_0 = -0.5$	26(2.0)	42(3.0)	20(5.0)	16(5.598)	18(3.0)	24(5.0)	20(6.0)	21(3.0)
$f_6, x_0 = 3.5$	18(2.0)	18(3.0)	20(5.0)	16(6.0)	18(3.0)	16(5.0)	16(6.0)	18(3.0)
$f_6, x_0 = 4.5$	22(2.0)	21(3.0)	52(4.998)	20(6.0)	21(3.0)	20(5.0)	20(6.0)	24(3.0)
$f_7, x_0 = 3.5$	32(2.0)	33(3.0)	352(5.0)	32(6.0)	27(3.0)	28(5.0)	24(5.998)	33(3.0)
$f_7, x_0 = 3.2$	24(2.0)	24(3.0)	32(5.0)	20(6.001)	21(3.0)	20(5.0)	20(6.0)	24(3.0)
$f_8, x_0 = 1.4$	746(1.297)	708(1.176)	div	688(1.104)	561(1.121)	640(1.09)	580(1.072)	792(1.205)
$f_8, x_0 = 1.2$	750(1.297)	711(1.176)	div	692(1.104)	564(1.121)	644(1.09)	584(1.072)	795(1.205)
$f_9, x_0 = 1.1$	744(1.297)	705(1.176)	div	688(1.104)	558(1.121)	636(1.09)	576(1.072)	789(1.205)
$f_9, x_0 = 0.9$	742(1.297)	705(1.176)	div	684(1.104)	558(1.121)	636(1.09)	576(1.072)	789(1.205)

$$f_7(x) = \mathrm{e}^{x^2+7x-30} - 1, \quad x_* = 3,$$
$$f_8(x) = (x^3 + 4x^2 - 10)^2, \quad x_* = 1.365230013414096845760806829 0,$$
$$f_9(x) = (x-1)^2\mathrm{e}^x, \quad x_* = 1.$$

We present some numerical test results for various iterative schemes and the Newton method in Table 1. Compared were the Newton method (NM), the method of Weerakoon and Fernando (WF3) defined by (43), the method of Homeier (HM3) defined by (49), the fifth-order variant of WF (45) (IW5), the sixth-order variant of WF3 (57) (IW6), the fifth-order variant of HM3 (51) (IH5), the sixth-order variant of HM3 (63) (IH6) and the third-order method defined by (39) (KM3) newly obtained in the present contribution. We note that these methods do not require the computation of second derivatives to carry out iterations.

Displayed in Table 1 are the number of function evaluations (NFE) required such that $|f(x_n)| < \epsilon$ and the computational order of convergence (COC) in parentheses. Here, COC is defined by [19]

$$\rho \approx \frac{\ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

In the table 'div' means that the sequence of approximate zeros produced from the corresponding method doesn't converge within the maximum iteration number.

The numerical results in Table 1 show that they are well in accordance with the theory developed in this paper; for almost all of the test functions, the proposed methods improve the corresponding third-order methods, achieve higher computational efficiency and can compete with Newton's method. Besides, we can see that the new third-order method KM3 has at least equal performance as the other existing third-order methods. Numerical results also confirm that the methods have local convergence property depending on choice of initial approximations. Moreover, it can be observed that for the functions f_8 and f_9 having repeated zeros, all the methods under consideration show linear convergence even if the initial guesses are rather close to the zero as in Newton's method, which is well known.

We also present some numerical test results for various fifth-order convergent iterative schemes in Table 2. Compared were Grau et al.'s method in [5] (GM), Grau-Sanchez's method [6] (SM) defined by

Table 2
Comparison of various fifth-order convergent iterative methods

$f(x)$	NFE							
	GM	SM	KM1	KM2	VM	NM	HM1	HM2
$f_1, x_0 = 1.5$	16	16	16	16	16	16	16	16
$f_1, x_0 = 1$	20	16	16	16	16	16	16	16
$f_2, x_0 = 2$	20	16	16	16	16	16	16	16
$f_2, x_0 = 1$	132	20	24	16	20	16	20	16
$f_3, x_0 = 0$	16	16	16	16	16	16	16	16
$f_3, x_0 = -0.5$	16	16	16	16	16	16	16	16
$f_4, x_0 = 1.5$	16	16	16	16	16	16	16	16
$f_4, x_0 = -0.5$	div	16	div	16	28	20	16	20
$f_5, x_0 = -1$	20	16	20	16	16	16	16	16
$f_5, x_0 = -1.4$	16	16	16	20	16	16	16	16
$f_6, x_0 = 5$	div	div	div	44	28	24	124	div
$f_6, x_0 = 4$	88	div	24	24	20	20	32	div
$f_7, x_0 = 3.5$	32	28	div	div	28	28	352	28
$f_7, x_0 = 2.9$	div	24	div	20	40	20	20	20

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \tag{66}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(y_n)}, \tag{67}$$

where y_n is defined by (5), the method of Kou et al. with $\beta = 1, \gamma = -1$ in [10] (KM1), Kou et al.'s method in with $\beta = 1$ [9] (KM2), Vy's method [18] (VM) defined by

$$x_{n+1} = y_n - \frac{f(y_n)}{f'\left(y_n - \frac{f(y_n)}{2f'(x_n)}\right)}, \tag{68}$$

where y_n is defined by (5), Noor's method [16] (NM) defined by

$$x_{n+1} = y_n - \frac{2f(x_n)f(y_n)f'(y_n)}{2f(x_n)f'^2(y_n) - f'^2(x_n)f(y_n) + f'(x_n)f'(y_n)f(y_n)}, \tag{69}$$

where y_n is defined by (5), and the methods (44) and (45) (HM1), and (50) and (51) (HM2) introduced in the present contribution.

Displayed in Table 2 are the number of function evaluations (NFE) required such that $|f(x_n)| < \epsilon$. The test result indicates that the proposed fifth-order methods have at least equal performance as compared with the other methods of the same order. It also confirms that the considered methods have local convergence depending on choice of initial approximations. We refer the reader to [9,10] for some numerical results showing that the Kou et al. methods can compete with Newton's method.

5. Conclusion

In this work we considered developing some higher-order modifications of Newton's method for solving nonlinear equations. The proposed methods add only one function evaluation at the point iterated by the existing iteration formula but they increase the order of the existing method to one, two or three units higher. Some of the presented methods were compared in their performance to some known methods of the same order; it was observed that they can be competitive to those methods and also improve the existing methods. Our approach can be continuously applied in order to improve any existing iteration formula.

Acknowledgements

The authors would like to thank the referees for their constructive comments and suggestions which substantially improved the quality of this paper. The work of the first author was supported by University of Northern Iowa. The work of the third author was supported by the SRC/ERC programme of MOST/KOSEF R11-1999-054 & BK21.

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