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On local Hermitian and skew-Hermitian splitting iteration methods for generalized saddle point problems☆

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ABSTRACT

In this paper, we first present a local Hermitian and skew-Hermitian splitting (LHSS) iteration method for solving a class of generalized saddle point problems. The new method converges to the solution under suitable restrictions on the preconditioning matrix. Then we give a modified LHSS (MLHSS) iteration method, and further extend it to the generalized saddle point problems, obtaining the so-called generalized MLHSS (GMLHSS) iteration method. Numerical experiments for a model Navier–Stokes problem are given, and the results show that the new methods outperform the classical Uzawa method and the inexact parameterized Uzawa method.

1. Introduction

We consider the solution of systems of linear equations of the block 2×2 form

$$\begin{bmatrix} A & B^* \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \text{or } \hat{\mathcal{A}} u = \hat{b}, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times m}$, $x, f \in \mathbb{C}^n$, $y, g \in \mathbb{C}^m$, and $m \leq n$. We further assume that the matrices A, B, and C are large and sparse, see [1–3].

The linear system (1.1) arises in a variety of scientific and engineering applications, including computational fluid dynamics, mixed finite element of elliptic PDEs, constrained optimization, constrained least-squares problem, and so on. In a large number of these applications such as the constrained optimization, A is Hermitian and positive definite, B has full row-rank, (namely, $r(B) = m$), and $C = 0$. In this case, the linear system (1.1) is called a saddle point problem, which has been studied in many papers on iterative methods, such as Uzawa-type methods [4–8], HSS iteration methods [9–15], preconditioned Krylov subspace iteration methods [16–18,3], restrictively preconditioned conjugate gradient methods [19–21]. However, there are other situations, most notably the numerical solution of the Navier–Stokes equations of fluid dynamics, where $A \neq A^*$, and its Hermitian part $H := \frac{1}{2}(A + A^*)$ is positive definite, $r(B) = m$, and C is Hermitian and positive semi-definite. In this case, the linear system (1.1) is called a generalized saddle point problem, and, specially, when $C = 0$, we call it a non-Hermitian saddle point problem. Numerical iterative methods for the generalized saddle point problem has been studied in many papers, including Uzawa-type methods[4,5,22,23], HSS iteration methods [9–13,24],

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preconditioned Krylov subspace iteration methods [25–27,3,8,28,29], and restrictively preconditioned conjugate gradient methods [19–21].

In this paper, we will focus on the numerical solution for the generalized saddle point problem and assume that the Hermitian part H of the non-Hermitian matrix A is dominant. It has been studied that the Hermitian part of $\hat{\mathcal{A}}$ is indefinite and, therefore, $\hat{\mathcal{A}}$ has eigenvalues on both sides of the imaginary axis. Such eigenvalue distributions are generally considered unfavorable for solutions by simple iterative methods. Therefore, instead of solving (1.1), Benzi and Golub [25] solved the following equivalent system

$$\begin{bmatrix} A & B^* \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix}, \quad \text{or } \mathcal{A}u = b, \tag{1.2}$$

and the coefficient matrix of (1.2) has the following desirable properties.

Lemma 1.1 ([25]). *Let $\mathcal{A} \in \mathbb{C}^{(m+n)\times(m+n)}$ be the coefficient matrix defined in (1.2). Assume that $H := \frac{1}{2}(A + A^*)$ is positive definite, B has full row-rank, C is Hermitian and positive semi-definite, and $ker(H) \bigcap ker(B) = 0$, where $ker(\cdot)$ denotes the null-space of the corresponding matrix. Let $\sigma(\mathcal{A})$ denote the spectrum of $\mathcal{A}$ and $\lambda \in \sigma(\mathcal{A})$ be an eigenvalue of $\mathcal{A}$. Then*

1. *$\mathcal{A}$ is nonsingular;*
2. *$\mathcal{A}$ is semi-positive: $\mathrm{Re}(v^*\mathcal{A}v) \geq 0$ for all $v \in \mathbb{C}^{(m+n)}$;*
3. *$\mathcal{A}$ is positive stable: $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \sigma(\mathcal{A})$, where $\mathrm{Re}(\lambda)$ denotes the real part of the complex number λ.*

Thus by changing the sign of the last m equations in (1.1), we can gain the positive definiteness. Then by appropriate translation, the equivalent generalized saddle point system can be equivalently seen as the non-Hermitian saddle point system. This will be discussed in Section 3 in detail. It is also interesting to notice that several approaches which have been devised for the Hermitian case, have also been explored to solve the non-Hermitian case, as in [4,28]. So under the condition of positive definiteness of the coefficient matrix of (1.2), by constructing special splitting we obtain a new iterative method which is similar to the method in [4,30] for non-Hermitian saddle point problem. Moreover, this new method can also be extended to the generalized saddle point problem.

The remainder of the paper is arranged as follows. In Section 2, from the Uzawa algorithm and the method given in [30] we present new iterative methods for non-Hermitian saddle point problems. Then by the splitting preconditioning, we deduce that the equivalent generalized saddle point problem is just the non-Hermitian saddle point problem and extend the new iterative methods to the generalized saddle point problem in Section 3. In Section 4, numerical experiments for a model Navier–Stokes problem are presented. The numerical results show that our new methods are powerful and better than those of the classical Uzawa method and an inexact parameterized Uzawa method.

2. Iteration methods

2.1. LHSS iteration

In this subsection, we generalize the classical Uzawa method to the non-Hermitian saddle point problem. At the beginning, we consider the matrix $\mathcal{A}$ defined in (1.2), with $C = 0$, and make the following special splitting:

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} = \begin{bmatrix} H & 0 \\ -B & Q_2 \end{bmatrix} - \begin{bmatrix} -S & -B^* \\ 0 & Q_2 \end{bmatrix},$$

where $H := \frac{1}{2}(A + A^*)$ and $S := \frac{1}{2}(A - A^*)$ are the Hermitian and the skew-Hermitian parts of A, respectively. In fact, $A = H + S$ induces the Hermitian and the skew-Hermitian splitting of the matrix A, see [9] and [14,15]. As assumed in Section 1, H is nonsingular. We choose Q_2 to be a Hermitian positive definite matrix ($Q_2 \in \mathbb{C}^{m\times m}$). By this special splitting, we obtain a new iterative method, called the local Hermitian and skew-Hermitian splitting (LHSS) iteration method, which is defined as follows:

$$\begin{bmatrix} H & 0 \\ -B & Q_2 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} -S & -B^* \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f \\ -g \end{bmatrix}.$$

The corresponding computational process is described below.

Algorithm 2.1 (*LHSS Iteration Method*).

$$\begin{cases} x_{n+1} = x_n + H^{-1}(f - Ax_n - B^*y_n), \\ y_{n+1} = y_n + Q_2^{-1}(Bx_{n+1} - g). \end{cases}$$

It is evident that the splitting given above is a special case of the parameterized inexact Uzawa (PIU) splitting used in [5]. Thus, the LHSS method is a special case of the PIU method. If $H = A$, $Q_2 = \frac{1}{\delta}I$, where δ is a relaxation parameter and I is the identity matrix, then the above method becomes the classical Uzawa method.

In the following, we deduce the convergence property for the LHSS iteration. Note that the iteration matrix of the LHSS iteration is

$$\Gamma = \begin{bmatrix} H & 0 \\ -B & Q_2 \end{bmatrix}^{-1} \begin{bmatrix} -S & -B^* \\ 0 & Q_2 \end{bmatrix}. \tag{2.1}$$

Let $\rho(\Gamma)$ denote the spectral radius of Γ. Then the LHSS iteration converges if and only if $\rho(\Gamma) < 1$. Let λ be an eigenvalue of Γ and $\binom{u}{v}$ be the corresponding eigenvector. Then we have

$$\begin{bmatrix} -S & -B^* \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} H & 0 \\ -B & Q_2 \end{bmatrix} \begin{bmatrix} \lambda u \\ \lambda v \end{bmatrix},$$

or equivalently,

$$(\lambda H + S)u + B^* v = 0, \tag{2.2}$$

$$\lambda B u + (1 - \lambda) Q_2 v = 0. \tag{2.3}$$

To get a convergence condition, we first assume that $\lambda \neq 0$ and give some lemmas.

Lemma 2.1. *Let A be a non-Hermitian matrix, with the Hermitian part $H := \frac{1}{2}(A + A^*)$ being positive definite, and the matrix B has full rank ($r(B) = m$). Let Γ be defined as in (2.1). If λ is an eigenvalue of the matrix Γ, then $\lambda \neq 1$.*

Proof. If $\lambda = 1$ and $\binom{u}{v}$ is the corresponding eigenvector, then from (2.2) and (2.3) we have

$$\begin{cases} Au + B^* v = 0, \\ Bu = 0. \end{cases} \tag{2.4}$$

It is easy to get that the coefficient matrix of (2.4) is nonsingular. Hence $u = 0$ and $v = 0$, which contradicts the assumption that $\binom{u}{v}$ is an eigenvector of the iteration matrix Γ. So $\lambda \neq 1$. □

Lemma 2.2. *If S is a skew-Hermitian matrix, then $\mathrm{i} \cdot S$ (i is the imaginary unit) is a Hermitian matrix and $u^* S u$ is a purely imaginary number or zero for all $u \in \mathbb{C}^n$. In particular, if S is a skew-symmetric matrix, then $u^* S u = 0$ for all $u \in \mathbb{C}^n$.*

Lemma 2.3. *Let A be a non-Hermitian matrix with the positive definite Hermitian part $H := \frac{1}{2}(A + A^*)$, and the skew-Hermitian part $S := \frac{1}{2}(A - A^*)$. Let the matrix B have full row-rank. If $\binom{u}{v}$ is an eigenvector of the matrix Γ corresponding to the eigenvalue λ, then $u \neq 0$. Moreover, if $v = 0$, then $|\lambda| < 1$.*

Proof. If $u = 0$, then from (2.2) we have $B^* v = 0$. Because B has full row-rank, we have $v = 0$, which contradicts the assumption that $\binom{u}{v}$ is an eigenvector. Therefore, $u \neq 0$.

If $v = 0$, then from (2.2) we have

$$(\lambda H + S)u = 0 \quad \text{and} \quad \lambda u^* H u + u^* S u = 0. \tag{2.5}$$

Define

$$\lambda = \alpha + \mathrm{i} \cdot \beta, \quad a = \frac{u^* H u}{u^* u}, \quad -b = \frac{u^* \mathrm{i} \cdot S u}{u^* u}.$$

Then we have from (2.5)

$$a\alpha + i \cdot (a\beta + b) = 0.$$

The real part is satisfied for $\alpha = 0$, since $a \neq 0$. It then follows that

$$\beta = -\frac{b}{a}.$$

By the assumption that the Hermitian part H of the non-Hermitian matrix A is dominant, we have $|b| < |a|$. Therefore

$$|\lambda| = |\alpha + i \cdot \beta| = \left| \frac{b}{a} \right| < 1. \quad \square$$

Lemma 2.4 ([5]). *Both roots of the complex quadratic equation $\lambda^2 + \phi\lambda + \varphi = 0$ have modulus less than one if and only if $|\phi - \bar{\phi}\varphi| + |\varphi|^2 < 1$, where $\bar{\phi}$ denotes the conjugate complex of ϕ.*

Let $\lambda_{\max}(W)$ and $\lambda_{\min}(W)$ denote the maximum and the minimum eigenvalues of a Hermitian matrix W, respectively. Then we have the following convergence result.

Theorem 2.1. *Let A be a non-Hermitian matrix with the positive definite Hermitian part $H := \frac{1}{2}(A+A^*)$, and the skew-Hermitian part $S := \frac{1}{2}(A-A^*)$. Let the matrix B have full row-rank and let Q_2 be a Hermitian positive definite matrix. Assume that $[u^*, v^*]^*$ is an eigenvector according to an eigenvalue of the iteration matrix Γ. Denote*

$$a := \frac{u^* H u}{u^* u}, \quad -b := \frac{u^* \mathrm{i} \cdot S u}{u^* u}, \quad c := \frac{u^* B^* Q_2^{-1} B u}{u^* u}.$$

Then the LHSS iteration is convergent if a, b, c satisfy the following condition:

$$0 \le c < \left(2 - \frac{4b^2}{a^2 + b^2}\right) a.$$

Proof. By Lemma 2.1, we have $\lambda \neq 1$. Then we can obtain from (2.2) and (2.3) that

$$(\lambda H + S)u - \frac{\lambda}{1-\lambda} B^* Q_2^{-1} B u = 0. \tag{2.6}$$

If $Bu = 0$, it follows from (2.6) that $(\lambda H + S)u = 0$, which is similar to the second part of Lemma 2.3, then $|\lambda| < 1$.
If $Bu \neq 0$, which means that $c > 0$ according to the definition of c. From (2.6) we have

$$a\lambda + \mathrm{i} \cdot b - \frac{\lambda}{1-\lambda} c = 0.$$

That is to say, λ satisfies the complex quadratic equation

$$\lambda^2 + \left(\frac{c - a + \mathrm{i} \cdot b}{a}\right)\lambda + \left(-\frac{b}{a} \cdot i\right) = 0. \tag{2.7}$$

Now, according to Lemma 2.4 we know that both roots of λ of the complex quadratic equation (2.7) satisfy $|\lambda| < 1$ if and only if

$$\left|\frac{c - a + \mathrm{i} \cdot b}{a} + \frac{b^2 + \mathrm{i} \cdot (c-a)b}{a^2}\right| + \left|\frac{b^2}{a^2}\right| < 1. \tag{2.8}$$

By straightforwardly solving (2.8) we immediately obtain the condition that we are demonstrating. □

For the real case, there are better results, which are summarized in the following corollaries.

Corollary 2.1. *Let A be a nonsymmetric matrix with the positive definite symmetric part $H := \frac{1}{2}(A+A^{\mathrm{T}})$, and the skew-symmetric part $S := \frac{1}{2}(A-A^{\mathrm{T}})$. Let the matrix B have full row-rank and let Q_2 be a symmetric positive definite matrix. Then the LHSS iteration converges provided that*

$$0 < \lambda_{\max}(B^{\mathrm{T}} Q_2^{-1} B) < 2\lambda_{\min}(H).$$

Corollary 2.2. *Under the assumptions of Corollary 2.1, the LHSS iteration is convergent if $2H - B^{\mathrm{T}} Q_2^{-1} B$ is positive definite.*

Corollary 2.3. *Under the assumptions of Corollary 2.1, if $Q_2 = \frac{1}{\delta} I$, then the LHSS iteration converges when $0 < \delta < \frac{2\lambda_{\min}(H)}{\lambda_{\max}(B^{\mathrm{T}}B)}$.*

Corollary 2.4. *Under the assumptions of Corollary 2.1, if $Q_2 = (BB^{\mathrm{T}})(BHB^{\mathrm{T}})^{-1}(BB^{\mathrm{T}})$, then $\rho(\Gamma)$ approaches an infinitesimal constant.*

Remark 2.1. Under the assumptions of Corollary 2.1, when we choose Q_2 to be the special matrix $(BB^{\mathrm{T}})(BHB^{\mathrm{T}})^{-1}(BB^{\mathrm{T}})$, $\rho(\Gamma)$ approaches an infinitesimal constant, which is a favorable result. In this way we need to compute the inverse of BHB^{T}, while in iterative algorithm we only need to compute $(BB^{\mathrm{T}})^{-1}$. This may be more effective in practice than choosing Q_2 to be the Schur complement. Moreover, in many engineering applications, $BB^{\mathrm{T}}(\in \mathbb{C}^{m\times m})$ is a sparse, Hermitian and nearly singular matrix. In this case, we may choose $Q_2 = \alpha(BB^{\mathrm{T}})(BHB^{\mathrm{T}})^{-1}(BB^{\mathrm{T}})$ with an appropriate α.

2.2. Modified LHSS (MLHSS) iteration

In this subsection, we still consider the matrix $\mathcal{A}$ defined in (1.2) with $C = 0$. Now, we focus on the following splitting

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} = \begin{bmatrix} Q_1 + H & 0 \\ -B & Q_2 \end{bmatrix} - \begin{bmatrix} Q_1 - S & -B^* \\ 0 & Q_2 \end{bmatrix},$$

where $H := \frac{1}{2}(A + A^*)$, $S := \frac{1}{2}(A - A^*)$ are the Hermitian and the skew-Hermitian parts of A, respectively. The matrix B still has full row-rank. $Q_1 \in \mathbb{C}^{n\times n}$ is a Hermitian positive semi-definite matrix. $Q_2 \in \mathbb{C}^{m\times m}$ is a Hermitian positive definite matrix. Then we have the modified LHSS (MLHSS) iteration:

$$\begin{bmatrix} Q_1 + H & 0 \\ -B & Q_2 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} Q_1 - S & -B^* \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f \\ -g \end{bmatrix}.$$

The corresponding computational process is described below.

Algorithm 2.2 (*MLHSS Iteration Method*).

$$\begin{cases} x_{n+1} = x_n + (Q_1 + H)^{-1}(f - Ax_n - B^* y_n), \\ y_{n+1} = y_n + Q_2^{-1}(Bx_{n+1} - g). \end{cases}$$

If $Q_1 = 0$, then the MLHSS iteration method is just the LHSS method proposed in Section 2.1. In fact, the LHSS method and the MLHSS method are both special cases of the PIU method, see, for example, [4,5,22,23,6–8]. It is worth pointing out that in [22] and [23], the authors also proposed more efficient nonlinear inexact Uzawa methods. Recently, there is a generalized inexact parameterized Uzawa method in [6], which is mainly about the Hermitian saddle point problems. For non-Hermitian saddle point problems, the generalized inexact parameterized Uzawa method may be taken as follows.

Algorithm 2.3.

$$\begin{cases} x_{n+1} = x_n + (Q_1 + H)^{-1}(f - Ax_n - B^* y_n), \\ y_{n+1} = y_n + Q_2^{-1}((1-t)Bx_{n+1} + tBx_n - g), \end{cases}$$

where t is an iteration parameter.

In what follows, we give some analysis on the convergence of the MLHSS iteration.

Let

$$\tilde{\Gamma} = \begin{bmatrix} Q_1 + H & 0 \\ -B & Q_2 \end{bmatrix}^{-1} \begin{bmatrix} Q_1 - S & -B^* \\ 0 & Q_2 \end{bmatrix} \tag{2.9}$$

denote the MLHSS iteration matrix. Let λ be an eigenvalue of $\tilde{\Gamma}$ and $\binom{u}{v}$ be a corresponding eigenvector. Then

$$\begin{bmatrix} Q_1 - S & -B^* \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Q_1 + H & 0 \\ -B & Q_2 \end{bmatrix} \begin{bmatrix} \lambda u \\ \lambda v \end{bmatrix},$$

or equivalently

$$[(\lambda - 1)Q_1 + \lambda H + S]u + B^* v = 0, \tag{2.10}$$

$$\lambda Bu + (1 - \lambda)Q_2 v = 0. \tag{2.11}$$

To obtain a convergence condition, we first give some lemmas to be used later.

Lemma 2.5. *Let the matrix $\tilde{\Gamma}$ be defined as in (2.9). If λ is an eigenvalue of $\tilde{\Gamma}$, then $\lambda \neq 1$.*

Lemma 2.6. *Let the matrix $\tilde{\Gamma}$ be defined as in (2.9). If $\binom{u}{v}$ is an eigenvector of $\tilde{\Gamma}$ corresponding to the eigenvalue λ, then $u \neq 0$. Moreover, if $v = 0$, then $|\lambda| < 1$.*

Theorem 2.2. *Suppose that A is a non-Hermitian matrix with the positive definite Hermitian part $H := \frac{1}{2}(A + A^*)$ and the skew-Hermitian part $S := \frac{1}{2}(A - A^*)$. Let B have full row-rank. Let Q_1 be Hermitian positive semi-definite and Q_2 be Hermitian positive definite. Assume that $[u^*, v^*]^*$ is an eigenvector according to an eigenvalue of the iteration matrix $\tilde{\Gamma}$. Denote*

$$a := \frac{u^* H u}{u^* u}, \qquad -b := \frac{u^* \mathrm{i} \cdot S u}{u^* u}, \qquad c := \frac{u^* B^* Q_2^{-1} B u}{u^* u}, \qquad d := \frac{u^* Q_1 u}{u^* u}.$$

Then the MLHSS iteration is convergent if a, b, c and d satisfy the following condition:

$$0 \le c < \frac{2a^3 + 4a^2 d - 2ab^2}{a^2 + b^2}.$$

Proof. By Lemma 2.5 we have $\lambda \neq 1$. As Q_2 is Hermitian positive definite, $(1-\lambda)Q_2$ is nonsingular. Hence, from (2.11) we obtain

$$v = -\frac{\lambda}{1-\lambda} Q_2^{-1} B u.$$

By eliminating v from (2.10), we have

$$[(\lambda - 1)Q_1 + \lambda H + S]u = \frac{\lambda}{1-\lambda} B^* Q_2^{-1} B u. \tag{2.12}$$

If $Bu = 0$, we have from (2.12) that $[(\lambda-1)Q_1 + \lambda H + S]u = 0$, which is similar to the second part of Lemma 2.6, then $|\lambda| < 1$.

If $Bu \neq 0$, we have from (2.12) that

$$(\lambda - 1)d + \lambda a + \mathrm{i} \cdot b - \frac{\lambda}{1-\lambda} c = 0$$

and, thus,

$$\lambda^2 + \frac{c - 2d - a + \mathrm{i} \cdot b}{d + a}\lambda + \frac{d - \mathrm{i} \cdot b}{d + a} = 0. \tag{2.13}$$

Now, according to Lemma 2.4 we know that both roots λ of the complex quadratic equation (2.13) satisfy $|\lambda| < 1$ if and only if

$$\left| \frac{c - a - 2d + \mathrm{i} \cdot b}{d + a} - \frac{(c - a - 2d - \mathrm{i} \cdot b)(d - \mathrm{i} \cdot b)}{(d+a)^2} \right| + \frac{d^2 + b^2}{(d+a)^2} < 1. \tag{2.14}$$

By straightforwardly solving (2.14) we immediately obtain the condition that we are demonstrating. □

For the real case, there are better results, too. We summarize them in the following corollaries.

Corollary 2.5. *Let A be a nonsymmetric matrix with the positive definite symmetric part $H := \frac{1}{2}(A + A^{\mathrm{T}})$, and the skew-symmetric part $S := \frac{1}{2}(A - A^{\mathrm{T}})$. Let B have full row-rank. Let Q_1 be symmetric positive semi-definite and Q_2 be symmetric positive definite. Then the MLHSS iteration converges provided that*

$$0 < \lambda_{\max}(B^{\mathrm{T}} Q_2^{-1} B) < 2\lambda_{\min}(H) + 4\lambda_{\min}(Q_1).$$

Corollary 2.6. *Under the assumptions of Corollary 2.5, the MLHSS iteration is convergent if $2H + 4Q_1 - B^{\mathrm{T}} Q_2^{-1} B$ is positive definite.*

Corollary 2.7. *Under the assumptions of Corollary 2.5, if $Q_1 = \alpha I$ and $Q_2 = \frac{1}{\delta} I$, then the MLHSS iteration converges when*

$$0 < \delta < \frac{2\lambda_{\min}(H) + 4\alpha}{\lambda_{\max}(B^{\mathrm{T}} B)}.$$

3. Iterative methods for generalized saddle point problems

In this section, by splitting preconditioning we first study the equivalent generalized saddle point problem (1.2) and deduce its equivalence to a non-Hermitian saddle point problem. The matrix $C \in \mathbb{C}^{m \times m}$ defined in (1.2) is assumed to be Hermitian positive semi-definite. Moreover, we assume that C has rank p $(0 < p < m)$, and $ker(C) \bigcap ker(B^*) = 0$. Then there exists an unitary matrix $\begin{bmatrix} E & F \end{bmatrix}$ such that

$$C = \begin{bmatrix} E & F \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E^* \\ F^* \end{bmatrix},$$

where $E \in \mathbb{C}^{m \times p}$, $F \in \mathbb{C}^{m \times (m-p)}$ is a basis of the null-space of C, and $D \in \mathbb{C}^{p \times p}$ is a diagonal matrix whose diagonal elements are the eigenvalues of C. By premultiplying (1.2) with the nonsingular and square matrix

$$R = \begin{bmatrix} I & 0 \\ 0 & E^* \\ 0 & F^* \end{bmatrix},$$

and postmultiplying it with R^*, we obtain

$$\begin{bmatrix} A & B^*E & B^*F \\ -E^*B & D & 0 \\ -F^*B & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y_e \\ y_f \end{bmatrix} = \begin{bmatrix} f \\ -g_e \\ -g_f \end{bmatrix}, \tag{3.1}$$

with $y = Ey_e + Fy_f$, $g_e = E^*g$, $g_f = F^*g$, where we have made use of the equality $CF = 0$. Then by appropriately partitioning, (3.1) can be seen as

$$\begin{bmatrix} \bar{A} & \bar{B}^* \\ -\bar{B} & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{f} \\ \bar{g} \end{bmatrix}, \quad \text{or } \bar{\mathcal{A}}\bar{u} = \bar{b}. \tag{3.2}$$

It is easy to see that $\bar{A}$ is positive stable and F^*B has full rank (see also [26]). Thus by the splitting preconditioning, the equivalent generalized saddle point problem (1.2) can be equivalently seen as the non-Hermitian saddle point problem (3.2), where $\bar{A}$ is non-Hermitian, with the positive definite Hermitian part $\bar{H} := \begin{bmatrix} H & 0 \\ 0 & D \end{bmatrix}$, and $\bar{B}$ has full rank.

We mainly consider to extend the MLHSS iteration to the generalized saddle point problem and to analyze the best convergence case of the obtained iteration scheme for the generalized saddle point problem. For (3.2), we present the splitting:

$$\begin{bmatrix} \bar{A} & \bar{B}^* \\ -\bar{B} & 0 \end{bmatrix} = \begin{bmatrix} \bar{Q}_1 + \bar{H} & 0 \\ -\bar{B} & \bar{Q}_2 \end{bmatrix} - \begin{bmatrix} \bar{Q}_1 - \bar{S} & -\bar{B}^* \\ 0 & \bar{Q}_2 \end{bmatrix}, \tag{3.3}$$

where

$$\bar{H} := \begin{bmatrix} H & 0 \\ 0 & D \end{bmatrix} \quad \text{and} \quad \bar{S} := \begin{bmatrix} S & B^*E \\ -E^*B & 0 \end{bmatrix}$$

are the Hermitian and the skew-Hermitian parts of $\bar{A}$, respectively. Let

$$\bar{Q}_1 := \begin{bmatrix} \bar{Q}_1^1 & 0 \\ 0 & \bar{Q}_1^2 \end{bmatrix}, \tag{3.4}$$

where $\bar{Q}_1^1 \in \mathbb{C}^{n\times n}$ and $\bar{Q}_1^2 \in \mathbb{C}^{p\times p}$ are Hermitian and positive semi-definite. Let $\bar{Q}_2 \in \mathbb{C}^{(m-p)\times(m-p)}$ be Hermitian and positive definite. Then the MLHSS iteration can be extended to the generalized saddle point problem, detaining the following generalized MLHSS (GMLHSS) iteration method:

$$\begin{bmatrix} \bar{Q}_1^1 + H & 0 & 0 \\ 0 & \bar{Q}_1^2 + D & 0 \\ -F^*B & 0 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ y_{n+1}^e \\ y_{n+1}^f \end{bmatrix} = \begin{bmatrix} \bar{Q}_1^1 - S & -B^*E & -B^*F \\ E^*B & \bar{Q}_1^2 & 0 \\ 0 & 0 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n^e \\ y_n^f \end{bmatrix} + \begin{bmatrix} f \\ -g_e \\ -g_f \end{bmatrix}.$$

The corresponding computational process is described below.

Algorithm 3.1 (*GMLHSS Iteration Method*).

$$\begin{cases} x_{n+1} = x_n + (\bar{Q}_1^1 + H)^{-1}(f - Ax_n - B^*Ey_n^e - B^*Fy_n^f), \\ y_{n+1}^e = y_n^e + (\bar{Q}_1^2 + D)^{-1}(E^*Bx_n - Dy_n^e - g_e), \\ y_{n+1}^f = y_n^f + (\bar{Q}_2)^{-1}(F^*Bx_{n+1} - g_f), \\ y_{n+1} = Ey_{n+1}^e + Fy_{n+1}^f. \end{cases}$$

Denote the GMLHSS iteration matrix by

$$\bar{\Gamma} = \begin{bmatrix} \bar{Q}_1 + \bar{H} & 0 \\ -\bar{B} & \bar{Q}_2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{Q}_1 - \bar{S} & -\bar{B}^* \\ 0 & \bar{Q}_2 \end{bmatrix}.$$

From Theorem 2.2 we can obtain a sufficient condition for the convergence of Algorithm 3.1, which is summarized as the following theorem.

Theorem 3.1. *Assume that A is a non-Hermitian matrix with the positive definite Hermitian part $H := \frac{1}{2}(A + A^*)$ and the skew-Hermitian part $S := \frac{1}{2}(A - A^*)$, B has full row-rank, C is Hermitian and positive semi-definite and has rank $p(0 < p < m)$, $F \in \mathbb{C}^{m\times(m-p)}$ is a basis of the null-space of C, $D \in \mathbb{C}^{p\times p}$ is a diagonal matrix whose diagonal elements are the eigenvalues of C. Let $\bar{H}$ be defined as in (3.3), and $\bar{Q}_1$ be defined as in (3.4). Let $\bar{Q}_2$ be Hermitian and positive definite. Assume that $[u^*, v^*]^*$ is an eigenvector according to an eigenvalue of the iteration matrix $\bar{\Gamma}$. Denote*

$$a := \frac{u^*\bar{H}u}{u^*u}, \qquad -b := \frac{u^*\mathrm{i}\cdot\bar{S}u}{u^*u}, \qquad c := \frac{u^*B^*F\bar{Q}_2^{-1}F^*Bu}{u^*u}, \qquad d := \frac{u^*\bar{Q}_1u}{u^*u}.$$

Then the GMLHSS iteration is convergent if a, b, c and d satisfy the following condition:

$$0 \le c < \frac{2a^3 + 4a^2 d - 2ab^2}{a^2 + b^2}.$$

Proof. The proof of Theorem 3.1 is similar to that of Theorem 2.2. Thus it is omitted. □

Corollary 3.1. *Assume that A is a nonsymmetric matrix with the positive definite symmetric part $H := \frac{1}{2}(A + A^{\mathrm{T}})$ and the skew-symmetric part $S := \frac{1}{2}(A - A^{\mathrm{T}})$, B has full row-rank, C is symmetric and positive semi-definite and has rank $p(0 < p < m)$, $F \in \mathbb{C}^{m\times(m-p)}$ is a basis of the null-space of C. Let $\bar{Q}_1^1$ and $\bar{Q}_1^2$ be symmetric and positive semi-definite, and $\bar{Q}_2$ be symmetric and positive definite. Then the GMLHSS iteration converges, provided*

$$0 < \lambda_{\max}(B^{\mathrm{T}}F\bar{Q}_2^{-1}F^{\mathrm{T}}B) < 2\min(\lambda_{\min}(H), \lambda_{\min}(D)) + 4\min(\lambda_{\min}(\bar{Q}_1^1), \lambda_{\min}(\bar{Q}_1^2)).$$

Corollary 3.2. *Under the assumptions of Corollary 3.1, the GMLHSS iteration is convergent if $2\bar{H} + 4\bar{Q}_1 - \bar{B}^{\mathrm{T}}\bar{Q}_2^{-1}\bar{B}$ is positive definite.*

Corollary 3.3. *Under the assumptions of Corollary 3.1, if $\bar{Q}_1$ is a zero matrix and $\bar{Q}_2 = (F^{\mathrm{T}}BB^{\mathrm{T}}F)(F^{\mathrm{T}}BHB^{\mathrm{T}}F)^{-1}(F^{\mathrm{T}}BB^{\mathrm{T}}F)$, then $\rho(\bar{\Gamma})$ approaches an infinitesimal constant.*

4. Numerical experiments

In this section, we present some numerical experiments to compare our new methods with the classical Uzawa method and an inexact parameterized Uzawa method for the nonsymmetric saddle point problem and the generalized saddle point problem, respectively. The problem under consideration is the classical incompressible steady state Navier–Stokes problem:

$$\begin{cases} -\nu\Delta u + (\omega\cdot\nabla)u + \nabla p = f, \\ \nabla\cdot u = 0, \end{cases} \quad \text{in } \Omega \tag{4.1}$$

with suitable boundary conditions on $\partial\Omega$. The test problem is a "leaky" two-dimensional lid-driven cavity problem in a square domain ($0 \le x \le 1, 0 \le y \le 1$). The boundary conditions are $u_x = u_y = 0$ on three fixed walls ($x = 0, x = 1, y = 0$), and $u_x = 1, u_y = 1$ on the moving wall ($y = 1$). On constructing the coefficient matrix $\hat{\mathcal{A}}$, we select the constant wind: $\omega_x = 1, \omega_y = 2$.

We use two methods to discretize (4.1). One is a "marker and cell"(MAC) finite difference scheme (cf. [16,17]) based on $ne \times ne$ uniform grids of square meshes. The second is a finite element subdivision based on $ne \times ne$ uniform grids of square elements. The mixed finite element (cf. [31,16]) used is the bilinear-constant velocity–pressure: $Q_1 - P_0$ pair with local stabilization. The resulting linear system for the discrete solution has the form

$$\begin{bmatrix} A & B^* \\ -B & C \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix}, \tag{4.2}$$

where $x = (u_x^*, u_y^*)^*$ represents the approximate velocities at the nodes and p stands for the approximate pressure at each grid. It should be pointed out that $C = 0$ (corresponding to nonsymmetric saddle point problems) in MAC finite difference scheme and C is semi-positive definite (corresponding to generalized saddle point problems) in $Q_1 - P_0$ finite element method.

In our numerical experiments, we use the zero vector as the initial guess, and choose the right-hand side vector $(f^*, -g^*)^*$ such that the exact solution of the saddle point problem is $(x^*, p^*)^* = (1, \dots, 1)^*$. We stop the iteration as soon as the error is less than 10^{-5}. By Theorems 2.1 and 2.2 and their corollaries, we can compute the upper bounds on the parameters δ for the algorithms we proposed above. Let $Q_i (i = 1, 2)$, IT, CPU and ER denote the preconditioning matrix, iterative steps, seconds needed for convergence, the error, respectively. Let δ, t and α be parameters of the modified iterative method. For comparison we choose the classical Uzawa method and Algorithm 2.3, where $Q_1 = \delta H$ and $Q_2 = BQ_1^{-1}B^*$.

The following Tables 1–5 give the numerical results of the Uzawa method, the LHSS method, Algorithm 2.3 and the MLHSS method for the nonsymmetric saddle point problem arising from the MAC method, with $\nu = 1$ and different ne.

The above numerical results show that for the nonsymmetric saddle point problem our methods have better convergence property. For LHSS iteration, when $Q_2 = \frac{1}{\delta}I$, the number of the iterative steps is almost the same as that of the Uzawa method. When choosing $Q_2 = 0.5(BB^*)(BHB^*)^{-1}(BB^*)$, the number of the LHSS iteration steps is much less than that of the Uzawa method although we have to compute the inverse of BB^* in the iteration process. For large linear systems, Algorithm 2.3 may have less iteration steps. But from the numerical results, we find that it takes more time since it needs to compute the inverse of the Schur complement matrix. In the MLHSS iteration, if $Q_1 = \alpha I$ and $Q_2 = \frac{1}{\delta}I$, we can obtain different ranges of δ by choosing different α. Then by taking appropriate δ, the number of iteration steps in Table 3 is much less than that

Table 1
Classical Uzawa method.

ne	*n*	*m*	δ	IT	CPU	ER (10^{-6})
8	112	63	121	884	0.204	9.9641
16	480	225	480	3 825	11.640	9.9851
24	1104	575	1050	9 094	135.906	9.9974
32	1984	1023	1900	16 141	770.075	9.9972
48	4512	2303	–	>40 000	–	–

Table 2
LHSS iteration method, $Q_2 = \frac{1}{\delta}I$.

ne	*n*	*m*	δ	IT	CPU	ER (10^{-6})
8	112	63	121	881	0.204	9.9641
16	480	225	480	3 822	11.610	9.9888
24	1104	575	1050	9 091	135.766	9.9980
32	1984	1023	1900	16 138	769.125	9.9978
48	4512	2303	–	>40 000	–	–

Table 3
LHSS iteration method, $Q_2 = 0.5(BB^*)(BHB^*)^{-1}(BB^*)$.

ne	*n*	*m*	IT	CPU	ER (10^{-6})
8	112	63	16	0.015	6.8599
16	480	225	16	0.046	7.9294
24	1104	575	50	0.781	9.4270
32	1984	1023	66	1.281	7.6748
48	4512	2303	50	31.25	8.6567

Table 4
Algorithm 2.3 with $\delta = 0.4$ and $t = 0.01$.

ne	*n*	*m*	IT	CPU	ER (10^{-6})
8	112	63	21	0.015	6.1206
16	480	225	28	0.296	9.1614
24	1104	575	29	1.641	8.6575
32	1984	1023	24	2.281	2.8422
48	4512	2303	27	31.84	6.3132

Table 5
MLHSS iteration method, $Q_1 = \alpha I$, $Q_2 = \frac{1}{\delta}I$.

ne	*n*	*m*	IT	CPU	ER (10^{-6})	IT	CPU	ER (10^{-6})
			$\alpha = 8, \delta = 271$			$\alpha = 12, \delta = 404$		
8	112	63	422	0.094	9.9668	372	0.110	8.1667
			$\alpha = 6, \delta = 781$			$\alpha = 7.5, \delta = 975$		
16	480	225	2788	8.625	9.9895	2507	7.8910	9.7874
			$\alpha = 60, \delta = 1739$			$\alpha = 62.5, \delta = 1812$		
24	1104	575	7463	107.984	9.3575	7351	106.156	9.4901
			$\alpha = 2, \delta = 24$			$\alpha = 3, \delta = 33$		
32	1984	1023	14423	552.344	9.2764	17189	637.281	9.5959
			$\alpha = 0.06, \delta = 4600$			$\alpha = 0.05, \delta = 4600$		
48	4512	2303	34296	3211.8	9.9980	35439	3321.9	9.8670

of the Uzawa method. All these results show that our new methods have faster convergence rate than the classical Uzawa method for the nonsymmetric saddle point problem.

We use CPU, CPU1 and CPU2 to represent the elapsed CPU times in seconds for the classical Uzawa method, eigenvalue decomposition of C and GMLHSS iteration. Tables 6 and 7 give the numerical results for the Uzawa method and the GMLHSS method for the generalized saddle point problem arising from the $Q_1 - P_0$ finite element method, with $\nu = 1$ and different *ne*.

Clearly, the above numerical results show that the GMLHSS iteration has faster convergence rate than that of the classical Uzawa method for the generalized saddle point problem. Moreover, the results show that the methods we proposed in this paper are powerful solvers for the large sparse generalized saddle point problems.

Table 6
Classical Uzawa method.

ne	n	m	IT	CPU	ER (10^{-6})
8	98	62	729	0.172	9.5106
16	450	254	6 808	21.297	9.9093
24	1058	574	8 228	108.079	9.9946
32	1922	1022	19 807	970.235	9.4358
48	4418	2302	>40 000	–	–

Table 7
GMLHSS iteration method, $\bar{Q}_2 = (F^*BB^*F)(F^*BHB^*F)^{-1}(F^*BB^*F)$.

ne	n	m	IT	CPU1	CPU2	ER (10^{-6})
8	98	62	176	0.015	0.047	9.5106
16	450	254	532	0.063	1.953	9.9093
24	1058	574	907	0.546	24.11	9.9959
32	1922	1022	1055	1.452	45.32	9.5658
48	4418	2302	2865	5.465	80.56	9.8796

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