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Identification of a memory kernel in a semilinear integrodifferential parabolic problem with applications in heat conduction with memory

K. Van Bockstal *, R.H. De Staelen, M. Slodička

Department of Mathematical Analysis, Research group of Numerical Analysis and Mathematical Modeling (NaM[2]), Ghent University, Galglaan 2 - S22, Gent 9000, Belgium

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ABSTRACT

In this contribution, the reconstruction of a solely time-dependent convolution kernel is studied in an inverse problem arising in the theory of heat conduction for materials with memory. The missing kernel is recovered from a measurement of the average of temperature. The existence, uniqueness and regularity of a weak solution is addressed. More specific, a new numerical algorithm based on Rothe's method is designed. The convergence of iterates to the exact solution is shown.

1. Introduction

Identification of missing memory kernels in partial integrodifferential equations is relatively new in inverse problems (IPs). Some references are [1–5]. For instance, Ref. [5] derives some local and global in time existence results for the recovery of solely time-dependent memory kernels in semilinear integrodifferential models. More specific, they studied the evolution equation for materials with memory. This equation is given by

$$\partial_t u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) + \int_0^t K(t-s)\Delta u(\mathbf{x}, s)\mathrm{d}s + F(u(\mathbf{x}, t)), \quad \mathbf{x} \in \Omega_0 \subset \mathbb{R}^3,\ t \in [0, T_0].$$

To determine the memory kernel K an additional measurement on u is needed; $\int_\Omega \phi(\mathbf{x})u(\mathbf{x}, t)\mathrm{d}x = G(t),\ \forall t \in [0, T_0]$. In these references, there is no description of constructive algorithms how to find a solution. The construction of a numerical algorithm for this type of problems is the central theme of this article. The following inverse problem for a semilinear parabolic equation with memory is considered: determine the unknown couple $\langle u, K\rangle$ obeying

$$\begin{cases} \partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + K(t)h(\mathbf{x}, t) - (K * \Delta u(\mathbf{x}))(t) = f(u(\mathbf{x}, t)), & \text{in } \Omega \times I, \\ \alpha(u(\mathbf{x}, t)) + \nabla u(\mathbf{x}, t) \cdot \nu = g(\mathbf{x}, t), & \text{on } \Gamma \times I, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a Lipschitz domain [6] in $\mathbb{R}^N$, $N \geq 1$, with $\partial\Omega = \Gamma$ and $I = [0, T]$, $T > 0$, is the time frame. The usual convolution in time is denoted by $K * u$, namely $(K * u(\mathbf{x}))(t) = \int_0^t K(t-s)u(\mathbf{x}, s)\mathrm{d}s$. The missing time-convolution kernel $K = K(t)$

* Corresponding author.
E-mail addresses: karel.vanbockstal@ugent.be (K. Van Bockstal), rob.destaelen@ugent.be (R.H. De Staelen), marian.slodicka@ugent.be (M. Slodička).
URLs: http://cage.ugent.be/~kvb (K. Van Bockstal), http://cage.ugent.be/~rds (R.H. De Staelen), http://cage.ugent.be/~ms (M. Slodička).

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will be recovered from the following integral-type measurement

$$\int_\Omega u(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = m(t), \quad t \in [0, T]. \tag{2}$$

Note that this equation is also used in the modeling of phenomena in viscoelasticity [7]. The integral type over-determination in IPs combined with evolutionary partial differential equations (PDEs) has been studied in several other papers, e.g. [8–10] and the references therein.

The main goal of this paper is to design a productive numerical scheme describing a way of retrieving the couple $\langle u, K\rangle$. This is achieved not by the minimization of a cost functional (which is typical for IPs) but by the semi-discretization in time by Rothe's method [11,12]. First, this introduction is finished with the derivation of a suitable variational formulation. Section 2 is devoted to the study of regularity of a weak solution and its uniqueness is addressed in Theorem 1. Section 3 deals with the time discretization, where (based on the backward Euler scheme) the continuous problem is approximated by a sequence of steady state settings at each point of a time partitioning. Stability analysis of approximates is performed in appropriate function spaces and convergence (based on compactness argument) is established in Theorem 2.

Notations. Denote by $(\cdot,\cdot)$ the standard inner product of $\mathrm{L}^2(\Omega)$ and $\|\cdot\|$ its induced norm. A similar notation is used when working at the boundary Γ, namely $(\cdot,\cdot)_\Gamma$, $\mathrm{L}^2(\Gamma)$ and $\|\cdot\|_\Gamma$. Consider an abstract Banach space X with norm $\|\cdot\|_X$. The set of continuous abstract functions $w : [0, T] \to X$ equipped with the usual norm $\max_{t\in[0,T]} \|\cdot\|_X$ is denoted by $\mathrm{C}([0, T], X)$. The space $\mathrm{L}^p((0, T), X)$ is furnished with the norm $\left(\int_0^T \|\cdot\|_X^p\right)^{\frac{1}{p}}$ with $p > 1$, cf. [13]. The symbol X^* stands for the dual space to X. Finally, as is usual in papers of this sort, C, ε and C_ε denote generic positive constants depending only on a priori known quantities, where ε is small and $C_\varepsilon = C\left(\varepsilon^{-1}\right)$ is large.

Derivation of the variational problem. First, the PDE in (1) is multiplied with a test function $\phi \in \mathrm{H}^1(\Omega)$ and integrated over Ω to obtain that

$$(\partial_t u, \phi) - (\Delta u, \phi) + K(h, \phi) - (K * \Delta u, \phi) = (f(u), \phi). \tag{3}$$

Secondly, using Green's first identity implies that

$$(\partial_t u, \phi) + (\nabla u, \nabla\phi) + K(h, \phi) + (K * \nabla u, \nabla\phi) = (f(u), \phi) + (g - \alpha(u), \phi)_\Gamma + (K * (g - \alpha(u)), \phi)_\Gamma . \tag{P}$$

Finally, we set $\phi = 1$ in (P) and obtain together with the measurement $\int_\Omega u(t) = m(t)$ that

$$m' + K\int_\Omega h = \int_\Omega f(u) + \int_\Gamma (g - \alpha(u)) + \int_\Gamma K * (g - \alpha(u)). \tag{MP}$$

The relations (P) and (MP) represent the variational formulation of (1) and (2).

2. Stability analysis of a solution, uniqueness

First, this section starts with a study of natural regularity of a solution $\langle u, K\rangle$. This helps us to choose appropriate function spaces for the variational framework. Uniqueness of a solution is addressed at the end of this section. Two frequently used estimates for the convolution term are [14, Lemma 1]:

Proposition 2.1. *Set $I = [0, \eta]$, $\eta > 0$. Suppose $\kappa \in \mathrm{L}^2(I)$ and $\upsilon \in \mathrm{L}^2(I, \mathrm{L}^2(\Omega))$, then it holds that*

$$\|\kappa * \upsilon\|^2 \le \kappa^2 * \|\upsilon\|^2 , \tag{$*$}$$

$$\int_0^\eta \|\kappa * \upsilon\|^2 \le \int_0^\eta |\kappa|^2 \int_0^\eta \|\upsilon\|^2 . \tag{$**$}$$

Remark. Note that the estimates ($*$) and ($**$) also hold when $\kappa \in \mathrm{L}^2(I)$ and $\upsilon \in \mathrm{L}^2(I, \mathrm{L}^2(\Gamma))$ in the appropriate norm.

Proposition 2.2. *Let f and α be bounded, i.e. $|f| \le C$ and $|\alpha| \le C$. Moreover, assume that $u_0 \in \mathrm{L}^2(\Omega)$, $g \in \mathrm{C}\left([0, T], \mathrm{L}^2(\Gamma)\right)$, $h \in \mathrm{C}\left([0, T], \mathrm{L}^2(\Omega)\right)$, $\min_{t\in[0,T]} \left|\int_\Omega h(t)\right| \ge \omega > 0$ and $m \in \mathrm{C}^1([0, T])$. If $\langle u, K\rangle$ is a solution of (1) and (2), then K is bounded on $[0, T]$, i.e.*

$$\max_{t\in[0,T]} |K(t)| \le C.$$

Proof. Take any $t \in [0, T]$. From (MP) it follows that

$$\left|K(t)\int_\Omega h(t)\right| \le \int_\Omega |f(u(t))| + \int_\Gamma |(g(t) - \alpha(u(t)))| + \int_\Gamma |(K * (g - \alpha(u)))(t)| + \left|m'(t)\right| .$$

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Involving the assumptions on the data, we see that

$$\begin{aligned}\omega\left|K(t)\right| &\leq \left|\int_{\Omega} h(t)\right|\left|K(t)\right| \\ &\leq C+\int_{\Gamma}\left|(K*(g-\alpha(u)))(t)\right| \leq C+\int_{\Gamma}\int_0^t \left|K(s)\right|\left|g(\mathbf{x},t-s)-\alpha(u(\mathbf{x},t-s))\right| \mathrm{d}s\, \mathrm{d}\mathbf{x} \\ &\leq C+\int_0^t \left|K(s)\right| \left\|(g-\alpha(u))(t-s)\right\|_{\mathrm{L}^1(\Gamma)} \mathrm{d}s \leq C+C\int_0^t \left|K(s)\right| \mathrm{d}s.\end{aligned}$$

The proof is concluded by Grönwall's argument, cf. [15]. □

Proposition 2.3. *Let the conditions of Proposition 2.2 be satisfied. If $\langle u, K\rangle$ is a solution of (1) and (2), then there exists $C > 0$ such that*

(i) $\max_{t\in[0,T]} \|u(t)\|^2 + \int_0^T \|\nabla u\|^2 \leq C$

(ii) $\int_0^T \|\partial_t u\|^2_{(\mathrm{H}^1(\Omega))^*} \leq C.$

Proof. (i) If we set $\phi = u$ in (P) and integrate in time over $(0, t)$, we obtain

$$\begin{aligned}&\int_0^t (\partial_t u, u) + \int_0^t \|\nabla u\|^2 + \int_0^t K\,(h, u) + \int_0^t (K*\nabla u, \nabla u) \\ &\quad = \int_0^t (f(u), u) + \int_0^t (g-\alpha(u), u)_\Gamma + \int_0^t \left(\left(K*(g-\alpha(u))\right)(\xi), u(\xi)\right)_\Gamma \mathrm{d}\xi.\end{aligned} \tag{4}$$

The first term on the left-hand side (LHS) can be rewritten as

$$\int_0^t (\partial_t u, u) = \frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|u_0\|^2.$$

For the third term, we get by the boundedness of K (see Proposition 2.2) that

$$\left|\int_0^t K(h,u)\right| \leq \int_0^t |K|\,\|h\|\,\|u\| \leq C\int_0^t \|h\|^2 + C\int_0^t \|u\|^2.$$

The fourth one is bounded by

$$\begin{aligned}\left|\int_0^t ((K*\nabla u)(\xi), \nabla u(\xi))\,\mathrm{d}\xi\right| &\leq C_\varepsilon \int_0^t \left\|\int_0^\xi K(\xi-s)\nabla u(s)\mathrm{d}s\right\|^2 \mathrm{d}\xi + \varepsilon\int_0^t \|\nabla u(\xi)\|^2 \mathrm{d}\xi \\ &\leq C_\varepsilon \int_0^t\int_0^\xi \|\nabla u(s)\|^2 \mathrm{d}s\mathrm{d}\xi + \varepsilon\int_0^t \|\nabla u(\xi)\|^2 \mathrm{d}\xi,\end{aligned}$$

due to Young's inequality, Jensen's inequality and the boundedness of K. The first term on the right-hand side (RHS) of (4) can be estimated as follows

$$\left|\int_0^t (f(u), u)\right| \leq \int_0^t \|f(u)\|\,\|u\| \leq \frac{1}{2}\int_0^t \|f(u)\|^2 + \frac{1}{2}\int_0^t \|u\|^2 \leq C + \frac{1}{2}\int_0^t \|u\|^2,$$

as f is bounded. The last term on the RHS of (4) is bounded by

$$\begin{aligned}\left|\int_0^t (K*(g-\alpha(u)), u)_\Gamma\right| &\leq C\int_0^t \|K*(g-\alpha(u))\|_\Gamma \|u\|_{\mathrm{H}^1(\Omega)} \\ &\leq C_\varepsilon \int_0^t \|K*(g-\alpha(u))\|^2_\Gamma + \varepsilon\int_0^t \|u\|^2_{\mathrm{H}^1(\Omega)} \\ &\leq C_\varepsilon \int_0^t |K|^2 \int_0^t \|(g-\alpha(u))\|^2_\Gamma + \varepsilon\int_0^t \|u\|^2_{\mathrm{H}^1(\Omega)} \\ &\leq C_\varepsilon + \varepsilon\int_0^t \|u\|^2_{\mathrm{H}^1(\Omega)}\end{aligned}$$

by Cauchy's inequality, the estimate (∗∗), the trace theorem and the assumptions on g and α. The estimation of the second term is similar to the last one. Putting all things together, fixing a sufficiently small $\varepsilon > 0$ and taking into account $\|u\|^2_{\mathrm{H}^1(\Omega)} = \|u\|^2 + \|\nabla u\|^2$, we obtain that

$$\|u(t)\|^2 + \int_0^t \|\nabla u(\xi)\|^2 \,\mathrm{d}\xi \le C + C\int_0^t \|u\|^2 \,\mathrm{d}\xi + C\int_0^t\int_0^\xi \|\nabla u(s)\|^2 \,\mathrm{d}s\mathrm{d}\xi,$$

which is valid for any $t \in [0, T]$. An application of Grönwall's lemma concludes the proof.

(ii) Starting from (P) and using the Cauchy inequality, the boundedness of K (see Proposition 2.2), the assumptions on the data, bound (∗), the trace theorem and point (i) of this proposition, we successively deduce that

$$\begin{aligned}
|(\partial_t u, \phi)| &= \left|(f(u), \phi) + (g - \alpha(u), \phi)_\Gamma + (K * (g - \alpha(u)), \phi)_\Gamma - (\nabla u, \nabla\phi) - K\,(h, \phi) - (K * \nabla u, \nabla\phi)\right| \\
&\le C\left(\|\phi\| + \|\phi\|_\Gamma + \|K * (g - \alpha(u))\|_\Gamma \|\phi\|_\Gamma + \|\nabla u\|\,\|\nabla\phi\| + \|K * \nabla u\|\,\|\nabla\phi\|\right) \\
&\le C\left(\|\phi\| + \|\phi\|_\Gamma + \sqrt{K^2 * \|g - \alpha(u)\|^2_\Gamma}\,\|\phi\|_\Gamma + \|\nabla u\|\,\|\nabla\phi\| + \sqrt{K^2 * \|\nabla u\|^2}\,\|\nabla\phi\|\right) \\
&\le C\left(\|\phi\| + \|\phi\|_\Gamma + \|\nabla u\|\,\|\nabla\phi\| + \sqrt{\int_0^t \|\nabla u\|^2}\,\|\nabla\phi\|\right) \\
&\le C\left(\|\nabla u\|\,\|\nabla\phi\| + \|\phi\|_{\mathrm{H}^1(\Omega)}\right).
\end{aligned}$$

Thus, $(\partial_t u, \phi)$ can be seen as a linear functional on $\mathrm{H}^1(\Omega)$ and we may write

$$\|\partial_t u\|_{(\mathrm{H}^1(\Omega))^*} = \sup_{\|\phi\|_{\mathrm{H}^1(\Omega)} \le 1} |(\partial_t u, \phi)| \le C\,(1 + \|\nabla u\|),$$

which implies by (i) that

$$\int_0^T \|\partial_t u\|^2_{(\mathrm{H}^1(\Omega))^*} \le C + C\int_0^T \|\nabla u\|^2 \le C. \quad \square$$

The Rellich–Kondrachov theorem [16, Section 5.8.1] implies that

$$\mathrm{H}^1(\Omega) \subset\subset \mathrm{L}^2(\Omega) \cong \left(\mathrm{L}^2(\Omega)\right)^* \subset\subset \left(\mathrm{H}^1(\Omega)\right)^*.$$

From the previous propositions and [17, Lemma 7.3], the following corollary follows immediately.

Corollary 2.4. *If $\langle u, K\rangle$ is a solution of (1) and (2), then $K \in \mathrm{L}^2(0, T)$ and $u \in \mathrm{C}\left([0, T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0, T), \mathrm{H}^1(\Omega)\right)$ with $\partial_t u \in \mathrm{L}^2\left((0, T), \left(\mathrm{H}^1(\Omega)\right)^*\right)$.*

Uniqueness. Now, it is possible to establish the uniqueness of a solution to (P)–(MP). The proof is by contradiction. Suppose that there are two solutions $\langle u_1, K_1\rangle$ and $\langle u_2, K_2\rangle$ solving (P)–(MP). By subtracting the corresponding variational formulations follows that

$$\begin{aligned}
&(\partial_t(u_1 - u_2), \phi) + (\nabla(u_1 - u_2), \nabla\phi) + (K_1 - K_2)\,(h, \phi) + (K_1 * \nabla u_1 - K_2 * \nabla u_2, \nabla\phi) \\
&\quad = (f(u_1) - f(u_2), \phi) + (\alpha(u_2) - \alpha(u_1), \phi)_\Gamma + (K_1 * (g - \alpha(u_1)) - K_2 * (g - \alpha(u_2)), \phi)_\Gamma
\end{aligned}$$

and

$$(K_1 - K_2)\int_\Omega h = \int_\Omega (f(u_1) - f(u_2)) + \int_\Gamma (\alpha(u_2) - \alpha(u_1)) + \int_\Gamma [K_1 * (g - \alpha(u_1)) - K_2 * (g - \alpha(u_2))].$$

Denote the difference of the solutions by $e_K(t) = K_1(t) - K_2(t)$ and $e_u(\mathbf{x}, t) = u_1(\mathbf{x}, t) - u_2(\mathbf{x}, t)$ in $\Omega \times I$. Then the previous equations can be rewritten as

$$\begin{aligned}
&(\partial_t e_u, \phi) + (\nabla e_u, \nabla\phi) + e_K\,(h, \phi) + (e_K * \nabla u_1 + K_2 * \nabla e_u, \nabla\phi) \\
&\quad = (f(u_1) - f(u_2), \phi) + (\alpha(u_2) - \alpha(u_1), \phi)_\Gamma + (e_K * g, \phi)_\Gamma + (K_2 * (\alpha(u_2) - \alpha(u_1)) - e_K * \alpha(u_1), \phi)_\Gamma
\end{aligned} \tag{5}$$

and

$$e_K\int_\Omega h = \int_\Omega (f(u_1) - f(u_2)) + \int_\Gamma (\alpha(u_2) - \alpha(u_1)) + \int_\Gamma e_K * g + \int_\Gamma [K_2 * (\alpha(u_2) - \alpha(u_1)) - e_K * \alpha(u_1)]. \tag{6}$$

In the proof of uniqueness, the Nečas inequality [18] is crucial, i.e.

$$\|z\|^2_\Gamma \le \varepsilon\,\|\nabla z\|^2 + C_\varepsilon\,\|z\|^2, \quad \forall z \in \mathrm{H}^1(\Omega),\ 0 < \varepsilon < \varepsilon_0. \tag{7}$$

Theorem 1 (*Uniqueness*). *Assume that $h \in \mathrm{C}\left([0, T], \mathrm{L}^2(\Omega)\right)$, $g \in \mathrm{C}\left([0, T], \mathrm{L}^2(\Gamma)\right)$, $\min_{t\in[0,T]}\left|\int_\Omega h(t)\right| \ge \omega > 0$, $u_0 \in \mathrm{L}^2(\Omega)$ and $m \in \mathrm{C}^1([0, T])$. The bounded functions f and α are supposed to be Lipschitz continuous. Then the problem (P)–(MP) has at most one solution $\langle u, K\rangle \in \left[\mathrm{C}\left([0, T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0, T), \mathrm{H}^1(\Omega)\right)\right] \times \mathrm{L}^2(0, T)$ with $\partial_t u \in \mathrm{L}^2\left((0, T), \left(\mathrm{H}^1(\Omega)\right)^*\right)$.*

Proof. Consider Eq. (6). The Lipschitz continuity of f and α, the boundedness of α and K_2 imply that

$$\begin{aligned}\omega\,|e_K| &\leqslant \left| e_K \int_\Omega h \right| \\ &\leqslant C\left(\|f(u_1)-f(u_2)\| + \|\alpha(u_2)-\alpha(u_1)\|_\Gamma + \|e_K * g\|_\Gamma + \|K_2 * (\alpha(u_2)-\alpha(u_1))\|_\Gamma + \|e_K * \alpha(u_1)\|_\Gamma\right) \\ &\leqslant C\left(\|e_u\| + \|e_u\|_\Gamma + \sqrt{\int_0^t |e_K(s)|^2\,\mathrm{d}s} + \sqrt{\int_0^t \|e_u(s)\|_\Gamma^2\,\mathrm{d}s}\right).\end{aligned}$$

Therefore, using the Nečas inequality (7) and the trace inequality, we get for $t \in (0, T]$ and γ small enough that

$$|e_K(t)|^2 \leqslant C_\gamma \|e_u(t)\|^2 + \gamma \|\nabla e_u(t)\|^2 + C\int_0^t |e_K(s)|^2\,\mathrm{d}s + C\int_0^t \|e_u(s)\|^2_{\mathrm{H}^1(\Omega)}\,\mathrm{d}s.$$

An application of Grönwall's lemma gives

$$\begin{aligned}|e_K(t)|^2 &\leqslant C_\gamma \|e_u(t)\|^2 + \gamma \|\nabla e_u(t)\|^2 + C\int_0^t \|e_u(s)\|^2_{\mathrm{H}^1(\Omega)}\,\mathrm{d}s \\ &\quad + C\int_0^t \left(C_\gamma \|e_u(\xi)\|^2 + \gamma\|\nabla e_u(\xi)\|^2 + C\int_0^\xi \|e_u(s)\|^2_{\mathrm{H}^1(\Omega)}\,\mathrm{d}s\right)\exp(CT)\mathrm{d}\xi\end{aligned}$$

and therefore

$$|e_K(t)|^2 \leqslant C_\gamma \|e_u(t)\|^2 + \gamma\|\nabla e_u(t)\|^2 + C_\gamma \int_0^t \|e_u(s)\|^2_{\mathrm{H}^1(\Omega)}\,\mathrm{d}s. \tag{8}$$

Now, we put $\phi = e_u(t)$ in (5) and integrate in time over $(0, \eta)$ to get

$$\begin{aligned}&\frac{1}{2}\|e_u(\eta)\|^2 + \int_0^\eta \|\nabla e_u\|^2 + \int_0^\eta e_K\,(h, e_u) + \int_0^\eta (e_K * \nabla u_1, \nabla e_u) + \int_0^\eta (K_2 * \nabla e_u, \nabla e_u) \\ &\quad = \int_0^\eta (f(u_1)-f(u_2), e_u) + \int_0^\eta (\alpha(u_2)-\alpha(u_1), e_u)_\Gamma \\ &\qquad + \int_0^\eta (e_K * g, e_u)_\Gamma + \int_0^\eta (K_2 * (\alpha(u_2)-\alpha(u_1)), e_u)_\Gamma - \int_0^\eta (e_K * \alpha(u_1), e_u)_\Gamma\,.\end{aligned} \tag{9}$$

This equality has to be estimated term by term. For the third term on the LHS, we get using the Cauchy and Young inequalities and $h \in \mathrm{C}\left([0, T], \mathrm{L}^2(\Omega)\right)$ that

$$\left|\int_0^\eta e_K\,(h, e_u)\right| \leqslant \int_0^\eta |e_K|\,\|h\|\,\|e_u\| \le C\int_0^\eta |e_K|^2 + C\int_0^\eta \|e_u\|^2\,.$$

For the fourth term on the LHS, we obtain due to $u_1 \in \mathrm{L}^2\left((0, T), \mathrm{H}^1(\Omega)\right)$ that

$$\left|\int_0^\eta (e_K * \nabla u_1, \nabla e_u)\right| \leqslant C_\varepsilon \int_0^\eta \|e_K * \nabla u_1\|^2 + \varepsilon\int_0^\eta \|\nabla e_u\|^2 \overset{(**)}{\le} C_\varepsilon \int_0^\eta |e_K|^2 + \varepsilon \int_0^\eta \|\nabla e_u\|^2\,.$$

By the boundedness of K_2, we get for the last term on the LHS that

$$\left|\int_0^\eta ((K_2 * \nabla e_u)(t), \nabla e_u(t))\,\mathrm{d}t\right| \leqslant C_\varepsilon \int_0^\eta\int_0^t \|\nabla e_u(s)\|^2\,\mathrm{d}s\mathrm{d}t + \varepsilon\int_0^\eta \|\nabla e_u(t)\|^2\,\mathrm{d}t.$$

For the first term on the RHS, we obtain by the Lipschitz continuity of f that

$$\left|\int_0^\eta (f(u_1)-f(u_2), e_u)\right| \le \int_0^\eta \|f(u_1)-f(u_2)\|\,\|e_u\| \le C\int_0^\eta \|e_u\|^2\,.$$

Analogously, by the Lipschitz continuity of α and the Nečas inequality (7), we have that

$$\left|\int_0^\eta (\alpha(u_2)-\alpha(u_1), e_u)_\Gamma\right| \le C\int_0^\eta \|e_u\|_\Gamma^2 \le C_\varepsilon \int_0^\eta \|e_u\|^2 + \varepsilon\int_0^\eta \|\nabla e_u\|^2\,.$$

The third term on the RHS obeys

$$\begin{aligned}\left|\int_0^\eta (e_K * g, e_u)_\Gamma\right| &\leqslant \frac{1}{2}\int_0^\eta \|e_K * g\|_\Gamma^2 + \frac{1}{2}\int_0^\eta \|e_u\|_\Gamma^2 \\ &\overset{(**),(7)}{\le} C\int_0^\eta |e_K|^2 + C_\varepsilon\int_0^\eta \|e_u\|^2 + \varepsilon\int_0^\eta \|\nabla e_u\|^2\,.\end{aligned}$$

For the fourth term, we get by the boundedness of K_2 that

$$\begin{aligned}\left|\int_0^\eta (K_2 * (\alpha(u_2)-\alpha(u_1)), e_u)_\Gamma\right| &\le \frac{1}{2}\int_0^\eta \|K_2 * (\alpha(u_2)-\alpha(u_1))\|_\Gamma^2 + \frac{1}{2}\int_0^\eta \|e_u\|_\Gamma^2 \\ &\overset{(**)}{\le} C\int_0^\eta \|e_u\|_\Gamma^2 \\ &\overset{(7)}{\le} C_\varepsilon \int_0^\eta \|e_u\|^2 + \varepsilon \int_0^\eta \|\nabla e_u\|^2 .\end{aligned}$$

The last term on the RHS can be estimated in the same way as the third term by the boundedness of α as follows

$$\left|\int_0^\eta (e_K * \alpha(u_1), e_u)_\Gamma\right| \le C\int_0^\eta |e_K|^2 + C_\varepsilon \int_0^\eta \|e_u\|^2 + \varepsilon\int_0^\eta \|\nabla e_u\|^2 .$$

Collecting all these estimates, we obtain

$$\begin{aligned}\|e_u(\eta)\|^2 + \int_0^\eta \|\nabla e_u(t)\|^2\,\mathrm{d}t &\le C_\varepsilon \int_0^\eta \|e_u(t)\|^2\,\mathrm{d}t + \varepsilon\int_0^\eta \|\nabla e_u(t)\|^2\,\mathrm{d}t \\ &\quad + C_\varepsilon \int_0^\eta\int_0^t \|\nabla e_u(s)\|^2\,\mathrm{d}s\mathrm{d}t + C_\varepsilon\int_0^\eta |e_K(t)|^2\,\mathrm{d}t.\end{aligned}$$

Now, using the estimate (8), we get that

$$\|e_u(\eta)\|^2 + \int_0^\eta \|\nabla e_u(t)\|^2\,\mathrm{d}t \le C_{\varepsilon,\gamma}\int_0^\eta \|e_u(t)\|^2\,\mathrm{d}t + (\varepsilon + C_\varepsilon\gamma)\int_0^\eta \|\nabla e_u(t)\|^2\,\mathrm{d}t + C_{\varepsilon,\gamma}\int_0^\eta\int_0^t \|\nabla e_u(s)\|^2\,\mathrm{d}s\mathrm{d}t.$$

From this, we can finally conclude that

$$\max_{t\in[0,T]} \|e_u(t)\|^2 + \int_0^T \|\nabla e_u(t)\|^2\,\mathrm{d}t = 0$$

by Grönwall's lemma when fixing first ε and then γ sufficiently small. Therefore, u is unique in $\mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)\cap\mathrm{L}^2\left((0,T),\mathrm{H}^1(\Omega)\right)$ with $\partial_t u \in \mathrm{L}^2\left((0,T),\left(\mathrm{H}^1(\Omega)\right)^*\right)$. The uniqueness of K in $\mathrm{L}^2(0,T)$ follows from (8). □

3. Time discretization, existence of a solution

Rothe's method [11,12] represents a constructive method suitable for solving evolution problems. Using a simple discretization in time, a time-dependent problem is approximated by a sequence of elliptic problems, which have to be solved successively with increasing time step. This standard technique is in our case more complicated by the unknown convolution kernel K. However, there exists a way to overcome this difficulty.

For ease of exposition, an equidistant time-partitioning is considered of the time frame $[0, T]$ with a step $\tau = T/n < 1$, for any $n \in \mathbb{N}$. The following notations are used: $t_i = i\tau$ and for any function z

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

In this section, a decoupled system is considered with unknowns $\langle u_i, K_i\rangle$ for $i = 1,\ldots,n$. At time t_i, from (3), the following backward Euler scheme is proposed

$$(\delta u_i,\phi) - (\Delta u_i,\phi) + K_i(h_i,\phi) - \left(\sum_{k=1}^i K_k\Delta u_{i-k}\tau,\phi\right) = (f_{i-1},\phi), \tag{10}$$

where $f_i := f(u_i)$. The choice of f_{i-1} in (10) makes the RHS of (10) independent of the solution such that the Lax–Milgram lemma can be applied in Proposition 3.1. Similarly, define $\alpha_i = \alpha(u_i)$. From (P) and (MP), one obtains for $\phi \in \mathrm{H}^1(\Omega)$ that

$$\begin{aligned}&(\delta u_i,\phi) + (\nabla u_i,\nabla\phi) + K_i(h_i,\phi) + \left(\sum_{k=1}^i K_k\nabla u_{i-k}\tau,\nabla\phi\right) \\ &\quad = (f_{i-1},\phi) + (g_i - \alpha_{i-1},\phi)_\Gamma + \left(\sum_{k=1}^i K_k(g_{i-k}-\alpha_{i-k})\tau,\phi\right)_\Gamma \end{aligned}\tag{DPi}$$

and

$$m_i' + K_i\int_\Omega h_i = \int_\Omega f_{i-1} + \int_\Gamma (g_i - \alpha_{i-1}) + \sum_{k=1}^i \tau K_k\int_\Gamma (g_{i-k}-\alpha_{i-k}). \tag{DMPi}$$

Note that for a given $i \in \{1, \dots, n\}$, first (DMPi) is solved and then (DPi). Further, the index i is increased to $i+1$. To begin, the existence of a solution on a single time step is to be proved.

Proposition 3.1. *Let f and α be bounded. Moreover, assume that $g \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Gamma)\right)$, $h \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right)$, $\min_{t\in[0,T]} \left|\int_\Omega h(t)\right| \geq \omega > 0$, $u_0 \in \mathrm{H}^1(\Omega)$ and $m \in \mathrm{C}^1([0,T])$. Then there exist $C > 0$ and $\tau_0 > 0$ such that for any $\tau < \tau_0$ and each $i \in \{1, \dots, n\}$ we have*

(i) *there exist $K_i \in \mathbb{R}$ and $u_i \in \mathrm{H}^1(\Omega)$ obeying (DMPi) and (DPi)*
(ii) $\max_{1\leq i\leq n} |K_i| \leq C$.

Proof. (i) Set $\tau_0 = \min\left\{1, \frac{\omega}{2\|g_0-\alpha(u_0)\|_{\mathrm{L}^1(\Gamma)}}\right\}$. Then for any $\tau < \tau_0$, we may write by the triangle inequality that

$$\begin{aligned} 0 < \omega - \tau_0 \int_\Gamma |g_0 - \alpha(u_0)| &\leq \omega - \tau \int_\Gamma |g_0 - \alpha(u_0)| \leq |(h_i, 1)| \\ &- \left|\tau \int_\Gamma (g_0 - \alpha(u_0))\right| \leq \left|(h_i, 1) - \tau \int_\Gamma (g_0 - \alpha(u_0))\right|. \end{aligned}$$

Then, we can apply the following recursive deduction for $i = 1, \dots, n$:

Step 1: Let $u_{i-1} \in \mathrm{H}^1(\Omega)$ be given. Then, (DMPi) implies the existence of $K_i \in \mathbb{R}$ such that

$$K_i\left[\int_\Omega h_i - \tau\int_\Gamma (g_0 - \alpha(u_0))\right] = \int_\Omega f_{i-1} - m_i' + \int_\Gamma (g_i - \alpha_{i-1}) + \sum_{k=1}^{i-1} \tau K_k \int_\Gamma (g_{i-k} - \alpha_{i-k}). \tag{11}$$

Step 2: Now, the relation (DPi) can be rewritten as

$$\begin{aligned} \left(\frac{u_i}{\tau}, \phi\right) + (\nabla u_i, \nabla\phi) = \left(\frac{u_{i-1}}{\tau}, \phi\right) + (f_{i-1}, \phi) + (g_i - \alpha_{i-1}, \phi)_\Gamma + \left(\sum_{k=1}^{i} K_k (g_{i-k} - \alpha_{i-k})\tau, \phi\right)_\Gamma \\ - K_i (h_i, \phi) - \left(\sum_{k=1}^{i} K_k \nabla u_{i-k}\tau, \nabla\phi\right). \end{aligned}$$

The LHS represents a continuous, elliptic and bilinear form on $\mathrm{H}^1(\Omega)$ and the RHS is a linear bounded functional on $\mathrm{H}^1(\Omega)$. The existence of $u_i \in \mathrm{H}^1(\Omega)$ follows from (DPi) by the Lax–Milgram lemma.

(ii) The relation (11) yields

$$|K_i| \leq C\left(1 + \sum_{k=1}^{i-1} |K_k|\,\tau\right),$$

which is valid for any $i = 1, \dots, n$. An application of the discrete Grönwall lemma gives the uniform bound of $|K_i|$. □

Proposition 3.2. *Let the conditions of Proposition 3.1 be satisfied. Then there exists $C > 0$ such that for any $\tau < \tau_0$*

$$\max_{1\leq j\leq n} \left\|u_j\right\|^2 + \sum_{i=1}^{n} \|\nabla u_i\|^2\,\tau + \sum_{i=1}^{n} \|u_i - u_{i-1}\|^2 \leq C.$$

Proof. If we set $\phi = u_i\tau$ in (DPi) and sum up for $i = 1, \dots, j$, we obtain

$$\begin{aligned} &\sum_{i=1}^{j} (\delta u_i, u_i)\,\tau + \sum_{i=1}^{j} \|\nabla u_i\|^2\,\tau + \sum_{i=1}^{j} K_i\,(h_i, u_i)\,\tau + \sum_{i=1}^{j}\left(\sum_{k=1}^{i} K_k \nabla u_{i-k}\tau, \nabla u_i\right)\tau \\ &\quad = \sum_{i=1}^{j} (f_{i-1}, u_i)\,\tau + \sum_{i=1}^{j} (g_i - \alpha_{i-1}, u_i)_\Gamma\,\tau + \sum_{i=1}^{j}\left(\sum_{k=1}^{i} K_k (g_{i-k} - \alpha_{i-k})\tau, u_i\right)_\Gamma \tau. \end{aligned} \tag{12}$$

The summation by parts formula says

$$\sum_{i=1}^{j} (\delta u_i, u_i)\tau = \sum_{i=1}^{j} (u_i - u_{i-1}, u_i) = \frac{1}{2}\left(\left\|u_j\right\|^2 - \|u_0\|^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2\right).$$

All the other terms in (12) need to be estimated. For the third term of the LHS of (12), we get

$$\left|\sum_{i=1}^{j} K_i(h_i, u_i)\tau\right| \le \sum_{i=1}^{j} |K_i| \, \|h_i\| \, \|u_i\| \, \tau \le C \sum_{i=1}^{j} \|h_i\|^2 \, \tau + C \sum_{i=1}^{j} \|u_i\|^2 \, \tau \le C + C \sum_{i=1}^{j} \|u_i\|^2 \, \tau,$$

as K_i is bounded, see Proposition 3.1. The last term in the LHS of (12) is bounded by

$$\begin{aligned}\left|\sum_{i=1}^{j}\left(\sum_{k=1}^{i} K_k \nabla u_{i-k}\tau, \nabla u_i\right)\tau\right| &\le C_\varepsilon \sum_{i=1}^{j}\left\|\sum_{k=1}^{i} K_k \nabla u_{i-k}\tau\right\|^2 \tau + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \, \tau \\ &\le C_\varepsilon \sum_{i=1}^{j}\left(\sum_{k=1}^{i} \|\nabla u_{i-k}\|^2 \, \tau\right)\tau + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \, \tau \\ &\le C_\varepsilon \sum_{i=1}^{j}\left(\sum_{k=0}^{i-1} \|\nabla u_k\|^2 \, \tau\right)\tau + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \, \tau\end{aligned}$$

again as K_i is bounded. The first term on the RHS of (12) can be estimated by the boundedness of f as follows

$$\left|\sum_{i=1}^{j}(f_{i-1}, u_i)\tau\right| \le \sum_{i=1}^{j} \|f_{i-1}\| \, \|u_i\| \, \tau \le C + C \sum_{i=1}^{j} \|u_i\|^2 \, \tau.$$

The second term in the RHS can be estimated by the trace theorem and the boundedness of α in the following way

$$\left|\sum_{i=1}^{j}(g_i - \alpha_{i-1}, u_i)_\Gamma \tau\right| \le C \sum_{i=1}^{j} \|g_i - \alpha_{i-1}\|_\Gamma \, \|u_i\|_{\mathrm{H}^1(\Omega)} \, \tau \le C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|u_i\|^2_{\mathrm{H}^1(\Omega)} \, \tau.$$

Analogously, for the last term on the RHS, we have that

$$\left|\sum_{i=1}^{j}\left(\sum_{k=1}^{i} K_k (g_{i-k} - \alpha_{i-k})\tau, u_i\right)_\Gamma \tau\right| \le C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|u_i\|^2_{\mathrm{H}^1(\Omega)} \, \tau.$$

Putting all things together, using $u_0 \in \mathrm{H}^1(\Omega)$, we obtain that

$$\|u_j\|^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j} \|\nabla u_i\|^2 \, \tau \le C_\varepsilon + C_\varepsilon \sum_{i=1}^{j} \|u_i\|^2 \, \tau + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \, \tau + C_\varepsilon \sum_{i=1}^{j}\left(\sum_{k=1}^{i-1} \|\nabla u_k\|^2 \, \tau\right)\tau.$$

Fixing a sufficiently small $\varepsilon > 0$ implies that

$$\begin{aligned}\|u_j\|^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j} \|\nabla u_i\|^2 \, \tau &\le C + C \sum_{i=1}^{j} \|u_i\|^2 \, \tau + C \sum_{i=1}^{j}\left(\sum_{k=1}^{i-1} \|\nabla u_k\|^2 \, \tau\right)\tau \\ &\le C + C \sum_{i=1}^{j}\left(\|u_i\|^2 + \sum_{k=1}^{i} \|\nabla u_k\|^2 \, \tau + \sum_{k=1}^{i} \|u_i - u_{i-1}\|^2\right)\tau.\end{aligned}$$

In the last inequality, we enlarged the RHS. Now, fixing τ sufficiently small and involving the discrete Grönwall lemma, we conclude the proof. □

Proposition 3.3. *Let the conditions of Proposition 3.1 be satisfied. Then there exists $C > 0$ such that for any $\tau < \tau_0$*

$$\sum_{i=1}^{n} \|\delta u_i\|^2_{(\mathrm{H}^1(\Omega))^*} \, \tau \le C.$$

Proof. The relation (DPi) can be rewritten for $\phi \in \mathrm{H}^1(\Omega)$ as

$$\begin{aligned}(\delta u_i, \phi) = (f_{i-1}, \phi) + (g_i - \alpha_{i-1}, \phi)_\Gamma + \left(\sum_{k=1}^{i} K_k (g_{i-k} - \alpha_{i-k})\tau, \phi\right)_\Gamma \\ - (\nabla u_i, \nabla \phi) - K_i \, (h_i, \phi) - \left(\sum_{k=1}^{i} K_k \nabla u_{i-k}\tau, \nabla \phi\right).\end{aligned}$$

Using the trace theorem, we obtain that

$$|(\delta u_i, \phi)| \leq C \left(1 + \|\nabla u_i\| + \sum_{k=1}^{i-1} \|\nabla u_k\| \, \tau\right) \|\phi\|_{\mathrm{H}^1(\Omega)},$$

which implies

$$\|\delta u_i\|_{(\mathrm{H}^1(\Omega))^*} = \sup_{\substack{\varphi \in \mathrm{H}^1(\Omega) \\ \|\varphi\|_{\mathrm{H}^1(\Omega)} \leqslant 1}} |(\delta u_i, \varphi)| \leq C \left(1 + \|\nabla u_i\| + \sum_{k=1}^{i-1} \|\nabla u_k\| \, \tau\right). \tag{13}$$

Then, taking the second power in (13), multiplying the inequality by τ, summing up for $i = 1, \dots, n$ and applying Proposition 3.2, we get the asked inequality. □

4. Existence of a solution

Let us introduce the following piecewise linear function in time

$$u_n : [0, T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 1 \leq i \leq n,$$

and a step function

$$\bar{u}_n : [0, T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0 \\ u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 1 \leq i \leq n.$$

Similarly, define $\bar{K}_n$, $\bar{h}_n$, $\bar{g}_n$, $\bar{m}_n$ and $\overline{m'}_n$. These prolongations are also called Rothe's (piecewise linear and continuous, or piecewise constant) functions. Using these Rothe's functions, (DP*i*) and (DMP*i*) can be rewritten on the whole time frame as[1]

$$\begin{aligned} &(\partial_t u_n(t), \phi) + (\nabla \bar{u}_n(t), \nabla \phi) + \bar{K}_n(t)(\bar{h}_n(t), \phi) + \left(\sum_{k=1}^{\lfloor t \rfloor_\tau} \bar{K}_n(t_k) \nabla \bar{u}_n(t - t_k)\tau, \nabla\phi\right) \\ &\quad = (f(\bar{u}_n(t - \tau)), \phi) + (\bar{g}_n(t) - \alpha(\bar{u}_n(t - \tau)), \phi)_\Gamma + \left(\sum_{k=1}^{\lfloor t \rfloor_\tau} \bar{K}_n(t_k) \left[\bar{g}_n(t - t_k) - \alpha(\bar{u}_n(t - t_k))\right] \tau, \phi\right)_\Gamma \end{aligned} \tag{DP}$$

and

$$\begin{aligned} \overline{m'}_n(t) + \bar{K}_n(t) \int_\Omega \bar{h}_n(t) &= \int_\Omega f(\bar{u}_n(t - \tau)) + \int_\Gamma (\bar{g}_n(t) - \alpha(\bar{u}_n(t - \tau))) \\ &\quad + \sum_{k=1}^{\lfloor t \rfloor_\tau} \tau \bar{K}_n(t_k) \int_\Gamma (\bar{g}_n(t - t_k) - \alpha(\bar{u}_n(t - t_k))). \end{aligned} \tag{DMP}$$

This puts us in a position to prove the existence of a weak solution to (P) and (MP).

Theorem 2 (*Existence*). *Suppose the conditions of Proposition 3.1 are fulfilled. Then there exists a weak solution $\langle u, K \rangle$ to the problem (P)–(MP), where $u \in \left[\mathrm{C}\left([0, T], \mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0, T), \mathrm{H}^1(\Omega)\right)\right]$, $\partial_t u \in \mathrm{L}^2\left((0, T), \left(\mathrm{H}^1(\Omega)\right)^*\right)$ and $K \in \mathrm{L}^2(0, T)$.*

Proof. From Propositions 3.2 and 3.3, we have that for all $n > 0$ it holds that

$$\int_0^t \|u_n(\xi)\|^2_{\mathrm{H}^1(\Omega)} \, \mathrm{d}\xi \leq C \quad \text{for all } t \in [0, T], \qquad \int_0^T \|\partial_t u_n(\xi)\|^2_{(\mathrm{H}^1(\Omega))^*} \, \mathrm{d}\xi \leq C.$$

Thanks to the compact embedding by the Rellich–Kondrachov theorem [16, Section 5.8.1], we have that

$$\mathrm{H}^1(\Omega) \subset\subset \mathrm{L}^2(\Omega) \cong \left(\mathrm{L}^2(\Omega)\right)^* \subset\subset \left(\mathrm{H}^1(\Omega)\right)^*.$$

Using the generalized Aubin–Lions lemma [17, Lemma 7.7], there exist $u \in \mathrm{L}^2\left((0, T), \mathrm{L}^2(\Omega)\right)$ and a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} u_{n_k} \to u, & \text{in } \mathrm{L}^2\left((0, T), \mathrm{L}^2(\Omega)\right) \Rightarrow u_{n_k} \to u, \text{ a.e. in } (0, T) \times \Omega, \quad \text{(a)} \\ u_{n_k} \rightharpoonup u, & \text{in } \mathrm{L}^2\left((0, T), \mathrm{H}^1(\Omega)\right), \quad \text{(b)} \\ \partial_t u_{n_k} \rightharpoonup \partial_t u, & \text{in } \mathrm{L}^2\left((0, T), \left(\mathrm{H}^1(\Omega)\right)^*\right), \quad \text{(c)} \end{cases} \tag{14}$$

[1] $\lfloor t \rfloor_\tau = i$ when $t \in (t_{i-1}, t_i]$.

which we denote again by u_n for ease of reading. Applying [17, Lemma 7.3], we get $u \in \mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right)$ because $u \in \mathrm{L}^2\left((0,T), \mathrm{H}^1(\Omega)\right)$ and $\partial_t u \in \mathrm{L}^2\left((0,T), \left(\mathrm{H}^1(\Omega)\right)^*\right)$. Note that $u_n(0) - \bar{u}_n(0) = 0$. For all $t \in (t_{i-1}, t_i]$ with $1 \le i \le n$, we have that

$$|u_n(t) - \bar{u}_n(t)| = |u_{i-1} + (t - t_{i-1})\delta u_i - u_i| = |(t - t_{i-1} - \tau)\delta u_i| = |(t - t_i)\delta u_i| \le \tau\,|\delta u_i| = |u_i - u_{i-1}|.$$

Employing Proposition 3.2 gives

$$\lim_{n\to\infty} \|u_n - \bar{u}_n\|^2_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))} \le \lim_{n\to\infty} \tau \sum_{i=1}^n \|u_i - u_{i-1}\|^2 \le \lim_{n\to\infty} \frac{C}{n} = 0,$$

such that u_n and $\bar{u}_n$ have the same limit in $\mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$, i.e.

$$\bar{u}_n \to u \quad \text{in } \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right) \Rightarrow \bar{u}_n \to u, \ \text{a.e. in } (0,T)\times\Omega. \tag{15}$$

Analogously, one can prove that

$$\lim_{n\to\infty} \int_0^T \|\bar{u}_n(t-\tau) - \bar{u}_n(t)\|^2 \,\mathrm{d}t = 0. \tag{16}$$

Using the Lipschitz continuity of α, the Nečas inequality (7), the fact that $\sum_{i=1}^n \|\nabla u_i\|^2 \tau$ is bounded (Proposition 3.2) and $u \in \mathrm{L}^2\left((0,T), \mathrm{H}^1(\Omega)\right)$, we obtain that

$$\begin{aligned}
\int_0^T \|\alpha(\bar{u}_n(t-\tau)) - \alpha(u(t))\|^2_\Gamma \,\mathrm{d}t &\le C \int_0^T \|\bar{u}_n(t-\tau) - u(t)\|^2_\Gamma \,\mathrm{d}t \\
&\le \varepsilon \int_0^T \|\nabla(\bar{u}_n(t-\tau) - u(t))\|^2 \,\mathrm{d}t + C_\varepsilon \int_0^T \|\bar{u}_n(t-\tau) - u(t)\|^2 \\
&\le \varepsilon + C_\varepsilon \int_0^T \|\bar{u}_n(t-\tau) \pm \bar{u}_n(t) - u(t)\|^2 .
\end{aligned}$$

Passing to the limit and applying (15) and (16), it holds

$$\lim_{n\to\infty} \int_0^T \|\alpha(\bar{u}_n(t-\tau)) - \alpha(u(t))\|^2_\Gamma \,\mathrm{d}t = 0 \tag{17}$$

and thus

$$\lim_{n\to\infty} \alpha(\bar{u}_n(t-\tau)) = \alpha(u(t)) \quad \text{in } \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Gamma)\right).$$

In fact, a same reasoning gives also

$$\lim_{n\to+\infty} \int_0^T \|\bar{u}_n - u\|^2_\Gamma \,\mathrm{d}\xi \le \varepsilon \Longrightarrow \bar{u}_n \to u, \quad \text{a.e. in } (0,T)\times\Gamma. \tag{18}$$

Using Proposition 3.1, we have that $\int_0^T \left|\bar{K}_n(t)\right|^2 \mathrm{d}t \le C$, which means that

$$\bar{K}_n \rightharpoonup K \quad \text{in } \mathrm{L}^2(0,T),$$

by the reflexivity of $\mathrm{L}^2(0,T)$. It is clear that $\lim_{n\to\infty} \overline{m'}_n(t) = m'(t)$ in $\mathrm{C}([0,T])$, $\lim_{n\to\infty} \bar{g}_n(t) = g(t)$ in $\mathrm{C}\left([0,T], \mathrm{L}^2(\Gamma)\right)$ and $\lim_{n\to\infty} \bar{h}_n(t) = h(t)$ in $\mathrm{C}\left([0,T], \mathrm{L}^2(\Omega)\right)$ because m, h and g are prescribed. Now, we integrate (DP) in time over $(0,\eta) \subset [0,T]$ to get

$$\begin{aligned}
&\int_0^\eta (\partial_t u_n(t), \phi) + \int_0^\eta (\nabla \bar{u}_n(t), \nabla\phi) + \int_0^\eta \bar{K}_n(t)(\bar{h}_n(t), \phi) + \int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\nabla\bar{u}_n(t-t_k)\tau, \nabla\phi \right) \\
&\quad = \int_0^\eta (f(\bar{u}_n(t-\tau)), \phi) + \int_0^\eta (\bar{g}_n(t) - \alpha(\bar{u}_n(t-\tau)), \phi)_\Gamma \\
&\qquad + \int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\left[\bar{g}_n(t-t_k) - \alpha(\bar{u}_n(t-t_k))\right]\tau, \phi \right)_\Gamma .
\end{aligned} \tag{19}$$

This expression is valid for any $\eta \in [0, T]$. We want to pass the limit $n \to \infty$ in (19). Using the stability result (14)(c), we have for $n \to \infty$ that

$$\int_0^\eta (\partial_t u_n, \varphi) \to \int_0^\eta (\partial_t u, \varphi) .$$

Take $\phi \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$, then

$$\int_0^\eta (\nabla \bar{u}_n(t), \nabla\phi)\mathrm{d}t = -\int_0^\eta (\bar{u}_n(t), \Delta\phi)\,\mathrm{d}t + \int_0^\eta (\bar{u}_n(t), \nabla\phi\cdot\nu)_\Gamma\,\mathrm{d}t.$$

We take the limit $n \to \infty$ in this equality and obtain by (15) and (18) that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\eta (\nabla \bar{u}_n(t), \nabla\phi)\mathrm{d}t &= -\int_0^\eta (u(t), \Delta\phi)\,\mathrm{d}t + \int_0^\eta (u(t), \nabla\phi\cdot\nu)_\Gamma\,\mathrm{d}t \\ &= \int_0^\eta (\nabla u(t), \nabla\phi)\,, \quad \forall\phi\in\mathrm{C}^\infty\left(\overline{\Omega}\right).\end{aligned}$$

Employing the density argument $\overline{\mathrm{C}^\infty\left(\overline{\Omega}\right)} = \mathrm{H}^1(\Omega)$, we get that

$$\lim_{n\to\infty}\int_0^\eta (\nabla \bar{u}_n(t), \nabla\phi)\mathrm{d}t = \int_0^\eta (\nabla u(t), \nabla\phi)\,, \quad \forall\phi\in\mathrm{H}^1(\Omega).$$

From the previous considerations, it is easy to see that

$$\lim_{n\to\infty}\int_0^\eta \bar{K}_n(\bar{h}_n, \phi)\mathrm{d}t = \int_0^\eta K(h, \phi)\mathrm{d}t.$$

We take again $\phi \in \mathrm{C}^\infty\left(\overline{\Omega}\right)$ and apply the Green theorem for the last term in the LHS of (19). We obtain

$$\begin{aligned}\int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\nabla\bar{u}_n(t-t_k)\tau, \nabla\phi\right)\mathrm{d}t &= -\int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\bar{u}_n(t-t_k)\tau, \Delta\phi\right)\mathrm{d}t \\ &\quad + \int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\bar{u}_n(t-t_k)\tau, \nabla\phi\cdot\nu\right)_\Gamma \mathrm{d}t.\end{aligned}$$

Due to $\bar{K}_n \rightharpoonup K$ in $\mathrm{L}^2(0, T)$, (15) and (18), we obtain for any $\phi \in \mathrm{C}^\infty(\overline{\Omega})$ that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\nabla\bar{u}_n(t-t_k)\tau, \nabla\phi\right)\mathrm{d}t &= -\int_0^\eta (K*u, \Delta\phi) + \int_0^\eta (K*u, \nabla\phi\cdot\nu)_\Gamma \\ &= \int_0^\eta (K*\nabla u, \nabla\phi)\,.\end{aligned}$$

Applying the above density argument once more, we conclude that

$$\lim_{n\to\infty}\int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\nabla\bar{u}_n(t-t_k)\tau, \nabla\phi\right)\mathrm{d}t = \int_0^\eta (K*\nabla u, \nabla\phi)\,, \quad \forall\phi\in\mathrm{H}^1(\Omega).$$

For the first term on the RHS of (19), we get

$$\lim_{n\to\infty}\left|\int_0^\eta (f(\bar{u}_n(t-\tau)) - f(u(t)), \phi)\,\mathrm{d}t\right| = \lim_{n\to\infty}\left|\int_0^\eta (f(\bar{u}_n(t-\tau)) \pm f(\bar{u}_n(t)) - f(u(t)), \phi)\,\mathrm{d}t\right| = 0,$$

as f is Lipschitz, (15) and (16). For the last two terms on the RHS of (19), we have due to $\bar{K}_n \rightharpoonup K$ in $\mathrm{L}^2(0, T)$, the Lipschitz continuity of α, (17) and (18) that

$$\lim_{n\to\infty}\int_0^\eta (\bar{g}_n(t) - \alpha(\bar{u}_n(t-\tau)), \phi)_\Gamma\mathrm{d}t = \int_0^\eta (g(t) - \alpha(u(t)), \phi)_\Gamma\mathrm{d}t,$$

$$\lim_{n\to\infty}\int_0^\eta \left(\sum_{k=1}^{\lfloor t\rfloor_\tau} \bar{K}_n(t_k)\left[\bar{g}_n(t-t_k) - \alpha(\bar{u}_n(t-t_k))\right]\tau, \phi\right)_\Gamma \mathrm{d}t = \int_0^\eta (K*(g-\alpha(u)), \phi)_\Gamma\,\mathrm{d}t.$$

Now, taking the limit $n \to \infty$ in (19) results in

$$\int_0^\eta (\partial_t u, \phi) + \int_0^\eta (\nabla u, \nabla \phi) + \int_0^\eta K(h, \phi) + \int_0^\eta (K * \nabla u, \nabla \phi)$$
$$= \int_0^\eta (f(u), \phi) + \int_0^\eta (g - \alpha(u), \phi)_\Gamma + \int_0^\eta (K * (g - \alpha(u)), \phi)_\Gamma .$$

Taking the derivative with respect to η, we arrive at (P). In the same way as before, we integrate (DMP) in time and pass the limit for $n \to \infty$. This follows the same line as passing the limit in (19), therefore we skip the details. Finally, we differentiate the result with respect to time and arrive at (MP). □

The convergences of Rothe's functions towards the weak solution (P)–(MP) (as stated in the proof of Theorem 2) have been shown for a subsequence. However, taking into account Theorem 1, it is clear that the whole Rothe's sequence converge against the solution.

Conclusion

A semilinear parabolic integro-differential problem of second order with an unknown solely time-dependent convolution kernel is considered. The missing information is compensated by an integral-type measurement over the domain. The existence and uniqueness of a weak solution for the IBVP is proved. A numerical procedure based on Rothe's method is developed and the convergence of approximations towards the exact solution is demonstrated.

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