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A reduced finite difference scheme based on singular value decomposition and proper orthogonal decomposition for Burgers equation[☆]

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ABSTRACT

In this article, a reduced optimizing finite difference scheme (FDS) based on singular value decomposition (SVD) and proper orthogonal decomposition (POD) for Burgers equation is presented. Also the error estimates between the usual finite difference solution and the POD solution of reduced optimizing FDS are analyzed. It is shown by considering the results obtained for numerical simulations of cavity flows that the error between the POD solution of reduced optimizing FDS and the solution of the usual FDS is consistent with theoretical results. Moreover, it is also shown that the reduced optimizing FDS is feasible and efficient.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected and polygonal domain. Consider the following Burgers equation.

Problem (I). Find (u, v) such that, for $T > 0$,

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{u\partial u}{\partial x} + \dfrac{v\partial u}{\partial y} = \dfrac{1}{Re}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + f_1, & (x, y, t) \in \Omega \times (0, T), \\ \dfrac{\partial v}{\partial t} + \dfrac{u\partial v}{\partial x} + \dfrac{v\partial v}{\partial y} = \dfrac{1}{Re}\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + f_2, & (x, y, t) \in \Omega \times (0, T), \\ u(x, y, t) = \varphi(x, y, t),\ v(x, y, t) = \psi(x, y, t), & (x, y, t) \in \partial\Omega \times (0, T), \\ u(x, y, 0) = v(x, y, 0) = 0, & (x, y) \in \Omega, \end{cases} \tag{1.1}$$

where (u, v) represents the velocity vector, T the total time, Re the Reynolds number, and $\varphi(x, y, t)$ and $\psi(x, y, t)$ are two given functions. For the sake of convenience, without loss of generality, we may as well suppose that $\varphi(x, y, t) = \psi(x, y, t) = 0$ in the following theoretical analysis.

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Burgers equation (1.1) is an important physical equation in atmospheric dynamics which has many applied backgrounds. Though the finite difference scheme (FDS) is one of the most effective approaches in computing a numerical solution for Burgers equation (see [1–3]), a fully discrete system of FDS solutions for Burgers equation includes many degrees of freedom. Thus, an important problem is how to simplify the computational load and save time-consuming calculations and resource demands in the actual computational process in a sense that guarantees a sufficiently accurate numerical solution. Proper orthogonal decomposition (POD), also known as Karhunen–Loève expansions in signal analysis and pattern recognition (see [4]), or principal component analysis in statistics (see [5]), or the method of empirical orthogonal functions in geophysical fluid dynamics (see [6,7]) or meteorology (see [8]), is a technique offering adequate approximation for representing fluid flow with reduced number of degrees of freedom, i.e., with lower dimensional models (see [9]) so as to alleviate the computational load and memory requirements savings.

Although the basic properties of the POD method have been well established and have widely been applied in computations of statistics and fluid dynamics [4–20], it is mainly used to perform principal component analysis and search the main behavior of a dynamic system. Some reduced optimizing order finite difference models and mixed finite element formulations and error estimates for the upper tropical Pacific ocean model based on POD (see, [21–25]) are developed. Also an FDS based on POD for the non-stationary Navier–Stokes equations has been presented but its error analysis has not been derived (see [26]). To the best of our knowledge, there are no published results to address when POD is used to reduce the formulation of FDS for Burgers equation and the error estimates between the solution of the usual FDS and POD solution of reduced optimizing FDS.

In this paper, singular value decomposition (SVD) and POD are used to reduce the formulation of FDS for Burgers equation and the error estimates between the solution of the usual FDS and POD solution of reduced optimizing FDS are derived. It is shown by considering the results obtained for numerical simulations of cavity flows that the error between the POD solution of the reduced optimizing FDS based POD technique and the solution of the usual FDS is consistent with theoretical results. Moreover, it is also shown that the reduced FDS based on SVD and POD is feasible and efficient in computing the numerical solution for Burgers equation.

Though Kunisch and Volkwein have presented some Galerkin POD methods for parabolic problems and a general equation in fluid dynamics in References [27,28], and the SVD approach combined with the POD technique is used to treat Burgers equation in Reference [29] and the cavity flow problem in Reference [12], the error estimates have not completely been derived, especially a reduced optimizing formulation of FDS for Burgers equation has not been derived. Our method here is different from their approaches, whose methods consist of Galerkin projection approaches where original variables are substituted for a linear combination of POD basis and their POD basis is generated with the solutions of the physical system at all time instances. While the basis ideal of our reduced optimizing technique is that ensembles of data are first compiled from transient solutions computed from the equation system derived with the usual FDS for Burgers equation or from physics system trajectories via drawing samples of experiments and interpolation (or date assimilation), a group of POD basis is next obtained with SVD, and then the unknowns of the usual FDS are substituted with the linear combination of POD basis to derive a reduced optimizing FDS for Burgers equation. Especially, we prove theoretically that it is unnecessary to take the solutions of the physical system at all time instances as snapshots in the reduced optimizing formulation of FDS. Nothing remains but to take the solutions of the physical system at a few time instances as snapshots so that our method could reduce the computational load finding the POD basis. It is shown that the present method has improved and innovated the existing methods.

The paper is organized as follows. Section 2 is to derive the usual FDS for Burgers equation and to generate snapshots from transient solutions computed from the equation system derived by the usual FDS. In Section 3, a group of optimal orthonormal basis is reconstructed from the elements of the snapshots with SVD and POD and a reduced optimizing FDS with lower dimensional number based on SVD and POD for Burgers equation is developed. In Section 4, error estimates between solutions of the usual FDS and POD solutions of the reduced optimizing FDS are derived. In Section 5, some numerical examples are presented illustrating that the errors between the reduced optimizing FDS solutions and the usual FDS solutions are consistent with previously obtained theoretical results, thus validating the feasibility and efficiency of the POD method. Section 6 provides the main conclusions and future tentative ideas.

2. Usual FDS for Burgers and generate snapshots

Let Δx and Δy be the spatial step increment in the x-direction and y-direction, respectively, and Δt be the time step increment, $u^n_{j+\frac{1}{2},k}$ and $v^n_{j,k+\frac{1}{2}}$ denote function values of u and v at point $(x_{j+\frac{1}{2}}, y_k, t_n)$ and $(x_j, y_{k+\frac{1}{2}}, t_n)(0 \le j \le J, 0 \le k \le K, 0 \le n \le N = T/\Delta t)$, respectively.

In the following, we apply staggered net (see Fig. 1) FDS to solving Problem (I).

(1) Discretizing all terms of the momentum equation

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f_1 \tag{2.1}$$

on the x-direction at point $(x_{j+\frac{1}{2}}, y_k, t_n)$ yields

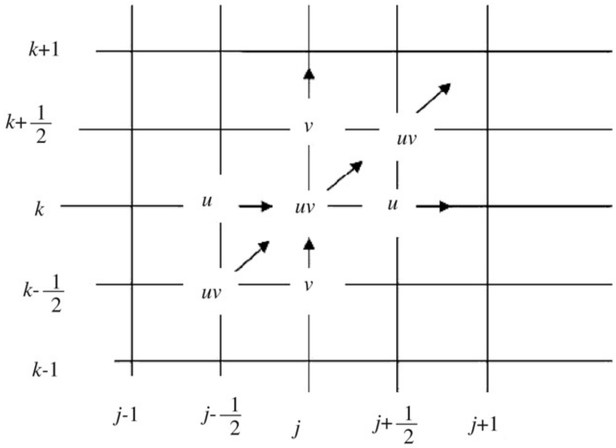


Fig. 1. Staggered mesh graphics.

$$u^{n+1}_{j+\frac{1}{2},k} = u^n_{j+\frac{1}{2},k} - F^n_{j+\frac{1}{2},k} + \Delta t f^n_{1,j+\frac{1}{2},k}, \tag{2.2}$$

where

$$\begin{aligned} F^n_{j+\frac{1}{2},k} = {} & \frac{\Delta t}{\Delta x} u^n_{j+\frac{1}{2},k}(u^n_{j+1,k} - u^n_{j,k}) + \frac{\Delta t}{\Delta y} v^n_{j+\frac{1}{2},k}(u^n_{j+\frac{1}{2},k+\frac{1}{2}} - u^n_{j+\frac{1}{2},k-\frac{1}{2}}) \\ & - \frac{\Delta t}{Re}\left[\frac{u_{j+\frac{1}{2},k-1} - 2u_{j+\frac{1}{2},k} + u_{j+\frac{1}{2},k+1}}{\Delta y^2} + \frac{u_{j-\frac{1}{2},k} - 2u_{j+\frac{1}{2},k} + u_{j+\frac{3}{2},k}}{\Delta x^2}\right]^n. \end{aligned} \tag{2.3}$$

(2) Expanding the momentum equation

$$\frac{\partial v}{\partial t} + \frac{u\partial v}{\partial x} + \frac{v\partial v}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + f_2 \tag{2.4}$$

on the y-direction at point $(x_j, y_{k+\frac{1}{2}}, t_n)$ yields

$$v^{n+1}_{j,k+\frac{1}{2}} = v^n_{j,k+\frac{1}{2}} - G^n_{j,k+\frac{1}{2}} + \Delta t f^n_{2,j,k+\frac{1}{2}}, \tag{2.5}$$

where

$$\begin{aligned} G^n_{j,k+\frac{1}{2}} = {} & \frac{\Delta t}{\Delta x} u^n_{j,k+\frac{1}{2}}(v^n_{j+\frac{1}{2},k+\frac{1}{2}} - v^n_{j-\frac{1}{2},k+\frac{1}{2}}) + \frac{\Delta t}{\Delta y} v^n_{j,k+\frac{1}{2}}(v^n_{j,k+1} - v^n_{j,k}) \\ & - \frac{\Delta t}{Re}\left[\frac{v_{j-1,k+\frac{1}{2}} - 2v_{j,k+\frac{1}{2}} + v_{j+1,k+\frac{1}{2}}}{\Delta x^2} + \frac{v_{j,k-\frac{1}{2}} - 2v_{j,k+\frac{1}{2}} + v_{j,k+\frac{3}{2}}}{\Delta y^2}\right]^n. \end{aligned} \tag{2.6}$$

Using the same approaches as the proof of the convergence and stability of finite difference equations of the non-stationary Navier–Stokes equation in [1] or [2], if $0.25(|u|^2 + |v|^2)\Delta t \cdot Re \le 1$, $\Delta t \le 0.25Re\Delta x^2$, and $\Delta t \le 0.25Re\Delta y^2$, it is not difficult to prove the convergence and stability of the usual FDS (2.2) and (2.5) for Burgers equation. We conclude the following result.

Theorem 1. *The usual FDS (2.2) and (2.5) for Burgers equation has the following error estimates*

$$|E_n(u^n_{j+\frac{1}{2},k}, v^n_{j,k+\frac{1}{2}})| = \|(u(x_{j+\frac{1}{2}}, y_k, t_n), v(x_j, y_{k+\frac{1}{2}}, t_n)) - (u^n_{j+\frac{1}{2},k}, v^n_{j,k+\frac{1}{2}})\| = O(\Delta t, \Delta x^2, \Delta y^2),\ 1 \le n \le N, \tag{2.7}$$

where $\|\cdot\|$ denotes the usual normal of the vector.

Proof. First, by expanding all terms of (2.2) at point $(x_{j+\frac{1}{2}}, y_k, t_n)$ and using the Taylor expansion, we obtain that

$$u^{n+1}_{j+\frac{1}{2},k} = u^n_{j+\frac{1}{2},k} + \Delta t\left(\frac{\partial u}{\partial t}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta t)^2}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta t)^3}{3!}\left(\frac{\partial^3 u}{\partial t^3}\right)^n_{j+\frac{1}{2},k} + \cdots, \tag{2.8}$$

$$[u_{j+\frac{1}{2},k-1} - 2u_{j+\frac{1}{2},k} + u_{j+\frac{1}{2},k+1}]^n = [u_{j+\frac{1}{2},k-1} - u_{j+\frac{1}{2},k}]^n + [u_{j+\frac{1}{2},k+1} - u_{j+\frac{1}{2},k}]^n$$

$$= -\Delta y\left(\frac{\partial u}{\partial y}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta y)^2}{2!}\left(\frac{\partial^2 u}{\partial y^2}\right)^n_{j+\frac{1}{2},k} - \frac{(\Delta y)^3}{3!}\left(\frac{\partial^3 u}{\partial y^3}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta y)^4}{4!}\left(\frac{\partial^4 u}{\partial y^4}\right)^n_{j+\frac{1}{2},k} + \cdots + \Delta y\left(\frac{\partial u}{\partial y}\right)^n_{j+\frac{1}{2},k}$$
$$+ \frac{(\Delta y)^2}{2!}\left(\frac{\partial^2 u}{\partial y^2}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta y)^3}{3!}\left(\frac{\partial^3 u}{\partial y^3}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta y)^4}{4!}\left(\frac{\partial^4 u}{\partial y^4}\right)^n_{j+\frac{1}{2},k} + \cdots$$
$$= (\Delta y)^2\left(\frac{\partial^2 u}{\partial y^2}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta y)^4}{12}\left(\frac{\partial^4 u}{\partial y^4}\right)^n_{j+\frac{1}{2},k} + \cdots, \tag{2.9}$$

$$[u_{j-\frac{1}{2},k} - 2u_{j+\frac{1}{2},k} + u_{j+\frac{3}{2},k}]^n = [u_{j-\frac{1}{2},k} - u_{j+\frac{1}{2},k}]^n + [u_{j+\frac{3}{2},k} - u_{j+\frac{1}{2},k}]^n$$
$$= -\Delta x\left(\frac{\partial u}{\partial x}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta x)^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)^n_{j+\frac{1}{2},k} - \frac{(\Delta x)^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)^n_{j+\frac{1}{2},k}$$
$$+ \frac{(\Delta x)^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)^n_{j+\frac{1}{2},k} + \cdots + \Delta x\left(\frac{\partial u}{\partial x}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta x)^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)^n_{j+\frac{1}{2},k}$$
$$+ \frac{(\Delta x)^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta x)^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)^n_{j+\frac{1}{2},k} + \cdots$$
$$= (\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta x)^4}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)^n_{j+\frac{1}{2},k} + \cdots, \tag{2.10}$$

$$\begin{aligned} u^n_{j+\frac{1}{2},k+\frac{1}{2}} - u^n_{j+\frac{1}{2},k-\frac{1}{2}} &= u^n_{j+\frac{1}{2},k+\frac{1}{2}} - u^n_{j+\frac{1}{2},k} + u^n_{j+\frac{1}{2},k} - u^n_{j+\frac{1}{2},k-\frac{1}{2}} \\ &= \frac{\Delta y}{2}\left(\frac{\partial u}{\partial y}\right)^n_{j+\frac{1}{2},k} + \frac{1}{2!}\left(\frac{\Delta y}{2}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right)^n_{j+\frac{1}{2},k} \\ &\quad + \frac{1}{3!}\left(\frac{\Delta y}{2}\right)^3\left(\frac{\partial^3 u}{\partial y^3}\right)^n_{j,k} + \cdots + \frac{\Delta y}{2}\left(\frac{\partial u}{\partial y}\right)^n_{j+\frac{1}{2},k} \\ &\quad - \frac{1}{2!}\left(\frac{\Delta y}{2}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right)^n_{j+\frac{1}{2},k} + \frac{1}{3!}\left(\frac{\Delta y}{2}\right)^3\left(\frac{\partial^3 u}{\partial y^3}\right)^n_{j+\frac{1}{2},k} - \cdots \\ &= \Delta y\left(\frac{\partial u}{\partial y}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta y)^3}{24}\left(\frac{\partial^3 u}{\partial y^3}\right)^n_{j+\frac{1}{2},k} + \cdots, \end{aligned} \tag{2.11}$$

$$\begin{aligned} u^n_{j+1,k} - u^n_{j,k} &= u^n_{j+1,k} - u^n_{j+\frac{1}{2},k} + u^n_{j+\frac{1}{2},k} - u^n_{j,k} \\ &= \frac{\Delta x}{2}\left(\frac{\partial u}{\partial x}\right)^n_{j+\frac{1}{2},k} + \frac{1}{2!}\left(\frac{\Delta x}{2}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^n_{j+\frac{1}{2},k} \\ &\quad + \frac{1}{3!}\left(\frac{\Delta x}{2}\right)^3\left(\frac{\partial^3 u}{\partial x^3}\right)^n_{j,k} + \cdots + \frac{\Delta x}{2}\left(\frac{\partial u}{\partial x}\right)^n_{j+\frac{1}{2},k} \\ &\quad - \frac{1}{2!}\left(\frac{\Delta x}{2}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^n_{j+\frac{1}{2},k} + \frac{1}{3!}\left(\frac{\Delta x}{2}\right)^3\left(\frac{\partial^3 u}{\partial x^3}\right)^n_{j+\frac{1}{2},k} - \cdots \\ &= \Delta x\left(\frac{\partial u}{\partial x}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta x)^3}{24}\left(\frac{\partial^3 u}{\partial x^3}\right)^n_{j+\frac{1}{2},k} + \cdots. \end{aligned} \tag{2.12}$$

Inserting (2.8)–(2.12) into (2.2) yields

$$\left[\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - f_1\right]^n_{j+\frac{1}{2},k}$$
$$= -\frac{\Delta t}{2!}\left(\frac{\partial^2 u}{\partial t^2}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta x)^2}{24}\left(\frac{u\partial^3 u}{\partial x^3}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta y)^2}{24}\left(\frac{v\partial^3 u}{\partial y^3}\right)^n_{j+\frac{1}{2},k}$$
$$+ \frac{(\Delta x)^2}{24}\left(\frac{\partial^3 p}{\partial x^3}\right)^n_{j+\frac{1}{2},k} + \frac{(\Delta x)^2}{12Re}\left(\frac{\partial^4 u}{\partial x^4}\right)^{n+1}_{j,k} + \frac{(\Delta y)^2}{12Re}\left(\frac{\partial^4 v}{\partial y^4}\right)^{n+1}_{j,k} + \cdots. \tag{2.13}$$

Therefore, the truncation error that (2.2) approximates to (2.1) is

$$TE_1 = O(\Delta t, \Delta x^2, \Delta y^2). \tag{2.14}$$

Next, using the same approach as in (2.14), the truncation errors that (2.5) approximates to (2.4) are given by

$$TE_2 = O(\Delta t, \Delta x^2, \Delta y^2). \tag{2.15}$$

Therefore, the error of numerical solutions $(u^n_{j+\frac{1}{2},k}, v^n_{j,k+\frac{1}{2}})$ $(1 \le n \le N, 0 \le j \le J, 0 \le k \le K)$ for Problem (I) obtained by (2.2) and (2.5) is that

$$\begin{aligned}|E_n(u^n_{j+\frac{1}{2},k}, v^n_{j,k+\frac{1}{2}})| &= \|(u(x_{j+\frac{1}{2}}, y_k, t_n), v(x_j, y_{k+\frac{1}{2}}, t_n)) - (u^n_{j+\frac{1}{2},k}, v^n_{j,k+\frac{1}{2}})\|_2 \\ &= O(\Delta t, \Delta x^2, \Delta y^2),\end{aligned} \tag{2.16}$$

which completes the proof of Theorem 1.

Thus, if Reynolds number Re, time step increment Δt, and spatial step increment Δx and Δy in the x-direction and y-direction are given, by solving (2.2) and (2.5) one could obtain $u^n_{j+\frac{1}{2},k}$ and $v^n_{j,k+\frac{1}{2}}$ $(0 \le j \le J, 0 \le k \le K, 1 \le n \le N)$.

Write $u^n_i = u^n_{j+\frac{1}{2},k}$ and $v^n_i = v^n_{j,k+\frac{1}{2}}$ $(i = k(J+1) + j + 1, m = JK, 1 \le i \le m, 0 \le j \le J, 0 \le k \le K, 1 \le n \le N)$. $L \times m$ group of values consisting of the ensemble $\{u^{n_l}_i, \ v^{n_l}_i\}^L_{l=1} (1 \le i \le m)$ (usually $L \ll N$), known as "snapshots" which is useful and of interest to us, are chosen from the $N \times m$ group of $\{u^n_i, \ v^n_i\}^N_{n=1} (1 \le i \le m)$.

Remark 1. When one computes actual problems, one may obtain the ensemble of snapshots from physical system trajectories via drawing samples of experiments and interpolation (or date assimilation). For example for a weather forecast, one can use the previous weather results to structure the ensemble of snapshots, then restructure the optimal basis for the ensemble of snapshots by the following SVD and POD, and finally combine with the POD projection to derive a reduced optimizing dynamical system. Thus, the situation of future weather change can be quickly simulated and the future weather change can be forecast, which is of major importance for actual real-life applications.

3. Reduced optimizing FDS based POD for Burgers equation

In this section, we first derive the POD basis to employ SVD, and then use the POD basis to develop a reduced optimizing FDS for Burgers equation.

3.1. Singular value decomposition and POD basis

The ensemble of snapshots $\{u^{n_l}_i, \ v^{n_l}_i\}^L_{l=1} (1 \le i \le m)$ can be expressed as two $m \times L$ matrices $\boldsymbol{A}_u$ and $\boldsymbol{A}_v$ as follows:

$$\boldsymbol{A}_u = \begin{pmatrix} u^{n_1}_1 & u^{n_2}_1 & \cdots & u^{n_L}_1 \\ u^{n_1}_2 & u^{n_2}_2 & \cdots & u^{n_L}_2 \\ \vdots & \vdots & \vdots & \vdots \\ u^{n_1}_m & u^{n_2}_m & \cdots & u^{n_L}_m \end{pmatrix}, \qquad \boldsymbol{A}_v = \begin{pmatrix} v^{n_1}_1 & v^{n_2}_1 & \cdots & v^{n_L}_1 \\ v^{n_1}_2 & v^{n_2}_2 & \cdots & v^{n_L}_2 \\ \vdots & \vdots & \vdots & \vdots \\ v^{n_1}_m & v^{n_2}_m & \cdots & v^{n_L}_m \end{pmatrix}. \tag{3.1}$$

In order to obtain optimal representation for $\boldsymbol{A}_u$ ($\boldsymbol{A}_v$ is similar), we employed SVD to research the FDS for Burgers equation, which is an important tool for finding optimal basis of optimization approximation. For matrix $\boldsymbol{A}_u \in R^{m\times L}$, there exists the SVD

$$\boldsymbol{A}_u = \boldsymbol{U}_u \begin{pmatrix} \boldsymbol{S}_u & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{V}_u^{\mathrm{T}}, \tag{3.2}$$

where $\boldsymbol{U}_u \in R^{m\times m}$ and $\boldsymbol{V}_u \in R^{L\times L}$ are all orthogonal matrices, $\boldsymbol{S}_u = \mathrm{diag}\{\sigma_{u1}, \sigma_{u2}, \ldots, \sigma_{u\ell}\} \in R^{\ell\times\ell}$ is the diagonal matrix corresponding to $\boldsymbol{A}_u$, and σ_{ui} $(i = 1, 2, \ldots, \ell)$ are the positive singular values. The matrices $\boldsymbol{U}_u = (\boldsymbol{\phi}_{u1}, \boldsymbol{\phi}_{u2}, \ldots, \boldsymbol{\phi}_{um}) \in R^{m\times m}$ and $\boldsymbol{V}_u = (\boldsymbol{\varphi}_{u1}, \boldsymbol{\varphi}_{u2}, \ldots, \boldsymbol{\varphi}_{uL}) \in R^{L\times L}$ contain the orthogonal eigenvectors to the $\boldsymbol{A}_u\boldsymbol{A}_u^{\mathrm{T}}$ and $\boldsymbol{A}_u^{\mathrm{T}}\boldsymbol{A}_u$, respectively. The columns of these eigenvector matrices are organized such that corresponding to the singular values σ_{ui} are comprised in $\boldsymbol{S}_u$ in a non-decreasing order. And the singular values of the decomposition are connected to the eigenvalues of the matrices $\boldsymbol{A}_u\boldsymbol{A}_u^{\mathrm{T}}$ and $\boldsymbol{A}_u^{\mathrm{T}}\boldsymbol{A}_u$ in a manner such that $\lambda_{ui} = \sigma^2_{ui}$ $(i = 1, 2, \ldots, \ell)$. Since the number of mesh points is far larger than that of transient moment points, i.e., $m \gg L$, that is also that the order m for matrix $\boldsymbol{A}_u\boldsymbol{A}_u^{\mathrm{T}}$ is far larger than the order L for matrix $\boldsymbol{A}_u^{\mathrm{T}}\boldsymbol{A}_u$. However, their null eigenvalues are identical, therefore, we may first solve the eigenequation corresponding to matrix $\boldsymbol{A}_u^{\mathrm{T}}\boldsymbol{A}_u$ to find the eigenvectors $\boldsymbol{\varphi}_{uj}$ $(j = 1, 2, \ldots, L)$, and then by the relationship

$$\boldsymbol{\phi}_{uj} = \frac{1}{\sigma_{uj}} \boldsymbol{A}_u \boldsymbol{\varphi}_{uj}, \quad j = 1, 2, \ldots, \ell, \tag{3.3}$$

we may obtain ℓ $(\ell \le L)$ eigenvectors corresponding to the non-null eigenvalues for matrix $\boldsymbol{A}_u\boldsymbol{A}_u^{\mathrm{T}}$.

Define matrix norm $\|\cdot\|_{\alpha,\beta}$ as $\|\boldsymbol{A}_u\|_{\alpha,\beta} = \sup_{\boldsymbol{x}\neq 0} \|\boldsymbol{A}\boldsymbol{x}\|_\alpha / \|\boldsymbol{x}\|_\beta$ (where $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are the norm of the vector). Let $\boldsymbol{A}_{M_u} = \sum_{i=1}^{M_u} \sigma_{ui}\boldsymbol{\phi}_{ui}\boldsymbol{\varphi}_{ui}^{\mathrm{T}}$, $\boldsymbol{\phi}_{ui}$ $(i = 1, 2, \ldots, M_u)$ and $\boldsymbol{\varphi}_{uj}$ $(j = 1, 2, \ldots, M_u)$ are M_u first column vectors of matrices $\boldsymbol{U}_u$ and $\boldsymbol{V}_u$, respectively. Then, by the relationship properties between spectral radius and $\|\cdot\|_{2,2}$ for the matrix, if $M_u < r = \mathrm{rank}\boldsymbol{A}_u$ $(r \leq \ell \leq L)$, then there is the following equation

$$\min_{\mathrm{rank}(\boldsymbol{B})\leq M_u} \|\boldsymbol{A}_u - \boldsymbol{B}\|_{2,2} = \|\boldsymbol{A}_u - \boldsymbol{A}_{M_u}\|_{2,2} = \sigma_{u(M_u+1)}, \tag{3.4}$$

which shows that $\boldsymbol{A}_{M_u}$ is an optimal representation of $\boldsymbol{A}_u$, i.e., $\boldsymbol{A}_{M_u}$ is an optimal approximation of $\boldsymbol{A}_u$ and the error is $\sigma_{u(M_u+1)} = \sqrt{\lambda_{u(M_u+1)}}$.

Denote the L column vectors of matrices $\boldsymbol{A}_u$ by $\boldsymbol{a}_u^l = (u_1^{n_l}, u_2^{n_l}, \ldots, u_m^{n_l})^{\mathrm{T}} (l = 1, 2, \ldots, L)$, and $\varepsilon_l (l = 1, 2, \ldots, L)$ by unit column vectors except that the lth component is 1, while the other components are 0. Then by the compatibility of the norm for matrices and vectors, we obtain that

$$\|\boldsymbol{a}_u^l - P_{M_u}(\boldsymbol{a}_u^l)\|_2 = \|(A_u - A_{M_u})\varepsilon_l\|_2 \leq \|A_u - A_{M_u}\|_{2,2}\|\varepsilon_l\|_2 = \sqrt{\lambda_{u(M_u+1)}}, \tag{3.5}$$

where $P_{M_u}(\boldsymbol{a}_u^l) = \sum_{j=1}^{M_u} (\boldsymbol{\phi}_{uj}, \boldsymbol{a}_u^l)\boldsymbol{\phi}_{uj}$, $(\boldsymbol{\phi}_{uj}, \boldsymbol{a}_u^l)$ are the canonical inner products for vector $\boldsymbol{\phi}_{uj}$ and vector $\boldsymbol{a}_u^l$. Inequality (3.5) shows that $P_{M_u}(\boldsymbol{a}_u^l)$ are the optimal approximations to $\boldsymbol{a}_u^l$, whose errors are all $\sqrt{\lambda_{u(M_u+1)}}$. Thus, a group of optimal basis is found in the construction of A_{M_u}. By the property of the eigenvector, it is well known that $\boldsymbol{\Phi}_u = (\boldsymbol{\phi}_{u1}, \boldsymbol{\phi}_{u2}, \ldots, \boldsymbol{\phi}_{M_u})(M_u \ll L)$ is an orthonormal matrix and $\{\boldsymbol{\phi}_{uj}\}_{j=1}^{M_u}$ is a group of optimal basis, which is known as a group of POD basis.

By the same approach as the above (3.5), if $\boldsymbol{a}_v^l = (v_1^{n_l}, v_2^{n_l}, \ldots, v_m^{n_l})^{\mathrm{T}} (l = 1, 2, \ldots, L)$ are the L column vectors of matrices $\boldsymbol{A}_v$, then $P_{M_v}(\boldsymbol{a}_v^l) = \sum_{j=1}^{M_v} (\boldsymbol{\phi}_{vj}, \boldsymbol{a}_v^l)\boldsymbol{\phi}_{vj}$ are the optimal approximations to $\boldsymbol{a}_v^l$, whose errors are all $\sqrt{\lambda_{v(M_v+1)}}$, i.e.,

$$\|\boldsymbol{a}_v^l - P_{M_v}(\boldsymbol{a}_v^l)\|_2 \leq \sqrt{\lambda_{v(M_v+1)}}, \tag{3.6}$$

where $\lambda_{v(M_v+1)}$ is the $(M_v + 1)$th eigenvalue for $\boldsymbol{A}_v\boldsymbol{A}_v^{\mathrm{T}}$, and $\boldsymbol{\Phi}_v = (\boldsymbol{\phi}_{v1}, \boldsymbol{\phi}_{v2}, \ldots, \boldsymbol{\phi}_{vM_v})$ is an orthonormal matrix and $\{\boldsymbol{\phi}_{vj}\}_{j=1}^{M_v}$ is a group of optimal basis corresponding to $\boldsymbol{A}_v$.

3.2. Reduced optimizing FDS based on POD for Burgers equation

In the following, we use the POD basis to derive a reduced optimizing FDS for Burgers equation.
Write

$$\begin{aligned} \boldsymbol{u}_m(t) &= (u_1(t), u_2(t), \ldots, u_m(t))^{\mathrm{T}}, \\ \boldsymbol{v}_m(t) &= (v_1(t), v_2(t), \ldots, v_m(t))^{\mathrm{T}}, \end{aligned} \tag{3.7}$$

where $u_i = u_{j+\frac{1}{2},k}$ and $v_i = v_{j,k+\frac{1}{2}}$ $(1 \leq i \leq m, i = k(J+1) + j + 1, m = KJ, 0 \leq j \leq J, 0 \leq k \leq K)$. Thus, (2.2) and (2.5) are written as the following vector formulation

$$\left(\boldsymbol{u}_m^{n+1}, \boldsymbol{v}_m^{n+1}\right)^{\mathrm{T}} = \left(\boldsymbol{u}_m^n, \boldsymbol{v}_m^n\right)^{\mathrm{T}} + \Delta t\tilde{\boldsymbol{F}}(\boldsymbol{u}_m^n, \boldsymbol{v}_m^n), \quad 0 \leq n \leq N, \tag{3.8}$$

where $\tilde{\boldsymbol{F}}(\boldsymbol{u}_m^n, \boldsymbol{v}_m^n) = (\tilde{F}_1(\boldsymbol{u}_m^n, \boldsymbol{v}_m^n), \tilde{F}_2(\boldsymbol{u}_m^n, \boldsymbol{v}_m^n))^{\mathrm{T}}$ is the vector function following from (2.2) and (2.5). Put

$$\left(\boldsymbol{u}_m^n, \boldsymbol{v}_m^n\right)^{\mathrm{T}} = \left(\boldsymbol{\Phi}_u\boldsymbol{\alpha}_{M_u}^n, \boldsymbol{\Phi}_v\boldsymbol{\beta}_{M_v}^n\right)^{\mathrm{T}}, \tag{3.9}$$

where $\boldsymbol{u}_m^n = (u_1^n, u_2^n, \ldots, u_m^n)^{\mathrm{T}}$ and $\boldsymbol{v}_m^n = (v_1^n, v_2^n, \ldots, v_m^n)^{\mathrm{T}}$. Inserting (3.9) into (3.8) and noting that $\boldsymbol{\Phi}_u$ and $\boldsymbol{\Phi}_v$ are orthogonal matrices, we may obtain a reduced optimizing model which has $M_u + M_v (M_u, M_v \ll L \ll m)$ unknown values:

$$\begin{pmatrix} \boldsymbol{\alpha}_{M_u}^{n+1} \\ \boldsymbol{\beta}_{M_v}^{n+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_{M_u}^{n} \\ \boldsymbol{\beta}_{M_v}^{n} \end{pmatrix} + \Delta t \begin{pmatrix} \boldsymbol{\Phi}_u^{\mathrm{T}}\tilde{F}_1(\boldsymbol{\Phi}_u\boldsymbol{\alpha}_{M_u}^n, \boldsymbol{\Phi}_v\boldsymbol{\beta}_{M_v}^n) \\ \boldsymbol{\Phi}_v^{\mathrm{T}}\tilde{F}_2(\boldsymbol{\Phi}_u\boldsymbol{\alpha}_{M_u}^n, \boldsymbol{\Phi}_v\boldsymbol{\beta}_{M_v}^n) \end{pmatrix}, \tag{3.10}$$

where $n = 0, 1, 2, \ldots, N$, initial values are $\boldsymbol{\alpha}_{M_u}^0 = \boldsymbol{\Phi}_u^{\mathrm{T}}\boldsymbol{u}_m^0$ and $\boldsymbol{\beta}_{M_v}^0 = \boldsymbol{\Phi}_v^{\mathrm{T}}\boldsymbol{v}_m^0$.

After one has obtained $\boldsymbol{\alpha}_{M_u}^n$ and $\boldsymbol{\beta}_{M_v}^n$ from (3.10), one obtains the POD optimal solutions which are written as $\boldsymbol{u}_i^{*n} = \boldsymbol{\Phi}_u\boldsymbol{\alpha}_{M_u}^n$ and $v_i^{*n} = \boldsymbol{\Phi}_v\boldsymbol{\beta}_{M_v}^n$ for Problem (I) by (3.9). Thus, we get the optimal numerical solutions which are written as $(u_{j+\frac{1}{2},k}^{*n}, v_{j,k+\frac{1}{2}}^{*n})$ $(0 \leq j \leq J-1, 0 \leq k \leq K-1, 0 \leq n \leq N)$ for Problem (I), where $u_{j+\frac{1}{2},k}^{*n} = u_i^{*n}$, $v_{j,k+\frac{1}{2}}^{*n} = v_i^{*n}$ $(j = i - 1 - k(J+1) \geq 0, 1 \leq i \leq m = KJ, 0 \leq k \leq K-1, 0 \leq n \leq N)$.

Remark 2. Formula (3.10) with (3.9) is the reduced optimizing FDS based on SVD and POD for Problem (I), since it only includes $(M_u + M_v) \times N$ $(M_u, M_v \ll L \ll m)$ degrees of freedom while the usual FDS (2.2) and (2.5) includes $2m \times N$. When one computes actual problems, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation), then restructures the POD basis for the ensemble of snapshots, and finally combines it with POD projection to derive a reduced optimizing FDS, i.e., one needs only to solve the above formula (3.10) with (3.9) which has only a few degrees of freedom, but it is unnecessary to solve the usual FDS (2.2) and (2.5). Thus, the computational load and memory requirements can be greatly alleviated.

4. Error analysis of reduced optimizing FDS

This section is devoted to discussing the error estimates of the reduced optimizing FDS (3.9) and (3.10) for Problem (I). Let

$$\mathcal{X}_u = \mathrm{span}\{\boldsymbol{\phi}_{u1}, \boldsymbol{\phi}_{u2}, \ldots, \boldsymbol{\phi}_{M_u}\},\\ \mathcal{X}_v = \mathrm{span}\{\boldsymbol{\phi}_{v1}, \boldsymbol{\phi}_{v2}, \ldots, \boldsymbol{\phi}_{vM_v}\}. \tag{4.1}$$

Then, for column vectors $\boldsymbol{a}_u^l$ $(1 \le l \le L)$ of $\boldsymbol{A}_u$, by (3.5) we have that $\boldsymbol{a}_u^l = \boldsymbol{u}_m^{n_l}$, and there is a $P_{M_u}(\boldsymbol{u}_m^{n_l}) = P_{M_u}(\boldsymbol{a}_u^l) = \sum_{j=1}^{M_u}(\boldsymbol{\phi}_{uj}, \boldsymbol{a}_u^l)\boldsymbol{\phi}_{uj} = \sum_{j=1}^{M_u}(\boldsymbol{\phi}_{uj}, \boldsymbol{u}_m^{n_l})\boldsymbol{\phi}_{uj} \in \mathcal{X}_u$ such that

$$\|\boldsymbol{u}_m^{n_l} - P_{M_u}(\boldsymbol{u}_m^{n_l})\|_2 \le \sqrt{\lambda_{u(M_u+1)}}, \quad 1 \le l \le L. \tag{4.2}$$

While $n \in \{n_1, n_2, \ldots, n_L\}$, $\boldsymbol{u}_m^{*n} = P_{M_u}(\boldsymbol{u}_m^n) = \sum_{j=1}^{M_u}(\boldsymbol{\phi}_{uj}, \boldsymbol{u}_m^n)\phi_{uj}$ obtained by (3.9) and (3.10), therefore, we obtain that

$$\|\boldsymbol{u}_m^n - \boldsymbol{u}_m^{*n}\|_2 \le \sqrt{\lambda_{u(M_u+1)}}, \quad n \in \{n_1, n_2, \ldots, n_L\}. \tag{4.3}$$

Using the same approach as (4.3), we could obtain that

$$\|\boldsymbol{v}_m^n - \boldsymbol{v}_m^{*n}\|_2 \le \sqrt{\lambda_{v(M_v+1)}}, \quad \text{if } n \in \{n_1, n_2, \ldots, n_L\}. \tag{4.4}$$

When $n \notin \{n_1, n_2, \ldots, n_L\}$, we may as well let $t_n \in (t_{n_l}, t_{n_{l+1}})$ and t_n be the nearest point to t_{n_l}. Comparing (3.9) and (3.10) with (3.8), (3.9) and (3.10) can be written similarly in forms of (2.2) and (2.5) as follows

$$u^{*n+1}_{j+\frac{1}{2},k} = u^{*n}_{j+\frac{1}{2},k} - F^{*n}_{j+\frac{1}{2},k} + \Delta t f^n_{1,j+\frac{1}{2},k}, \tag{4.5}$$

where

$$\begin{aligned} F^{*n}_{j+\frac{1}{2},k} = {} & \frac{\Delta t}{\Delta x}u^{*n}_{j+\frac{1}{2},k}(u^{*n}_{j+1,k} - u^{*n}_{j,k}) + \frac{\Delta t}{\Delta y}v^{*n}_{j+\frac{1}{2},k}(u^{*n}_{j+\frac{1}{2},k+\frac{1}{2}} - u^{*n}_{j+\frac{1}{2},k-\frac{1}{2}}) \\ & - \frac{\Delta t}{Re}\left[\frac{u^{*n}_{j+\frac{1}{2},k-1} - 2u^{*n}_{j+\frac{1}{2},k} + u^{*n}_{j+\frac{1}{2},k+1}}{\Delta y^2} + \frac{u^{*n}_{j-\frac{1}{2},k} - 2u^{*n}_{j+\frac{1}{2},k} + u^{*n}_{j+\frac{3}{2},k}}{\Delta x^2}\right] \end{aligned} \tag{4.6}$$

and

$$v^{*n+1}_{j,k+\frac{1}{2}} = v^{*n}_{j,k+\frac{1}{2}} - G^{*n}_{j,k+\frac{1}{2}} + \Delta t f^n_{2,j,k+\frac{1}{2}}, \tag{4.7}$$

where

$$\begin{aligned} G^{*n}_{j,k+\frac{1}{2}} = {} & \frac{\Delta t}{\Delta x}u^{*n}_{j,k+\frac{1}{2}}(v^{*n}_{j+\frac{1}{2},k+\frac{1}{2}} - v^{*n}_{j-\frac{1}{2},k+\frac{1}{2}}) + \frac{\Delta t}{\Delta y}v^{*n}_{j,k+\frac{1}{2}}(v^{*n}_{j,k+1} - v^{*n}_{j,k}) \\ & - \frac{\Delta t}{Re}\left[\frac{v^{*n}_{j-1,k+\frac{1}{2}} - 2v^{*n}_{j,k+\frac{1}{2}} + v^{*n}_{j+1,k+\frac{1}{2}}}{\Delta x^2} + \frac{v^{*n}_{j,k-\frac{1}{2}} - 2v^{*n}_{j,k+\frac{1}{2}} + v^{*n}_{j,k+\frac{3}{2}}}{\Delta y^2}\right]. \end{aligned} \tag{4.8}$$

If $|u^n_{j+\frac{1}{2},k}|, |v^n_{j+\frac{1}{2},k}|, |u^n_{j,k+\frac{1}{2}}|, |v^n_{j,k+\frac{1}{2}}|, |u^{*n}_{j+\frac{1}{2},k}|, |v^{*n}_{j+\frac{1}{2},k}|, |u^{*n}_{j,k+\frac{1}{2}}|$, and $|v^{*n}_{j,k+\frac{1}{2}}|$ are all bounded, then by subtracting (4.5) and (4.8) from (2.2) and (2.5), respectively, and writing the vector we obtain that

$$\|\boldsymbol{u}_m^{n+1} - \boldsymbol{u}_m^{*n+1}\|_2 + \|\boldsymbol{v}_m^{n+1} - \boldsymbol{v}_m^{*n+1}\|_2 \le M(\|\boldsymbol{u}_m^n - \boldsymbol{u}_m^{*n}\|_2 + \|\boldsymbol{v}_m^n - \boldsymbol{v}_m^{*n}\|_2), \tag{4.9}$$

where $M = 1 + C\Delta t/\min(\Delta x, \Delta y, Re\Delta x^2, Re\Delta y^2)$, C is a constant independent of Δt, Δx^2, and Δy^2. Summing (4.9) from $n_l, n_l + 1, \ldots, n - 1$ can yield that

$$\begin{aligned} \|\boldsymbol{u}_m^n - \boldsymbol{u}_m^{*n}\|_2 + \|\boldsymbol{v}_m^n - \boldsymbol{v}_m^{*n}\|_2 \le {} & \|\boldsymbol{u}_m^{n_l} - \boldsymbol{u}_m^{*n_l}\|_2 + \|\boldsymbol{v}_m^{n_l} - \boldsymbol{v}_m^{*n_l}\|_2 \\ & + C\Delta t/\min(\Delta x, \Delta y, Re\Delta x^2, Re\Delta y^2)\sum_{j=n_l}^{n-1}(\|\boldsymbol{u}_m^j - \boldsymbol{u}_m^{*j}\|_2 + \|\boldsymbol{v}_m^j - \boldsymbol{v}_m^{*j}\|_2). \end{aligned} \tag{4.10}$$

If $\Delta t = O(\Delta x^2, \Delta y^2)$, $Re^{-2} \le \Delta t$, by the discrete Gronwall Lemma (see [30,31]), we get that

$$\|\boldsymbol{u}_m^n - \boldsymbol{u}_m^{*n}\|_2 + \|\boldsymbol{v}_m^n - \boldsymbol{v}_m^{*n}\|_2 \le (\|\boldsymbol{u}_m^{n_l} - \boldsymbol{u}_m^{*n_l}\|_2 + \|\boldsymbol{v}_m^{n_l} - \boldsymbol{v}_m^{*n_l}\|_2)\exp[C\Delta t^{\frac{1}{2}}(n - n_l - 1)]. \tag{4.11}$$

If t_l $(1 \le l \le L)$ are uniformly chosen from t_n $(1 \le l \le N)$, then $(n - n_l) \le N/(2L)$. If $L^{-2} = O(\Delta t)$, we obtain from (4.11) and (4.3) and (4.4) that

$$\|\boldsymbol{u}_m^n - \boldsymbol{u}_m^{*n}\|_2 + \|\boldsymbol{v}_m^n - \boldsymbol{v}_m^{*n}\|_2 \le C(\sqrt{\lambda_{u(M_u+1)}} + \sqrt{\lambda_{v(M_v+1)}}). \tag{4.12}$$

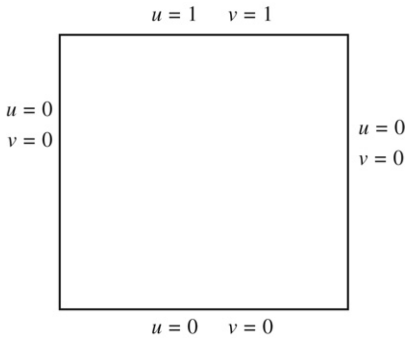


Fig. 2. Physics model of the cavity flows: $t = 0$, i.e., $n = 0$ initial values on boundary.

Synthesizing the above discussion could get the following result.

Theorem 2. *Let* $(\boldsymbol{u}_m^n, \boldsymbol{v}_m^n)(n = 1, 2, \ldots, N)$ *be vectors constituted with solutions of FDS (2.2) and (2.5),* $(\boldsymbol{u}_m^{*n}\boldsymbol{v}_m^{*n})$ *be the vectors of the reduced optimizing FDS (3.9) and (3.10), if* $n \in \{n_1, n_2, \ldots, n_L\}$, *then the following error estimates hold*

$$\|\boldsymbol{u}_m^n - \boldsymbol{u}_m^{*n}\|_2 \le \sqrt{\lambda_{u(M_u+1)}}, \qquad \|\boldsymbol{v}_m^n - \boldsymbol{v}_m^{*n}\|_2 \le \sqrt{\lambda_{v(M_v+1)}}. \tag{4.13}$$

Moreover, if $n \notin \{n_1, n_2, \ldots, n_L\}$, $\Delta t = O(\Delta x^2, \Delta y^2)$, $Re^{-2} \le \Delta t$, *and* $|u^n_{j+\frac{1}{2},k}|$, $|v^n_{j+\frac{1}{2},k}|$, $|u^n_{j,k+\frac{1}{2}}|$, $|v^n_{j,k+\frac{1}{2}}|$, $|u^{*n}_{j+\frac{1}{2},k}|$, $|v^{*n}_{j+\frac{1}{2},k}|$, $|u^{*n}_{j,k+\frac{1}{2}}|$, *and* $|v^{*n}_{j,k+\frac{1}{2}}|$ *are all bounded, snapshots* $\{u^{n_l}_{j+\frac{1}{2},k}, v^{n_l}_{j+\frac{1}{2},k}\}_{l=1}^{L}$ *are uniformly chosen from* $\{u^n_{j+\frac{1}{2},k}, v^n_{j+\frac{1}{2},k}\}_{n=1}^{N}$, $L^{-2} = O(\Delta t)$, *then the following error estimates hold*

$$\|\boldsymbol{u}_m^n - \boldsymbol{u}_m^{*n}\|_2 + \|\boldsymbol{v}_m^n - \boldsymbol{v}_m^{*n}\|_2 \le C(\sqrt{\lambda_{u(M_u+1)}} + \sqrt{\lambda_{v(M_v+1)}}) \tag{4.14}$$

where $N = T/\Delta t$ *and L is the number of snapshots.*

Note that the absolute value of each component of a vector is not more than any of its norm. Combining Theorems 1 and 2 could yield the following result.

Theorem 3. *Under the assumptions of Theorem 2, the following error estimates hold*

$$\begin{aligned}&|u(x_{j+\frac{1}{2}}, y_k, t_n) - u^{*n}_{j+\frac{1}{2},k}| + |v(x_j, y_{k+\frac{1}{2}}, t_n) - v^{*n}_{j,k+\frac{1}{2}}|\\ &\quad \le O(\sqrt{\lambda_{u(M_u+1)}} + \sqrt{\lambda_{v(M_v+1)}}, \Delta t, \Delta x^2, \Delta y^2), \quad 1 \le n \le N.\end{aligned} \tag{4.15}$$

Remark 3. The conditions $\Delta t = O(\Delta x^2, \Delta y^2)$ and $Re^{-2} \le \Delta t$ are reasonable. The condition $L^{-2} = O(\Delta t)$ in Theorem 2 implies $L^2 = O(N)$ and shows the relation between the number L of snapshots and the number N of all time instances. Therefore, it is unnecessary to take total transient solutions at all time instances t_n as snapshots (see [27,28]). Theorems 2 and 3 have presented the error estimates between the solution of the reduced optimizing FDS (3.9) and (3.10) and the solution of the usual FDS (2.2) and (2.5), and Problem (I), respectively. Since our methods employ some FDS solutions $(u^n_{j+\frac{1}{2},k}, v^n_{j,k+\frac{1}{2}})(n = 1, 2, \ldots, N)$ for Problem (I) as assistant analysis, the error estimates in Theorem 3 are correlated to the gridding scale Δx and Δy, and time step size Δt. However, when one computes actual problems, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). Thus, the assistant $(u^n_{j+\frac{1}{2},k}, v^n_{j,k+\frac{1}{2}})$ $(n = 1, 2, \ldots, N)$ could be substituted with the interpolation functions of experimental and previous results.

5. Some numerical experiments

In this section, we present some numerical examples of the physics model of the cavity flows with the reduced optimizing FDS (3.9) and (3.10) validating the feasibility and efficiency of the POD method.

Let the side length of the cavity be 1 (see Fig. 2). We take the spatial step increment as $\Delta x = \Delta y = \frac{1}{32}$ and the time step increment as $\Delta t = 0.001$. Except that u is equal to 1 on the upper boundary, other initial values, boundary values, and (f_1, f_2) are all taken as 0. Put $Re = 2000$ or 5000.

We obtain 20 values (i.e., snapshots) outputting at time $t = 10, 20, 30, \ldots, 200$ by solving the usual FDS, i.e., Eqs. (2.2) and (2.5). It is shown by computing that eigenvalues $\sqrt{\lambda_{u6}} + \sqrt{\lambda_{v6}} \le 3 \times 10^{-3}$.

When $t = 200$, we obtain the solutions of the reduced optimizing FDS (3.9) and (3.10) based POD method depicted graphically in Figs. 3 and 4 on the right-hand side used 5 optimal POD bases if $Re = 2000$ and also used 5 optimal POD bases

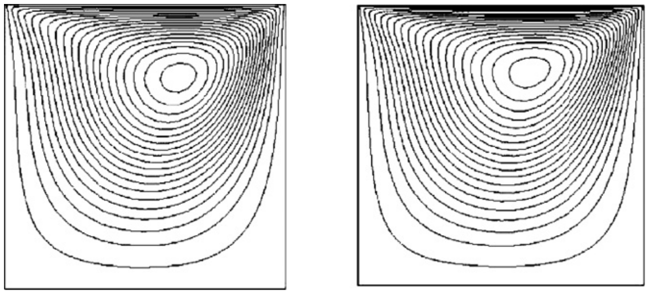


Fig. 3. When $Re = 2000$, velocity stream line figure for the usual FDS solution (on the left-hand side figure) and when $M_u = M_v = 5$, solution of the reduced FDS based on POD (on the right-hand side figure).

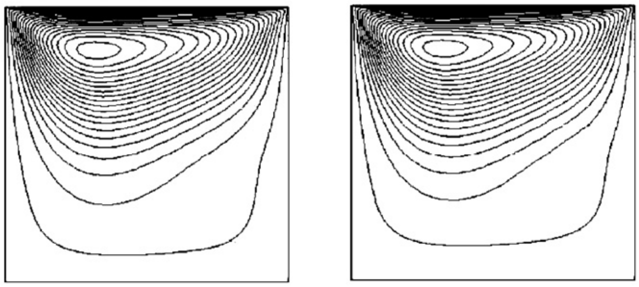


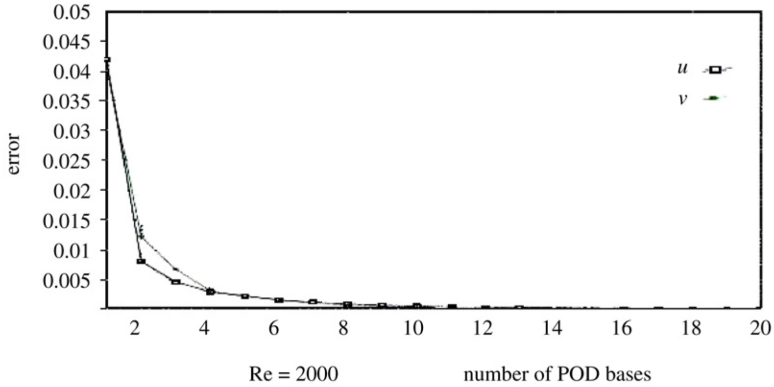
Fig. 4. When $Re = 5000$, velocity stream line figure for the usual FDS solutions (on the left-hand side figure) and when $M_u = M_v = 5$, solution of the reduced FDS based on POD (on the right-hand side figure).

if $Re = 5000$, but the solutions obtained with the usual FDS, i.e., (2.2) and (2.5) are depicted graphically in Figs. 3 and 4 on the left-hand side (since these figures are equal to solutions obtained with 20 bases, they are also known as the figures of solution with full bases).

Fig. 5 shows the errors between solutions obtained with different numbers of optimal POD bases and solutions obtained with full bases. Comparing the usual FDS, i.e., (2.2) and (2.5) with the reduced optimizing FDS (3.10) with (3.9) based POD method containing five optimal bases implementing 3000 times numerical simulation computations, we find that time-consuming calculations with the usual FDS, i.e., (2.2) and (2.5) are five minutes, while those with the reduced optimizing FDS (3.9) and (3.10) with five time-consuming optimal bases expend only three seconds, i.e., the usual FDS consumes 120 times as much computing time as the reduced optimizing FDS (3.9) and (3.10) with five time-consuming optimal bases, and the errors between their solutions are not more than 3×10^{-3}. Though our examples are in a sense recomputing what we have already computed by the usual FDS, when we compute actual problems, we may structure the snapshots and POD basis with interpolation or data assimilation by drawing samples from experiments, then solve directly the reduced optimizing FDS (3.10) with (3.9), while it is unnecessary to solve the usual FDS, thus, the time-consuming calculations and resource demands in the computational process will be greatly saved. It is also shown that finding the approximate solutions for Burgers equation with the reduced optimizing FDS (3.9) and (3.10) is very effective and that the results for numerical examples are consistent with those theoretical results.

6. Conclusions

In this paper, we have employed the SVD and the POD techniques to derive a reduced optimizing FDS for Burgers equation. We first compile ensembles of data from transient solutions computing an equation system derived with the usual FDS for Burgers equation, while in actual applications, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). Next we employ SVD to deal with ensembles of data obtaining the POD basis. And then the unknowns of the usual FDS are substituted with the linear combination of POD basis to derive the reduced optimizing FDS for Burgers equation. Since there are few bases in the POD basis, the reduced FDS based on POD is optimal. We have proceeded to derive error estimates between the our reduced optimizing finite difference approximate solutions and the usual finite difference solution which are consistent with the theoretical error results, thus validating both the feasibility and efficiency of our reduced optimizing FDS. Future work in this area will aim to extend the reduced optimizing FDS, implementing it for a realistic atmosphere quality forecast system and more complicated PDEs. From theoretical analysis and numerical examples, we have shown that the reduced optimizing FDS presented herein has extensive perspective applications.



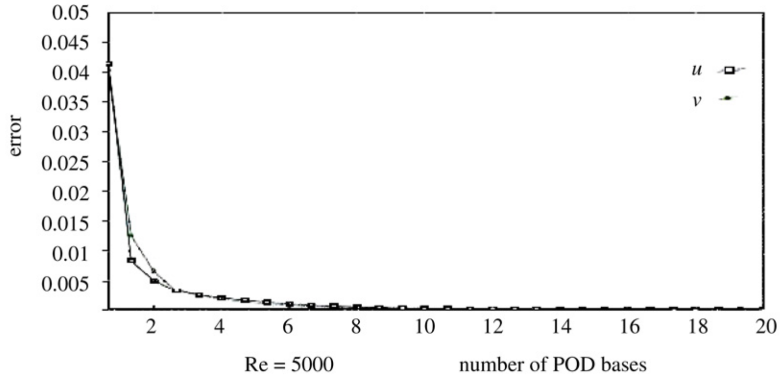


Fig. 5. Error for $Re = 2000$ on upper figure, error for $Re = 5000$ on lower figure.

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