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Maximum cut in fuzzy nature: Models and algorithms

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ABSTRACT

The maximum cut (Max-Cut) problem has extensive applications in various real-world fields, such as network design and statistical physics. In this paper, a more practical version, the Max-Cut problem with fuzzy coefficients, is discussed. Specifically, based on credibility theory, the Max-Cut problem with fuzzy coefficients is formulated as an expected value model, a chance-constrained programming model and a dependent-chance programming model respectively according to different decision criteria. When these fuzzy coefficients are represented by special fuzzy variables like triangular fuzzy numbers and trapezoidal fuzzy numbers, the crisp equivalents of the fuzzy Max-Cut problem can be obtained. Finally, a genetic algorithm combined with fuzzy simulation techniques is designed for the general fuzzy Max-Cut problem under these models and numerical experiment confirms the effectiveness of the designed genetic algorithm.

1. Introduction

The Max-Cut problem is a well-known combinatorial optimization problem which aims to find a division of a vertex set into two parts maximizing the sum of weights over all the edges across the two vertex subsets in a given edge-weighted graph. For an unweighted graph, the weights over the edges are equal to 1. Such a division is called a maximum cut. The worst-case complexity of the Max-Cut problem has been studied in a few papers, some of them dealing with weighted graphs and some with unweighted graphs. Determining the maximum cut of a graph is an NP-hard problem, though it is solvable in polynomial time for some special classes of graphs [1]. Besides its theoretical importance, the Max-Cut problem has many applications in various fields such as network design, statistical physics and VLSI design, circuit layout design [2], and data clustering [3]. For a comprehensive survey of the Max-Cut problem, the reader is referred to the paper of Poljak and Tuza [4].

The Max-Cut problem is one of the first problems proved to be NP-hard, meaning that there are no strongly efficient exact algorithms [5]. So many authors turn their attention to find approximate algorithms and heuristic algorithms. For example, Delorme and Poljak [6] and Poljak and Rendl [7] designed eigenvalue relaxation algorithms for the problem. Goemans and Williamson [8] developed a randomized algorithm based on semi-definite programming with the performance guarantee of 0.878. In contrast to these approximate algorithms with performance guarantee, several efficient heuristic algorithms have also been developed [9,10]. The striking characteristics of these heuristic algorithms are their efficiency in terms of implementing time and possibility of finding global optimal solutions.

When applied in various real-world fields, the weight of the cut in the Max-Cut problem has real and concrete implications. For example, for network design, the weight may represent the cost of some infrastructure. Owing to the effects

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of many factors in the real life, these parameters are not so deterministic. The imprecision may come from multiple sources such as linguistic definition of importance, inexact values of measurements, integration of opinions of multiple experts, and personal bias in evaluating a quantity [11–13]. At this time, we need to deal with decision-making problems in uncertain environments and design models suitable for a more practical situation. The concept of fuzzy set theory developed in [14,15] and well studied by many other researchers is suited to formulate vague or subjective nature quantities. Based on this theory, many optimization problems including various fuzzy programming models [16,17], fuzzy shortest path problems [18] and fuzzy max-flow problems [19] have been studied in imprecise environments. Recently, Liu [20,21] developed a credibility theory where the expected value criterion, the optimistic value criterion and the credibility criterion are used as fuzzy ranking methods. Based on this uncertain theory, several optimization models in stochastic and fuzzy environments have been studied [22–24].

In this paper, the Max-Cut problem is investigated in an uncertain environment based on Liu's credibility theory. Specifically, the Max-Cut problem with fuzzy coefficients is formulated as an expected value model, a chance-constrained programming model and a dependent-chance programming model respectively according to different decision criteria. Then their crisp equivalents are also discussed when the fuzzy costs are characterized by special fuzzy variables such as triangular fuzzy numbers and trapezoidal fuzzy numbers. Finally, since the Max-Cut problem is NP-hard even in a deterministic environment, this paper will examine the design of efficient heuristic algorithms. A hybrid genetic algorithm combined with fuzzy simulation techniques is designed for the general fuzzy Max-Cut problem under these three models.

The paper is organized as follows: Some preliminaries on credibility theory are described in Section 2, then Section 3 introduces the Max-Cut problem with fuzzy coefficients. Three types of optimization models for the Max-Cut problem with fuzzy coefficients are developed in Section 4, where their crisp equivalents under special fuzzy variables are also discussed. In Section 5, a genetic algorithm combined with fuzzy simulation techniques is designed for solving the general fuzzy Max-Cut problem. Finally, numerical experiments are done to test the effectiveness of the designed algorithm in Section 6. Section 7 concludes the paper.

2. Preliminaries for credibility theory

Fuzzy set theory, developed in [14] and well studied by many other researchers, is well suited for formulating vague or subjective quantities. Like the role of random variables in probability theory to describe many stochastic phenomena, a fuzzy variable is a type of mathematical tools to describe fuzzy or vague uncertainty. Based on fuzzy set theory, possibility theory was first developed in [15], and later extended by many researchers including Dubois and Prade [25].

Let $\mathbb{X}$ denote a nonempty set and $\mathscr{P}(\mathbb{X})$ denote the power set of $\mathbb{X}$. Pos is a possibility measure on $\mathscr{P}(\mathbb{X})$. Then $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \text{Pos})$ represents a possibility space. Let ξ be a fuzzy variable with membership function $\mu : R \to [0, 1]$, and let x, x_0 be real numbers. According to possibility theory [25], the possibility of a fuzzy event $\xi \geq x_0$ is defined as

$$\text{Pos}\{\xi \geq x_0\} = \sup_{x \geq x_0} \mu(x)$$

and the necessity of this event is defined as

$$\text{Nec}\{\xi \geq x_0\} = 1 - \text{Pos}\{\xi < x_0\} = 1 - \sup_{x < x_0} \mu(x).$$

Liu and Liu [22] introduced a credibility measure which is an average of possibility measure and necessity measure:

$$\text{Cr}\{\xi \geq x_0\} = \frac{1}{2}(\text{Pos}\{\xi \geq x_0\} + \text{Nec}\{\xi \geq x_0\}).$$

Credibility measure Cr is a monotone regularized non-additive measure and vanishes at an empty set. For each fuzzy event A its possibility to occur is $\text{Pos}\{A\}$, while its necessity to occur is defined by $\text{Nec}\{A\} = 1 - \text{Pos}\{A^c\}$. It is obvious that a fuzzy event may fail even if its possibility is 1, and hold even if its necessity is 0. Credibility measure, as the mean of possibility and necessity measures, can characterize fuzzy phenomena appropriately. Credibility theory was founded for studying the behavior of fuzzy phenomena [26] and has been widely used to describe fuzzy phenomena in many real problems [20,21, 27,28].

Let ξ be a fuzzy variable defined on the credibility space $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \text{Cr})$. Then its membership function is derived from the credibility measure Cr by

$$\mu(x) = (2\text{Cr}\{\xi = x\}) \wedge 1, \quad x \in R.$$

It is clear that a fuzzy variable has a unique membership function [26]. The membership function represents the degree of possibility that the fuzzy variable ξ takes some prescribed value. For example, the membership degree $\mu(x) = 0$ if x is an impossible point, and $\mu(x) = 1$ if x is the most possible point that ξ takes. With the credibility measure, the expected value of a fuzzy variable ξ is defined as follows:

Definition 2.1 (*Liu and Liu [21]*). Let ξ be a fuzzy variable defined on the credibility space $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \text{Cr})$. The expected value of ξ is defined as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

As indicated by the following proposition, the expected value operator is linear.

Proposition 2.1 (*Liu [20]*). *Let ξ and η be independent fuzzy variables defined on the credibility space* $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \mathrm{Cr})$. *Then for any real numbers a and b, the following equality*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$

holds.

Definition 2.2 (*Liu [20]*). Let ξ be a fuzzy variable defined on the credibility space $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \mathrm{Cr})$, and $\alpha \in (0, 1]$. Then

$$\xi_{\sup}(\alpha) = \sup\{r|\mathrm{Cr}\{\xi \geq r\} \geq \alpha\}$$

is called the α-optimistic value of ξ.

3. The Max-Cut problem with fuzzy coefficients

Let $G = (V, E)$ be an undirected edge-weighted graph with nonnegative weights $\xi : E \to R^+$. A cut C of G is any nontrivial subset of V and the weight of a cut is the sum of weights of edges crossing C and $\bar{C}$, where $\bar{C} = V - C$. A max-cut is then defined as a cut of G with maximum weight. The goal of the Max-Cut problem is just to find such a cut. Assume that the vertices in V are labeled as $\{v_1, v_2, \dots, v_n\}$ and the edges in E are labeled as $e_{ij} = (v_i, v_j)$, where $n = |V|$, $i, j = 1, 2, \dots, n$. ξ_{ij} is the weight associated with the edge e_{ij}. We define a binary variable x_i for each vertex v_i to denote whether it is in a cut C or not:

$$x_i = \begin{cases} 1 & \text{if } v_i \in C \\ -1 & \text{if } v_i \in V - C. \end{cases}$$

Then any cut of the graph G can be denoted by an n-dimensional binary vector $(x_1, x_2, \dots, x_n)^T$ over $\{1, -1\}$. Similarly, any n-dimensional binary vector $(x_1, x_2, \dots, x_n)^T$ over $\{1, -1\}$ corresponds to a cut of G. So the weight of a cut $(x_1, x_2, \dots, x_n)^T$ can be denoted as

$$W(x, \xi) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{ij}(1 - x_i x_j).$$

Obviously, if and only if v_i and v_j respectively belong to two parts of the vertex set V ($x_i x_j = -1$), the weight of the edge (if any) linking them is considered. Let $\mathcal{C}$ be the set of all cuts of G. Then a cut x^* is called a maximum cut if and only if

$$W(x^*, \xi) \geq W(x, \xi)$$

for any $x \in \mathcal{C}$. As mentioned in the Introduction, the Max-Cut problem has wide practical applications. When applied in the real world, the weights of a graph have real and concrete implications. They are often not so deterministic since decision makers are often faced with some uncertain situations due to the vagueness or subjective nature of these parameters. In other words, usually precise weight parameters can be hardly obtained. For these cases, the weights on edges can be formulated as fuzzy variables which are more proper to describe the quantities in the real world. An important problem in fuzzy set theory is how to rank two fuzzy variables. Many methods for ranking fuzzy numbers have been proposed [29,21]. Among them a credibility theory including the credibility criterion, the optimistic value criterion and the expected value criterion has recently been developed as a fuzzy ranking method [21], which is also the basis of our fuzzy Max-Cut models. For this paper, assume that ξ_{ij} are all fuzzy variables. In real-world decision systems, different decision makers may have different preferences. According to these ranking methods of fuzzy variables, we can have several more practical versions of the Max-Cut problem. Some decision makers want to find a cut maximizing the expected value of the total weight, then the concept of expected maximum cuts can be adopted.

Definition 3.1. A cut x^* is called an expected Max-Cut (EMC) if and only if

$$E[W(x^*, \xi_{ij})] \geq E[W(x, \xi_{ij})]$$

for any $x \in \mathcal{C}$. At this time, $E[W(x^*, \xi_{ij})]$ is called the expected weight of x^*.

Some others may prefer to find a critical value of the maximum weight, leading to the following criterion:

Definition 3.2. A cut x^* is called an α-optimistic Max-Cut (α-OMC) if and only if

$$\sup\{\bar{W}|\mathrm{Cr}\{W(x^*, \xi_{ij}) \geq \bar{W}\} \geq \alpha\} \geq \sup\{\bar{W}|\mathrm{Cr}\{W(x, \xi_{ij}) \geq \bar{W}\} \geq \alpha\}$$

for any $x \in \mathcal{C}$, where α is a predetermined confidence level. At this time, $\sup\{\bar{W}|\mathrm{Cr}\{W(x^*, \xi_{ij}) \geq \bar{W}\} \geq \alpha\}$ is called the α-optimistic weight of x^*.

There is also such a case, where decision makers prefer to provide a predetermined infimum $\bar{W}$ and hope to find a cut such that the credibility that its weight is not lower than $\bar{W}$ will be as maximized as possible. For this case, the following way can be adopted:

Definition 3.3. A cut x^* is called a most Max-Cut (MMC) if and only if

$$\mathrm{Cr}\{W(x^*, \xi_{ij}) \geq \bar{W}\} \geq \mathrm{Cr}\{W(x, \xi_{ij}) \geq \bar{W}\}$$

for any $x \in \mathcal{C}$, where $\bar{W}$ is a predetermined infimum. At this time, $\mathrm{Cr}\{W(x^*, \xi_{ij}) \geq \bar{W}\}$ is called the most credibility of x^* at $\bar{W}$.

These practical versions of the max-cut problem, on the one hand, incorporates crisp max-cut problems (with precise weights) as a special case. On the other hand, it generalizes the max-cut problem and its applicability.

4. Models for Max-Cut in fuzzy nature

In this section, we will develop three models for the fuzzy Max-Cut problem according to the three decision criteria introduced in Section 3. They are respectively the expected value model, the chance-constrained programming model and the dependent-chance programming model.

Like the important role of the expected value model in stochastic programming, the expected valued model of fuzzy programming, first developed in [20], is a straightforward and easily understandable method for modeling fuzzy/vague phenomena in mathematical programming. Its goal is to optimize the expected objective subject to some constraints. Here, if decision makers prefer finding a cut with the maximum expected weight, we can construct the following expected value model for the fuzzy Max-Cut problem:

$$\begin{array}{ll} \max & E\left[\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1-x_i x_j)\right] \\ s.t. & x_i^2 = 1, \\ & x_i \in Z, \\ & i = 1, 2, \ldots, n \end{array} \tag{1}$$

where the constraint $x_i^2 = 1,\ x_i \in Z$ implies $x_i \in \{-1, 1\}$, meaning $x \in \mathcal{C}$. This model indicates that among all the possible cuts of G, the one with the maximum expected weight is preferred. We call this model as the Expected Max-Cut (EMC) problem.

In contrast to the expected value model, chance-constrained programming developed by in [30] has been a powerful strategy for modeling stochastic phenomena in decision systems. Liu and Iwamura [23] presented chance-constrained programming to model the fuzzy case in an uncertain environment. The main idea underlying it is to optimize the critical value of the fuzzy objective with certain confidence level subject to some constraints, which is well suited for finding an α-optimistic Max-Cut mentioned in Section 3:

$$\begin{array}{ll} \max & \bar{W} \\ s.t. & \mathrm{Cr}\left\{\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1-x_i x_j) \geq \bar{W}\right\} \geq \alpha \\ & x_i^2 = 1, \\ & x_i \in Z, \\ & i = 1, 2, \ldots, n \end{array} \tag{2}$$

where α is a confidence level provided by decision makers. This model is referred as the α-Optimistic Max-Cut (α-OMC) problem.

As mentioned in Section 3, some decision makers prefer providing a predetermined infimum and hope to maximize the credibility that the weight of the cut wanted is not lower than this infimum. For this case, we can use dependent-chance programming developed in [24] for the third criteria in decision-making. The main idea underlying it is to select a solution that can meet a fuzzy event with maximum credibility. Therefore, to find a most Max-Cut of G, we can construct the following dependent-chance programming model for the fuzzy Max-Cut problem:

$$\begin{array}{ll} \max & \mathrm{Cr}\left\{\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1-x_i x_j) \geq \bar{W}\right\} \\ s.t. & x_i^2 = 1, \\ & x_i \in Z, \\ & i = 1, 2, \ldots, n \end{array} \tag{3}$$

where $\bar{W}$ is a predetermined infimum provided by decision makers. This model is referred as the Most Max-Cut (MMC) problem.

With the three models discussed above, the next important thing we should do is to find the best solutions for them. Since there are no simple analytic expressions for the objective functions or constraints in these models, we need to estimate the expected value, the critical value or the credibility measure in them, which is not a straightforward and easy work. From the following part of this section, we can see that there are actually several simple cases when the fuzzy variables in these models are special.

Firstly, from the linearity of the expected value of a fuzzy variable, we can directly have the following theorem:

Theorem 4.1. *Let ξ_{ij}, $i, j = 1, 2, \ldots, n$ be independent fuzzy variables defined on the possibility space $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \mathrm{Pos})$. Then the EMC model (1) is equivalent to the following crisp optimization model:*

$$\begin{array}{ll} \max & \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} E[\xi_{ij}](1 - x_i x_j) \\ s.t. & x_i^2 = 1, \\ & x_i \in Z, \\ & i = 1, 2, \ldots, n. \end{array} \tag{4}$$

Although the credibility of a fuzzy variable is not linear and thus the left two models do not have the corresponding crisp equivalents, they indeed have crisp equivalents when the fuzzy variables are special fuzzy numbers such as triangular fuzzy numbers and trapezoidal fuzzy numbers which are widely used in modeling fuzziness. A trapezoidal fuzzy variable is often denoted by $\tilde{r} = (r_1, r_2, r_3, r_4)$ whose membership function has the following expression:

$$\mu_{\tilde{r}}(x) = \begin{cases} \dfrac{x - r_1}{r_2 - r_1} & \text{if } r_1 \le x \le r_2, \\ 1 & \text{if } r_2 \le x \le r_3, \\ \dfrac{x - r_4}{r_3 - r_4} & \text{if } r_3 \le x \le r_4, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

From its membership function expression, we can see that a trapezoidal fuzzy variable is fully determined by the quadruple (r_1, r_2, r_3, r_4). When $r_2 = r_3$, the trapezoidal fuzzy variable becomes a triangular fuzzy number. From Zadeh's extension principle, it is easy to check that a nonnegative linear combination of two trapezoidal fuzzy variables is still a trapezoidal fuzzy number. In addition, from the special membership function and the definition of the credibility Cr, it is easy to compute the credibility of the fuzzy event $\{\tilde{r} \ge x_0\}$ based on a trapezoidal fuzzy number $\tilde{r} = (r_1, r_2, r_3, r_4)$ as follows:

$$\mathrm{Cr}(\tilde{r} \ge x_0) = \begin{cases} 1 & \text{if } x_0 \le r_1, \\ \dfrac{2r_1 - r_2 - x_0}{2(r_2 - r_1)} & \text{if } r_1 \le x_0 \le r_2, \\ \dfrac{1}{2} & \text{if } r_2 \le x_0 \le r_3, \\ \dfrac{r_3 - x_0}{2(r_4 - r_3)} & \text{if } r_3 \le x_0 \le r_4, \\ 0 & \text{if } x_0 \ge r_4. \end{cases} \tag{6}$$

Lemma 4.1. *Let $\xi = (r_1, r_2, r_3, r_4)$ be a trapezoidal fuzzy variable defined on the credibility space $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \mathrm{Cr})$, and let α be a given confidence level. Then* (a) *when $\alpha \le 1/2$, $\mathrm{Cr}\{\xi \ge \bar{W}\} \ge \alpha$ if and only if $(1 - 2\alpha)r_1 + 2\alpha r_2 \ge \bar{W}$;* (b) *when $\alpha > 1/2$, $\mathrm{Cr}\{\xi \ge \bar{W}\} \ge \alpha$ if and only if $(2 - 2\alpha)r_3 + (2\alpha - 1)r_4 \ge \bar{W}$.*

Theorem 4.2. *Let $\xi_{ij} = (\xi_{ij}^1, \xi_{ij}^2, \xi_{ij}^3, \xi_{ij}^4)$ be independent trapezoidal fuzzy numbers defined on the credibility space $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \mathrm{Cr})$. Then the crisp equivalent of the α-OMC model (2) is given by*

$$\begin{array}{ll} \max & f(x) \\ s.t. & x_i^2 = 1, \\ & x_i \in Z, \\ & i = 1, 2, \ldots, n \end{array} \tag{7}$$

where $f(x)$ is such a real function

$$f(x) = \begin{cases} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\frac{1}{2} - \alpha\right)\xi_{ij}^1 + \alpha\xi_{ij}^2\right](1 - x_i x_j) & \text{if } \alpha \le \dfrac{1}{2} \\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(1 - \alpha)\xi_{ij}^3 + \left(\alpha - \frac{1}{2}\right)\xi_{ij}^4\right](1 - x_i x_j) & \text{if } \alpha > \dfrac{1}{2}. \end{cases} \tag{8}$$

Proof. On the one hand, $1 - x_i x_j \geq 0$ and ξ_{ij} are independent, so according to the properties of trapezoidal fuzzy variables, $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j)$ is also a trapezoidal fuzzy number and can be denoted by a quadruple

$$\begin{pmatrix} r_1(x) \\ r_2(x) \\ r_3(x) \\ r_4(x) \end{pmatrix}^T = \begin{pmatrix} \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}^1(1 - x_i x_j) \\ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}^2(1 - x_i x_j) \\ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}^3(1 - x_i x_j) \\ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}^4(1 - x_i x_j) \end{pmatrix}^T. \tag{9}$$

On the other hand, from Lemma 4.1, the constraint in α-OMC model (2)

$$\mathrm{Cr}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j) \geq \bar{W}\right\} \geq \alpha$$

is equivalent to $(1 - 2\alpha)r_1(x) + 2\alpha r_2(x) \geq \bar{W}$ when $\alpha \leq \frac{1}{2}$ and $(2 - 2\alpha)r_3(x) + (2\alpha - 1)r_4(x) \geq \bar{W}$ when $\alpha > \frac{1}{2}$. In other words, this constraint is equivalent to $f(x) \geq \bar{W}$, where $f(x)$ is defined by (8). Since maximizing $\bar{W}$ is actually maximizing $f(x)$, the α-OMC model (2) can be transformed into a crisp equivalent (7) when ξ_{ij} are independent trapezoidal fuzzy variables. ¶

Note that at this time, the α-OMC model becomes a general nonlinear optimization model without fuzzy variables.

Theorem 4.3. *Let $\xi_{ij} = (\xi_{ij}^1, \xi_{ij}^2, \xi_{ij}^3, \xi_{ij}^4)$ be independent trapezoidal fuzzy numbers defined on the credibility space $(\mathbb{X}, \mathscr{P}(\mathbb{X}), \mathrm{Cr})$. Then the crisp equivalent of the MMC model (3) is given by*

$$\begin{array}{ll} \max & f(x) \\ s.t. & x_i^2 = 1, \\ & x_i \in Z, \\ & i = 1, 2, \ldots, n \end{array} \tag{10}$$

where $f(x)$ is such a real function

$$f(x) = \begin{cases} 1 & \text{if } \bar{W} \leq r_1(x), \\ \dfrac{2r_1(x) - r_2(x) - \bar{W}}{2(r_2(x) - r_1(x))} & \text{if } r_1(x) \leq \bar{W} \leq r_2(x), \\ \dfrac{1}{2} & \text{if } r_2(x) \leq \bar{W} \leq r_3(x), \\ \dfrac{r_3(x) - \bar{W}}{2(r_4(x) - r_3(x))} & \text{if } r_3(x) \leq \bar{W} \leq r_4(x), \\ 0 & \text{if } \bar{W} \geq r_4(x). \end{cases} \tag{11}$$

Proof. Since $1 - x_i x_j \geq 0$ and ξ_{ij} are independent, according to the properties of trapezoidal fuzzy variables, $\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j)$ is also a trapezoidal fuzzy number and can be denoted by the quadruple $(r_1(x), r_2(x), r_3(x), r_4(x))$ defined by (9). From formula (6), the objective function in the MMC model (3)

$$\mathrm{Cr}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j) \geq \bar{W}\right\}$$

is equivalent to the real function $f(x)$ defined by (11). Therefore, the MMC model can be transformed into a crisp equivalent (10). ¶

Again this is also a crisp nonlinear optimization model.

5. A hybrid genetic algorithm

As mentioned in the Introduction, since the crisp Max-Cut problem is NP-hard, there are no efficient exact algorithms for solving its large-scale instances. The fuzzy Max-Cut problem will be more difficult to solve since it is an extension to the Max-Cut problem. So we do not try to design exact algorithms such as branch-and-bound algorithms and semi-definite programming algorithms. We examine the design of heuristic algorithms for efficiently solving the models EMC, α-OMC, MMC. Genetic algorithm (GA) is a widely used global optimization method with a random search mechanism [31] and has been successfully used in various fields. Another reason for using GA is that the coding of GA for the fuzzy Max-Cut problem is simple and can be achieved by binary vectors. Binary vectors form a moderate search space and thus make GA able to find global optimal solutions.

5.1. Construction of the solution space

We use a binary vector to express an individual code which represents a cut of the graph G (a feasible solution to EMC, α-OMC and MMC). The length of individuals (feasible solutions) in the solution space is the size of the vertex set in G. After labeling the vertices by $v_1, v_2, \ldots, v_n$, the value -1 or 1 on the ith position of an individual characterizes the membership of the ith vertex. Thus, all binary vectors over $\{-1, 1\}$ with length of n constitute the whole solution space:

$$S = \{(x_1, x_2, \ldots, x_n)^T | x_i \in \{-1, 1\}, i = 1, 2, \ldots, n\}.$$

5.2. Designation of the fitness function

For each individual (chromosome) in a population of GA, we need to assign a fitness value. According to the optimization models EMC, α-OMC and MMC, the fitness of an individual is dependent on the objective values of these models with this individual as a feasible solution. Hence, the following fitness functions are designed respectively for EMC, α-OMC and MMC:

$$U_1(x) = E\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j)\right], \tag{12}$$

$$U_2(x) = \sup\left\{\bar{W} \,|\, \mathrm{Cr}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j) \geq \bar{W}\right\} \geq \alpha\right\}, \tag{13}$$

$$U_3(x) = \mathrm{Cr}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j) \geq \bar{W}\right\}. \tag{14}$$

Obviously, the fitness of an individual is positively proportional to the objective value of the corresponding model. The goal of GA is to find an individual with highest fitness.

Note that the expected value E and the credibility measure Cr have no simple analytic expressions for general fuzzy variables. So we have to resort to other means in order to compute the fitness of an individual. Fuzzy simulation technique for computing E and Cr presented in [20] is an approximate way and has been widely used to compute the objective values of expected value models, chance-constraint programming models and dependent-chance programming models. The motivation for these fuzzy simulation techniques is that, according to the definition of the credibility, for any number $r \geq 0$, $\mathrm{Cr}\{f(x, \xi) \geq r\}$ can be approximated by

$$L_{\max}(x, r) = \frac{1}{2}\left(\max_{1\leq k\leq N}\{v_k | f(x, u_k) \geq r\} + \min_{1\leq k\leq N}\{1 - v_k | f(x, u_k) < r\}\right) \tag{15}$$

and for any number $r < 0$, $\mathrm{Cr}\{f(x, \xi) \leq r\}$ can be approximated by

$$L_{\min}(x, r) = \frac{1}{2}\left(\max_{1\leq k\leq N}\{v_k | f(x, u_k) \leq r\} + \min_{1\leq k\leq N}\{1 - v_k | f(x, u_k) > r\}\right), \tag{16}$$

where N is a very large positive number, $v_k = \mu_\xi(u_k) \wedge 1$, and u_k, $k = 1, 2, \ldots, N$ are randomly generated from the level set of ξ. Here we define

$$f(x, \xi) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1 - x_i x_j).$$

Fuzzy simulation for expected value

The following algorithm is for computing $U_1(x)$ by fuzzy simulation:

Step 1. Set $E = 0$;

Step 2. Randomly generate vector u_k from the ε-level sets of fuzzy vectors ξ, $k = 1, 2, \ldots, N$, where ε is a sufficiently small number;

Step 3. Let $a = \min_{1\leq k\leq N} f(x, u_k)$ and $b = \max_{1\leq k\leq N} f(x, u_k)$;

Step 4. Randomly generate a random number r from $[a, b]$;

Step 5. If $r \geq 0$, then $E \leftarrow E + \mathrm{Cr}\{f(x, \xi) \geq r\}$; If $r < 0$, then $E \leftarrow E - \mathrm{Cr}\{f(x, \xi) \leq r\}$;
Step 6. Repeat the fourth to fifth steps for N times.
Step 7. $U_1(x) = a \vee 0 + b \wedge 0 + E(b - a)/N$.

In this algorithm, N is a large positive integer; Generally, a larger N corresponds to a better approximation to expected values. The principle in this algorithm is based on the expected value definition of fuzzy variables. Note that computing Cr in this algorithm also needs the fuzzy simulation for credibility (in the following).

Fuzzy simulation for α-optimistic value

The following algorithm is for computing $U_2(x)$ by fuzzy simulation:
Step 1. Randomly generate $u_{ij,k}$ from the ε-level set of ξ_{ij}, and, where $k = 1, 2, \ldots, N$ and ε is a sufficiently small positive number.
Step 2. Set $\upsilon_k = \min_{ij} \mu_{\xi_{ij}}(u_{ij,k})$ for $k = 1, 2, \ldots, N$.
Step 3. Compute $L_{\max}(x, \bar{W})$ by using υ_k, $u_{ij,k}$ according to the definition (15);
Step 4. Find a maximum value $\bar{W}$ such that $L(x, \bar{W}) \geq \alpha$ holds.
Step 5. $U_2(x) = \bar{W}$.

Fuzzy simulation for credibility

The following algorithm is for computing $U_3(x)$ by fuzzy simulation:
Step 1. Randomly generate $u_{ij,k}$ from the ε-level set of ξ_{ij}, and, where $k = 1, 2, \ldots, N$ and ε is a sufficiently small positive number;
Step 2. Set $\upsilon_k = \min_{ij} \mu_{\xi_{ij}}(u_{ij,k})$ for $k = 1, 2, \ldots, N$;
Step 3. Compute $L_{\max}(x, \bar{W})$ by using υ_k, $u_{ij,k}$ according to the definition (15);
Step 4. $U_3(x) = L_{\max}(x, \bar{W})$.

5.3. Genetic operators

There are several kinds of selection operators for GA, such as tournament selection, rank selection and roulette wheel selection. Among them roulette wheel selection is very popular in GA algorithm. However, in order to avoid the crowding phenomenon and to yield a more diverse population, we adopt both tournament selection and roulette wheel selection in different parts of our algorithm. A combination of single-point crossover and uniform crossover is adopted. In addition, according to the principle that the fittest survives, we use roulette wheel selection operator to select individuals to crossover and generate offspring. Similarly, we adopt the combination of single-point mutation and swap mutation.

5.4. Flow steps of hybrid genetic algorithm

The details of our implementation of GA for the fuzzy Max-Cut problem are described as follows:
Step 0. Give proper parameter settings, e.g. population size *popsize*, crossover rate p_c, mutation rate p_m and the maximum number of population generation GN.
Step 1. Randomly generate *popsize* individuals as an initial population P_0, $k = 0$.
Step 2. Evaluate P_k, i.e. compute the fitness of every individual in P_k using the functions (12), (13) or (14) and retain the individual with the highest fitness.
Step 3. If $k < GN$, stop, return the individual with the highest fitness in the history, otherwise create a new generation P_{k+1} using the following method:

(1) Select $(1 - p_c)$*popsize* members of P_k and add them to P_{k+1} using tournament selection operator.

(2) Probabilistically select $P_c \cdot$ *popsize*/2 pairs of individuals from P_k using roulette wheel selection operator. For each pair (x_1, x_2), produce two offspring by randomly applying single-point crossover operator and uniform crossover operator. Add all offspring to P_{k+1}.

(3) Select p_m percentage of the individuals in P_{k+1} with uniform probability. For each, invert the value at a randomly selected position, or swap the values at two randomly selected positions.

Step 4. $k = k + 1$, go to Step 2.

6. Numerical experiments

In this section, we consider several numerical examples to illustrate the proposed fuzzy Max-Cut models and verify the effectiveness of the designed hybrid genetic algorithm. The algorithm is coded in Microsoft Visual C++ 6.0 and implemented on PC.

6.1. A simulated example

In the example shown in Fig. 1, there are totally 8 vertices and 13 edges. The weight on each edge is given in Table 1. Some of these weights are trapezoidal fuzzy variables with the membership function (5) and denoted by the quadruple

Table 1
The weights on the edges of the graph, where Tri. denotes triangular fuzzy variables, Trp. denotes trapezoidal fuzzy variables and Nor. denotes normal fuzzy variables.

No.	Edges (i, j)	Type	Weights ξ_{ij}
1	(1, 2)	Tri.	(1, 2, 3)
2	(1, 3)	Tri.	(0.5, 1, 1.8)
3	(1, 4)	Nor.	(5, 1)
4	(2, 3)	Trp	(0.2, 0.8, 1.2, 2)
5	(2, 5)	Nor.	(4, 1)
6	(3, 6)	Tri.	(6, 7, 8)
7	(3, 7)	Trp.	(9, 10, 11, 12)
8	(4, 6)	Nor.	(12, 1)
9	(4, 7)	Trp.	(12, 13, 14, 15)
10	(5, 6)	Nor.	(16, 1)
11	(5, 8)	Nor.	(3, 1)
12	(6, 8)	Trp.	(2, 3, 4, 5)
13	(7, 8)	Tri.	(7, 8, 9.5)

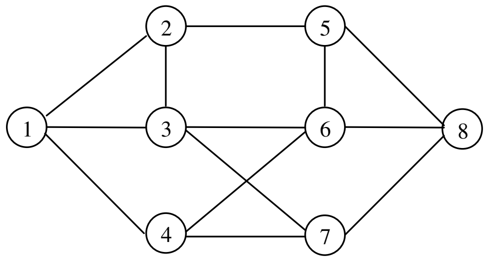

Fig. 1. A graph with 8 vertices and 13 edges.

(r_1, r_2, r_3, r_4) of crisp numbers with $r_1 < r_2 < r_3 < r_4$. Some of them are triangular fuzzy variables which are special trapezoidal fuzzy variables with $r_2 = r_3$ and thus denoted by a triplet (r_1, r_2, r_3) of crisp numbers with $r_1 < r_2 < r_3$. While other weights are normal fuzzy variables with the following membership function

$$\mu(x) = \exp(-(x - \bar{r})^2/\sigma)$$

and denoted by $(\bar{r}, \sigma)$.

In the GA implementation for solving the EMC, α-OMC and MMC models on this example, *popsize* is set as 30 and GN is set as 50. In addition, crossover rate $p_c = 0.5$ and mutation rate $p_m = 0.2$. In fact, GA is very robust with respect to these parameters. To solve the expected model of the fuzzy Max-Cut problem EMC, we use GA algorithm with the fitness function (12). The iteration process of GA on EMC is illustrated in Fig. 2(a). We obtain an expected Max-Cut $C = \{v_3, v_4, v_5, v_8\}$ with expected weight 82.35. Assume that the predetermined credibility level α given by decision makers is 0.8. The iteration process of GA on the chance-constraint programming model (α-OMC) is shown in Fig. 2(b). At this time, the fitness function (13) is used. We obtain an α-optimistic Max-Cut $C = \{v_3, v_4, v_5, v_8\}$ with an α-optimistic weight 78.78. Fig. 3(a) illustrates the effect of the predetermined credibility level α on α-optimistic Max-Cut weights. As expected, with the increase of α, the constraint of α-OMC becomes stronger and the weight W satisfying the constraint decreases. Finally, if a predetermined infimum $\bar{W}$ given by decision makers is 80, we use GA with the fitness function (14) to solve the corresponding dependent-chance programming model (MMC, See Fig. 2(c)) and obtain a most Max-Cut $C = \{v_3, v_4, v_5, v_8\}$ with a most credibility 0.755. Fig. 3(b) shows the effect of the predetermined infimum $\bar{W}$ on the most credibility. Again, as expected, with the increase of $\bar{W}$, the requirement of decision makers becomes stricter, so the credibility that the best solution can satisfy the requirement decreases. From Fig. 2 we can see that the designed hybrid genetic algorithm is able to converge to a good solution with high fitness (a solution with optimal or near optimal objective value) in only a few iterations, demonstrating its effectiveness.

6.2. A practical example

Cluster analysis aims to partition a set of data points into groups of closely related observations. The points in a particular group or cluster should be more similar to each other than to points in other clusters. Graph cut-based clustering methods have been widely used, including Max-Cut, Min-Cut, Min-Max-Cut and normalized cut [32,3,33]. When the objective function of clustering is to minimize the within-cluster sums of dissimilarities or distances [34], the clustering problem can be formulated as a Max-Cut problem by creating a graph with vertices representing data points, edges with weights denoting the dissimilarity or distance between a pair of data points. An online software for max-cut clustering can be found at http://riot.ieor.berkeley.edu/riot/Applications/Clustering/.

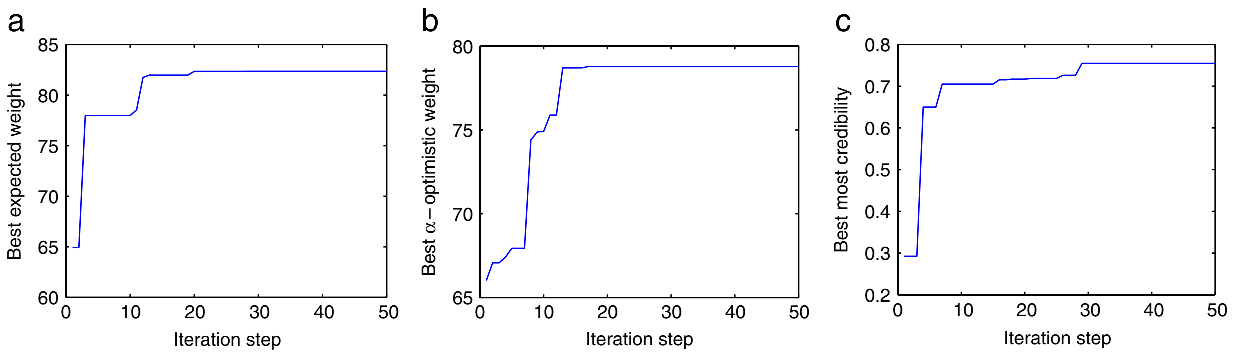

Fig. 2. (a) The iteration process of GA on the expected value model (EMC). (b)The iteration process of GA on the chance-constraint programming model (α-OMC). (c) The iteration process of GA on the dependent-chance programming model (MMC).

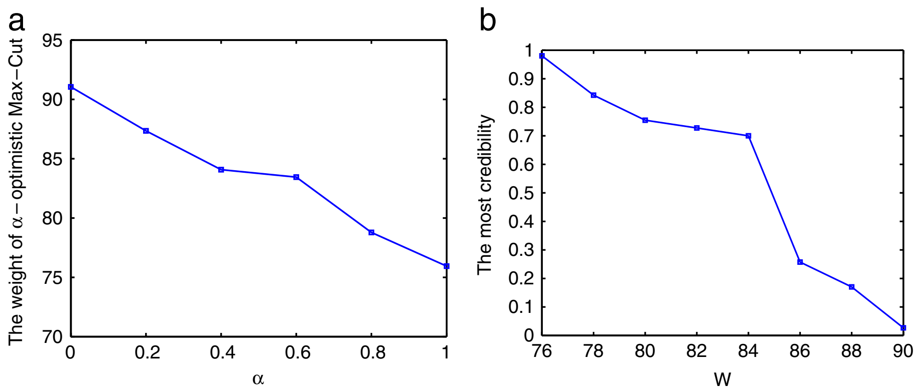

Fig. 3. (a) The plot of the weight of the α-optimistic Max-Cut with respect to α. (b) The plot of the most credibility with respect to the predetermined infimum W.

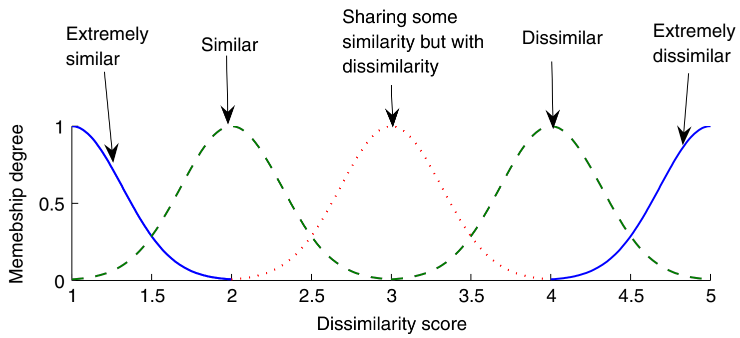

Fig. 4. Membership functions of linguistic distance measures.

We give an example where max-cut-based clustering needs to handle imprecise weights (dissimilarities) between data points. With the rapid progress in the internet technology, online marketing is becoming a promising way for selling diverse kinds of commodities. In order to make marketing more effective such as advertising and promotion, it is often demanded to collect the characteristics of consumers and classify them into different types according to their consumption records. Assume there is a consumer population who visit an online market often and the marketing manager wants to cluster the consumers into two groups. The similarity between consumers, which is critical for the classification, needs to be determined according to multiple factors such as age, fashion preference, and past shopping records. A simple real number is hard to describe the similarity between consumers comprehensively. The expert scoring method is usually used [13], which integrates the scores or suggestions (sometime, they are linguistic such as 'very similar', 'similar', and 'not similar at all') of individual experts into a fuzzy number represented by a membership function. Here five linguistic similarity (dissimilarity) degrees are used, with their membership functions shown in Fig. 4.

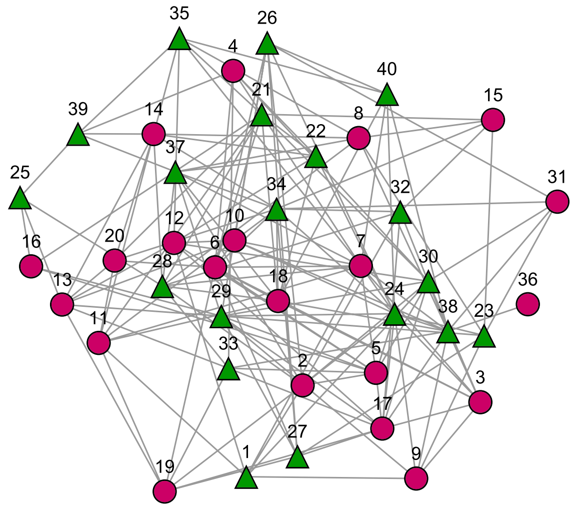

Fig. 5. The clustering result by solving the expected Max-Cut problem.

A graph with 40 consumers as vertices is generated. To test if the fuzzy max-cut-based clustering is able to classify the consumers correctly, we design the graph in such way: the relationships within the first 20 consumers and the last 20 consumers are randomly set as 'extremely similar' or 'similar' with 80 edges preserved, and the relationships between these two groups are randomly set as 'extremely dissimilar' or 'dissimilar' with 80 edges preserved. The goal is to test if the fuzzy max-cut-based clustering can cluster these consumers into two groups correctly. We applied our GA algorithm (with $popsize = 150$ and $GN = 500$) to solve the expected max-cut model (EMC) and cluster these consumers into two groups. The result is illustrated in Fig. 5, where the cut is denoted by different vertex shapes. We can see that most of consumers with high dissimilarities are clustered into different groups. Only three vertices are misclassified: vertex 1, vertex 31 and vertex 36. The misclassification of vertex 36 is because vertex 36 is a leaf node and has only a single connection. Thus putting it into the other group will increase the weight of maximum cut. The results given by the α-OMC and MMC models depend on the preferences of decision makers such as predetermined confidence level α and predetermined infimum $\bar{W}$, which are not shown here.

6.3. Benchmark examples

In order to further examine the effectiveness of GA algorithm and test our models on larger instances, we adapt some benchmark examples for the classical Max-Cut problem for our use [35]. Specifically, topology structure in this graph is preserved, and each edge is assigned with a normal fuzzy weight with $r = 1$ and σ being a random number in [0, 0.5]. If ignoring σ, GA is actually solving the classical Max-Cut problem. When considering fuzzy weights, GA is used to solve the fuzzy Max-Cut problem through the EMC, α-OMC and MMC models. For these larger instances, the parameters in GA algorithm are set as those in the last subsection. The results are listed in Tables 2 and 3. Since GA may return differential solutions upon different initial settings, the results shown here are the averaged solutions over ten execution times. As we know, the Max-Cut problem is NP-hard. Exact algorithms for even such instances of medium size take unacceptable time. From Tables 2 and 3, we can see when GA is used to solve the classical Max-Cut problem, it can obtain solutions quite near to the optima. The average running time for each instance is about 30 seconds. When solving the EMC, α-OMC and MMC models, the main part of running time is spent on fuzzy simulation for calculating expected values, α-optimistic values and credibilities, so GA takes more than one hour for each instance. With the optimal solutions as reference, we can see that the obtained solutions for EMC are also nearly optimal. The solutions of α-OMC indicate that when optimal solutions are hard to obtain, the solutions with α confidence level to optima, which are obviously smaller than optimal solutions, are preferred. The solutions of MMC indicate that for a preferred maximum cut infimum, what their confidence levels are in comparison with optima.

7. Conclusion and discussion

In this paper, we investigate the Max-Cut problem with fuzzy weights. The fuzzy Max-Cut problem was formulated into an expected valued model, a chance-constraint programming model and a dependent-chance programming model according to different decision criteria/fuzzy ranking methods. Their crisp equivalents under some special conditions are also discussed based on credibility theory. In addition, a genetic algorithm combined fuzzy simulation techniques is designed for solving the fuzzy Max-Cut problem under the presented models. Numerical experiments illustrate the algorithm and confirm its effectiveness. As we know, fuzziness is not the only kind of uncertain phenomena in the real world. Randomness

Table 2
The results of GA for the classical max-cut problem and the fuzzy max-cut problem on 60-node instances, where $\alpha = 0.85$ for α-OMC and $\bar{W} = 515$ for MMC.

Example name	Optimum	GA solution	EMC	α-OMC	MMC
G05_60.0	536	534.8	533.6	519.6	1.000
G05_60.1	532	529.5	532.1	517.9	0.801
G05_60.2	529	526.3	528.2	514.4	0.802
G05_60.3	538	535.0	535.5	520.1	1.000
G05_60.4	527	525.5	526.2	513.3	0.600
G05_60.5	533	529.6	528.9	517.7	0.899
G05_60.6	531	528.3	530.3	516.0	0.901
G05_60.7	535	532.5	533.3	520.1	1.000
G05_60.8	530	526.3	527.8	516.2	0.898
G05_60.9	533	531.6	531.3	517.8	0.921

Table 3
The results of GA for the classical max-cut problem and the fuzzy max-cut problem on 100-node instances, where $\alpha = 0.85$ for α-OMC and $\bar{W} = 1410$ for MMC.

Example name	Optimum	GA solution	EMC	α-OMC	MMC
G05_100.0	1430	1423.2	1421.2	1396.0	0.811
G05_100.1	1425	1420.2	1421.5	1397.5	0.804
G05_100.2	1432	1422.7	1424.9	1398.3	0.863
G05_100.3	1424	1416.0	1412.8	1389.8	0.600
G05_100.4	1440	1427.2	1432.3	1408.5	1.000
G05_100.5	1436	1424.6	1433.0	1412.8	0.987
G05_100.6	1434	1426.2	1430.9	1408.4	0.870
G05_100.7	1431	1422.0	1425.2	1402.0	0.801
G05_100.8	1432	1420.6	1417.4	1398.1	0.802
G05_100.9	1430	1417.8	1420.5	1396.2	0.737

and mixed uncertainness of randomness and fuzziness are also common in real decision systems. We may investigate the Max-Cut problem under these kinds of uncertain environments in the future research work. In addition, although we adopt the credibility measure to characterize fuzzy phenomena, the framework and algorithm provided in this paper also suit well for the models using possibility or necessity measures.

There are some limitations in the current study. Although the algorithm discussed in this work is efficient for instance of small and medium size, its running time will become very intense for very large-scale problems due to the fuzzy simulation part for evaluating objective functions. This is the price for the ability of the algorithms and models to handle any type of fuzzy variables. Other ranking methods of fuzzy variables may be explored in the future research [11]. In addition, some local search techniques can be incorporated into GA to further improve its efficiency. Finally, in this study, we only consider fuzzy Max-Cut problems in a moderate dimension. To handle the large-scale instances of Max-Cut problems in fuzzy nature, it will be better to build on the new progresses on heuristic algorithms for the Max-Cut problem such as the discrete filled function algorithm in [36].

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