



Newton's algorithm for magnetohydrodynamic equations with the initial guess from Stokes-like problem



Sang Dong Kim^a, Eunjung Lee^{b,*}, Wonjoon Choi^b

^a Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea

^b Department of Computational Science and Engineering, Yonsei University, Seoul 120-749, Republic of Korea

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ABSTRACT

The magnetohydrodynamic equations are second order nonlinear partial differential equations which are coupled by fluid velocity and magnetic fields and we consider to apply the Newton's algorithm to solve them. It is well known that the choice of a proper initial guess is critical to assure the convergence of Newton's iterations in solving nonlinear partial differential equations. In this paper, we provide a good initial guess for Newton's algorithm when it is applied for solving magnetohydrodynamic equations.

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1. Introduction

It is known that the magnetohydrodynamic equations have many applications in industry and engineering fields which may require understanding of plasmas, liquid metals, astrophysical flows and etc. (see [1,2] for example). Further, one may get a justification of using simplified magnetohydrodynamic equations to model magnetohydrodynamic flows in terrestrial [3] for example. A reduced stationary magnetohydrodynamic equations with magnetic and electric fields are studied by finite element approximation in [4,5]. Further, such stationary magnetohydrodynamics equations are extended to a time-dependent magnetohydrodynamics equations in [6], where two partitioned IMEX methods in finite element approximation are introduced to solve such an evolutionary magnetohydrodynamic equations. Some works on time-dependent magnetohydrodynamic may be found in [7,3,6,8–10].

The incompressible magnetohydrodynamic equations considered here, that are second order in the pair of fluid velocity and magnetic fields, describe the cosmic fluids which carry electric currents and their magnetic fields [11]. By omitting the displacement current, the magnetohydrodynamics equations can be simplified as

$$\begin{aligned} \rho (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) &= -\nabla p - \lambda \mathbf{b} \times (\nabla \times \mathbf{b}) + \nu \Delta \mathbf{u}, & \partial_t \mathbf{b} &= \nabla \times (\mathbf{u} \times \mathbf{b}) + \eta \Delta \mathbf{b}, \\ \nabla \cdot \mathbf{u} &= 0, & \nabla \cdot \mathbf{b} &= 0, \end{aligned} \quad (1.1)$$

where $\mathbf{u}$ and $\mathbf{b}$ are respective fluid velocity and magnetic fields; ρ is a constant mass density; λ , ν and η are conventional parameters which are circumstantially described in [11]. Obviously the magnetohydrodynamic equations are composed of nonlinear partial differential equations and solving this system requires a special treatment.

In this paper, without considering any time and space discretization to solve the targeting magnetohydrodynamic equations, we are focused on its linearization process. The Newton's algorithm is one of the most widely used linearization schemes to solve nonlinear partial differential equations. By using the first Fréchet derivative of a solution operator, it

* Corresponding author.

E-mail addresses: skim@knu.ac.kr (S.D. Kim), eunjunglee@yonsei.ac.kr (E. Lee), kgarrett0810@gmail.com (W. Choi).

linearizes the process and constructs a sequence that is supposed to converge to the exact solution. It is well known that the initial guess is required to be chosen close enough to the exact solution in order to guarantee the quadratic convergence of Newton's iteration [12]. The goal of this paper is to provide a proper initial guess of Newton's algorithm for magnetohydrodynamic equations. We first recast the magnetohydrodynamic equations into familiar Navier–Stokes like equations. It has been studied in a frame work of finite element approximations that proper Stokes equations may be chosen to search for initial guess (see [13–15]) of the Newton's iteration for Navier–Stokes equations. Based on these studies, we then find the corresponding evolutionary Stokes-like equations and construct an approximate solution by interrelated Galerkin method using a passage to the limit and compactness theorem. The existence and uniqueness of the evolution Stokes-like equations are shown following the techniques in [12]. At last, we will show that the solution of the evolution Stokes-like problem is a proper initial guess to guarantee the convergence of Newton's iteration in solving the magnetohydrodynamic equations.

2. Magnetohydrodynamic equations in a vector form

The magnetohydrodynamic (MHD) equations with a constant mass density ρ can be described as

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p - \lambda \mathbf{b} \times (\nabla \times \mathbf{b}) + \nu \Delta \mathbf{u} \quad \text{in } Q := \Omega \times [0, T] \quad (2.1)$$

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \eta \Delta \mathbf{b} \quad \text{in } Q \quad (2.2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0 \quad \text{in } Q \quad (2.3)$$

and $\mathbf{u}(\mathbf{x}, t_0) = \mathbf{u}_{IC}$, $\mathbf{b}(\mathbf{x}, t_0) = \mathbf{b}_{IC}$ in Ω , $\mathbf{u}(\Gamma, t) = \mathbf{u}_b$, $\mathbf{b}(\Gamma, t) = \mathbf{b}_b$ in $[0, T]$, where $\Gamma = \partial\Omega$, $\mathbf{u}$ and $\mathbf{b}$ are the velocity and magnetic fields respectively; p is the pressure which is divided by the constant mass density; ν and η are the kinematic viscosity and the magnetic resistivity, respectively, with $\lambda = \frac{1}{\mu\rho}$ and $\eta = \frac{1}{\mu\sigma}$ in which μ is a susceptibility and σ is an electric conductivity. Here, the spatial domain Ω is an open and bounded Lipschitz set in $\mathbb{R}^3$.

According to [11], by introducing new variables

$$\begin{bmatrix} \mathbf{z}^+ \\ \mathbf{z}^- \end{bmatrix} = \mathbf{z}^\pm := \mathbf{u} \pm \lambda^{\frac{1}{2}} \mathbf{b} = \begin{bmatrix} \mathbf{u} + \lambda^{\frac{1}{2}} \mathbf{b} \\ \mathbf{u} - \lambda^{\frac{1}{2}} \mathbf{b} \end{bmatrix}, \quad \begin{bmatrix} v^+ \\ v^- \end{bmatrix} = v^\pm := \frac{1}{2}(\nu \pm \eta) = \frac{1}{2} \begin{bmatrix} \nu + \eta \\ \nu - \eta \end{bmatrix} \quad (2.4)$$

and

$$q = p + \frac{1}{8} |\mathbf{z}^+ - \mathbf{z}^-|^2, \quad (2.5)$$

the MHD equations in (2.1)–(2.3) can be expressed such that

$$\partial_t \mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla) \mathbf{z}^+ + \nabla q - \Delta(v^+ \mathbf{z}^+ + v^- \mathbf{z}^-) = \mathbf{0} \quad \text{in } Q := \Omega \times [0, T] \quad (2.6)$$

$$\partial_t \mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla) \mathbf{z}^- + \nabla q - \Delta(v^- \mathbf{z}^+ + v^+ \mathbf{z}^-) = \mathbf{0} \quad \text{in } Q \quad (2.7)$$

$$\nabla \cdot \mathbf{z}^\pm = 0 \quad \text{in } Q \quad (2.8)$$

$$\mathbf{z}^\pm(\mathbf{x}, 0) = \mathbf{z}_{IC}^\pm(\mathbf{x}) \quad \text{in } \Omega \quad (2.9)$$

$$\mathbf{z}^\pm = \mathbf{z}_{BC}^\pm \quad \text{on } \Gamma \times [0, T]. \quad (2.10)$$

From now on, in all the sequel we assume that

$$v^- > 0, \quad \int_{\Omega} q d\Omega = 0 \quad \text{and} \quad \int_{\Gamma} \mathbf{z}^\pm \cdot \mathbf{n} ds = 0.$$

Let

$$\mathbf{M} = \begin{bmatrix} v^+ I_3 & v^- I_3 \\ v^- I_3 & v^+ I_3 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{0} & I_3 \\ I_3 & \mathbf{0} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q \\ q \end{bmatrix},$$

in which I_3 is the 3×3 identity matrix. For a $\mathbf{w} = [\mathbf{u}, \mathbf{v}]^T$, we denote

$$\begin{aligned} \partial_t \mathbf{w} &= \begin{bmatrix} \partial_t \mathbf{u} \\ \partial_t \mathbf{v} \end{bmatrix}, \quad \Delta \mathbf{w} = \begin{bmatrix} \Delta \mathbf{u} \\ \Delta \mathbf{v} \end{bmatrix}, \quad \mathbf{w} \cdot \nabla = \begin{bmatrix} \mathbf{u} \cdot \nabla \\ \mathbf{v} \cdot \nabla \end{bmatrix}, \\ \nabla \mathbf{q} &= \begin{bmatrix} \nabla q \\ \nabla q \end{bmatrix}, \quad \mathbf{Mw} = \begin{bmatrix} v^+ \mathbf{u} + v^- \mathbf{v} \\ v^- \mathbf{u} + v^+ \mathbf{v} \end{bmatrix} \quad \text{and} \quad \mathbf{Pw} = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}. \end{aligned}$$

With these notations and definitions, it should be understood as

$$\begin{bmatrix} (\mathbf{u}^- \cdot \nabla) \mathbf{u}^+ \\ (\mathbf{u}^+ \cdot \nabla) \mathbf{u}^- \end{bmatrix} = (\mathbf{P} \mathbf{u}^\pm \cdot \nabla) \mathbf{u}^\pm, \quad \Delta(\mathbf{M} \mathbf{z}^\pm) = \begin{bmatrix} \Delta v^+ \mathbf{z}^+ + \Delta v^- \mathbf{z}^- \\ \Delta v^- \mathbf{z}^+ + \Delta v^+ \mathbf{z}^- \end{bmatrix} = \mathbf{M} \Delta \mathbf{z}^\pm. \quad (2.11)$$

Define the operator K by

$$K\mathbf{u}^\pm := K(\mathbf{u}^+, \mathbf{u}^-) = (P\mathbf{u}^\pm \cdot \nabla)\mathbf{u}^\pm.$$

Then, Eqs. (2.6)–(2.10) can be written as

$$\partial_t \mathbf{z}^\pm + K\mathbf{z}^\pm + \nabla \mathbf{q} - M\Delta \mathbf{z}^\pm = \mathbf{0} \quad \text{in } Q \quad (2.12)$$

$$\nabla \cdot \mathbf{z}^\pm = 0 \quad \text{in } Q \quad (2.13)$$

$$\mathbf{z}^\pm(\mathbf{x}, 0) = \mathbf{z}_{\text{IC}}^\pm(\mathbf{x}) \quad \text{in } \Omega \quad (2.14)$$

$$\mathbf{z}^\pm = \mathbf{z}_{\text{BC}}^\pm \quad \text{on } \Gamma \times [0, T]. \quad (2.15)$$

Let us use the standard notations and definitions for the Sobolev spaces $H^k(\Omega)$, $L^2(\Omega)$ and $H^{1/2}(\Gamma)$ associated with the proper inner products and norms. The space $H_F^1(\Omega)$ is also a Sobolev space such that

$$H_F^1(\Omega) := \{w \in H^1(\Omega) \mid w(\mathbf{x}) = \hat{w}(\mathbf{x}) \text{ on } \Gamma\}.$$

Denote $H_0^1(\Omega)$ as the subspace of $H^1(\Omega)$ which consists of the functions vanishing on the boundaries. The dual space $H^{-1}(\Omega)$ is used with the usual standard norm $\|\cdot\|_{-1}$. For a given α , $1 \leq \alpha < \infty$, $L^\alpha(a, b; X)$ denotes the space of L^α -integrable functions from $[a, b]$ into X with the norm

$$\left(\int_a^b \|f(t)\|_X^\alpha dt \right)^{\frac{1}{\alpha}}.$$

For example, the space $L^2(0, T; H^k(\Omega))$ is the space of all functions $u(\mathbf{x}, t)$ such that $\int_0^T \|u(\mathbf{x}, t)\|_{H^k}^2 dt < \infty$ and the space $L^2(0, T; H^{1/2}(\Gamma))$ is the space of all functions $u(\mathbf{x}, t)$ such that $\int_0^T \|u(\mathbf{x}, t)\|_{H^{1/2}}^2 dt < \infty$.

3. Unsteady Stokes-like equations

In this section we discuss the existence and uniqueness of the following unsteady Stokes-like equations

$$\partial_t \mathbf{z}^\pm + \nabla \mathbf{q} - M\Delta \mathbf{z}^\pm = \mathbf{f} \quad \text{in } Q \quad (3.1)$$

$$\nabla \cdot \mathbf{z}^\pm = 0 \quad \text{in } Q \quad (3.2)$$

$$\mathbf{z}^\pm(\mathbf{x}, 0) = \mathbf{z}_{\text{IC}}^\pm(\mathbf{x}) \quad \text{in } \Omega \quad (3.3)$$

$$\mathbf{z}^\pm = \mathbf{z}_{\text{BC}}^\pm \quad \text{on } \Gamma. \quad (3.4)$$

For analytical simplicity, we suppose the boundary condition

$$\mathbf{z}^\pm = \mathbf{z}_{\text{BC}}^\pm = \mathbf{0} \quad \text{on } \Gamma. \quad (3.5)$$

Let $\mathcal{D}(\Omega)$ be the space of C^∞ functions with a compact support contained in Ω and $\mathcal{U} = \{\mathbf{u} \in \mathcal{D}(\Omega) \mid \nabla \cdot \mathbf{u} = 0\}$. Let $\mathcal{V}$ and $\mathcal{H}$ be the closure of $\mathcal{U}$ in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. Note that $\mathcal{V} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{V}^*$ with $\mathcal{V}^*$ the dual space of $\mathcal{V}$. Then one may define the weak formulation of (3.1)–(3.4) with (3.5) as follows: for $\mathbf{f} \in L^2(0, T; \mathcal{V}^*)$ and $\mathbf{z}_{\text{IC}}^\pm \in \mathcal{H}$, find $\mathbf{z}^\pm \in L^2(0, T; \mathcal{V})$ satisfying

$$\frac{d}{dt}(\mathbf{z}^\pm, \mathbf{w}^\pm) + (M\nabla \mathbf{z}^\pm, \nabla \mathbf{w}^\pm) = (\mathbf{f}, \mathbf{w}^\pm), \quad \text{for all } \mathbf{w}^\pm \in \mathcal{V} \quad (3.6)$$

$$\mathbf{z}^\pm(0) = \mathbf{z}_{\text{IC}}^\pm. \quad (3.7)$$

In this paper, we assume that the solution $\mathbf{z}^\pm \in L^2(0, T; \mathcal{V})$ of (3.6) and (3.7) is a weak solution to (3.1)–(3.5) (see a similar argument in [16] for evolution Stokes equations). The existence and uniqueness of such a weak solution can be shown as follows. We first introduce the following lemma which can be found from section 3.1 Chapter III in [16].

Lemma 3.1. *Let $\mathcal{V}$ be a given Banach space with dual $\mathcal{V}^*$ and let $\mathbf{u}$ and $\mathbf{g}$ be in $L^1(0, T; \mathcal{V})$. If for each $\mathbf{w} \in \mathcal{V}^*$, $\frac{d}{dt}(\mathbf{u}, \mathbf{w}) = (\mathbf{g}, \mathbf{w})$ in the scalar distribution sense on $(0, T)$ then $\mathbf{u}$ is almost everywhere equal to a continuous function from $[0, T]$ into $\mathcal{V}$.*

Let the operator $\Lambda : \mathcal{V} \rightarrow \mathcal{V}^*$ be defined by $(\Lambda \mathbf{v}, \mathbf{w}) = (M\nabla \mathbf{v}, \nabla \mathbf{w})$, for all $\mathbf{w} \in \mathcal{V}$. Then (3.6) can be written $\frac{d}{dt}(\mathbf{z}^\pm, \mathbf{w}^\pm) = (\mathbf{f} - \Lambda \mathbf{z}^\pm, \mathbf{w}^\pm)$ for all $\mathbf{w}^\pm \in \mathcal{V}$. Since Λ is linear and continuous from $\mathcal{V}$ into $\mathcal{V}^*$ and $\mathbf{z}^\pm \in L^2(0, T; \mathcal{V})$, we have $\mathbf{f} - \Lambda \mathbf{z}^\pm \in L^2(0, T; \mathcal{V}^*)$. Hence, it follows that, by Lemma 3.1, $\frac{d}{dt} \mathbf{z}^\pm \in L^2(0, T; \mathcal{V}^*)$ and that $\mathbf{z}^\pm$ is almost everywhere equal to a continuous function from $[0, T]$ into $\mathcal{V}^*$. Therefore the condition (3.7) makes sense.

Now, we follow the arguments in Theorem 3.1 Chapter III in [16] for the proof of existence and uniqueness of solutions to the weak formulations (3.6)–(3.7).

Theorem 3.2. For a given M and $\mathbf{f} \in L^2(0, T; \mathcal{V}^*)$ and $\mathbf{z}_{IC}^\pm \in \mathcal{H}$, the evolution Stokes-like equations (3.1)–(3.4) have a unique solution $\mathbf{z}^\pm \in L^2(0, T; \mathcal{V})$ satisfying (3.6)–(3.7).

Proof. Since $\mathcal{V}$ is separable, there exists a countable sequence of linearly independent $\{\mathbf{w}_j = \mathbf{w}_j(\mathbf{x})\}_{j=1}^\infty$ which is total in $\mathcal{V}$. For each m , define an approximation solution $\mathbf{u}_m^\pm$ of (3.6) satisfying

$$\mathbf{u}_m^+ = \sum_{j=1}^m g_{jm}^+(t) \mathbf{w}_j^+ \quad \text{and} \quad \mathbf{u}_m^- = \sum_{j=1}^m g_{jm}^-(t) \mathbf{w}_j^- \quad (3.8)$$

and

$$\left(\frac{d}{dt} \mathbf{u}_m^\pm(t), \mathbf{w}_j^\pm \right) + (M \nabla \mathbf{u}_m^\pm(t), \nabla \mathbf{w}_j^\pm) = (\mathbf{f}(t), \mathbf{w}_j^\pm), \quad j = 1, 2, \dots, m \quad (3.9)$$

$$\mathbf{u}_m^\pm(0) = \mathbf{u}_{IC,m}^\pm, \quad (3.10)$$

where g_{jm}^+ and g_{jm}^- are scalar functions defined on $[0, T]$ for $j = 1, 2, \dots, m$ and $\mathbf{u}_{IC,m}^\pm$ is the orthogonal projection in $\mathcal{H}$ of $\mathbf{z}_{IC}^\pm$ on the space spanned by $\{\mathbf{w}_j\}_{j=1}^m$. Because of the orthogonality in L^2 sense, we have

$$\|\mathbf{u}_{IC,m}^\pm\| \leq \|\mathbf{z}_{IC}^\pm\| \quad \text{and} \quad \lim_{m \rightarrow \infty} \mathbf{u}_m^\pm(0) = \mathbf{z}^\pm(0). \quad (3.11)$$

Note that (3.9)–(3.10) with (3.8) can be written as

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} \mathbf{g}_m^+(t) \\ \frac{d}{dt} \mathbf{g}_m^-(t) \end{bmatrix} + \begin{bmatrix} \nu^+ \mathbf{S}_1 & \nu^- \mathbf{S}_2 \\ \nu^- \mathbf{S}_2 & \nu^+ \mathbf{S}_3 \end{bmatrix} \begin{bmatrix} \mathbf{g}_m^+(t) \\ \mathbf{g}_m^-(t) \end{bmatrix} = \begin{bmatrix} (\mathbf{f}_1(t), \mathbf{w}_j^+) \\ (\mathbf{f}_2(t), \mathbf{w}_j^-) \end{bmatrix}, \quad (3.12)$$

where $\mathbf{M}_1 := ((\mathbf{w}_i^+, \mathbf{w}_j^+))$, $\mathbf{M}_2 := ((\mathbf{w}_i^-, \mathbf{w}_j^-))$, $\mathbf{S}_1 := ((\nabla \mathbf{w}_i^+, \nabla \mathbf{w}_j^+))$, $\mathbf{S}_2 := ((\nabla \mathbf{w}_i^-, \nabla \mathbf{w}_j^+))$, $\mathbf{S}_3 := ((\nabla \mathbf{w}_i^-, \nabla \mathbf{w}_j^-))$ and

$$\frac{d}{dt} \mathbf{g}_m^\pm(t) := \left[\frac{d}{dt} g_{1m}^\pm(t), \dots, \frac{d}{dt} g_{mm}^\pm(t) \right]^T, \quad \mathbf{g}_m^\pm(t) := [g_{1m}^\pm(t), \dots, g_{mm}^\pm(t)]^T.$$

The linear independent property for $\{\mathbf{w}_j\}$ leads to the invertibility of matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, so that (3.12) reduces to

$$\begin{bmatrix} \frac{d}{dt} \mathbf{g}_m^+(t) \\ \frac{d}{dt} \mathbf{g}_m^-(t) \end{bmatrix} + \begin{bmatrix} \nu^+ \mathbf{M}_1^{-1} \mathbf{S}_1 & \nu^- \mathbf{M}_1^{-1} \mathbf{S}_2 \\ \nu^- \mathbf{M}_2^{-1} \mathbf{S}_2 & \nu^+ \mathbf{M}_2^{-1} \mathbf{S}_3 \end{bmatrix} \begin{bmatrix} \mathbf{g}_m^+(t) \\ \mathbf{g}_m^-(t) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2^{-1} \end{bmatrix} \begin{bmatrix} (\mathbf{f}_1(t), \mathbf{w}_j^+) \\ (\mathbf{f}_2(t), \mathbf{w}_j^-) \end{bmatrix} \quad (3.13)$$

with the initial conditions

$$g_{km}^\pm(0) = \text{the } k\text{th component of } \mathbf{u}_{IC,m}^\pm. \quad (3.14)$$

Since the elements of the matrices $\mathbf{M}_l$ ($l = 1, 2$) and $\mathbf{S}_l$ ($l = 1, 2, 3$) are real, the system of differential equations (3.13) with such initial conditions (3.14) has a unique solution on the whole interval $[0, T]$ (see Theorem 5.1 [17]). Since the scalar functions $t \rightarrow (\mathbf{f}_1(t), \mathbf{w}_j^+)$ and $t \rightarrow (\mathbf{f}_2(t), \mathbf{w}_j^-)$ are square integrable functions, so are the functions $g_{km}^\pm(t)$ and therefore, for each m

$$\mathbf{u}_m^\pm, \quad \frac{d}{dt} \mathbf{u}_m^\pm \in L^2(0, T; \mathcal{V}). \quad (3.15)$$

From (3.9), we obtain

$$\left(\frac{d}{dt} \mathbf{u}_m^\pm, \mathbf{u}_m^\pm(t) \right) + (M \nabla \mathbf{u}_m^\pm(t), \nabla \mathbf{u}_m^\pm(t)) = (\mathbf{f}, \mathbf{u}_m^\pm(t)).$$

We note that the matrix M is symmetric positive definite with two positive eigenvalues

$$\lambda_+ = \nu^+ + \nu^- = 2\nu, \quad \lambda_- = \nu^+ - \nu^- = 2\eta. \quad (3.16)$$

(i) The properties (3.15) and (3.16) imply

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m^\pm(t)\|^2 + \lambda_- \|\mathbf{u}_m^\pm\|_1^2 &\leq \left(\frac{d}{dt} \mathbf{u}_m^\pm(t), \mathbf{u}_m^\pm(t) \right) + (\mathbf{M} \nabla \mathbf{u}_m^\pm(t), \nabla \mathbf{u}_m^\pm(t)) = (\mathbf{f}(t), \mathbf{u}_m^\pm(t)) \\ &\leq \|\mathbf{f}(t)\|_{-1} \|\mathbf{u}_m^\pm\|_1 \\ &\leq \frac{\lambda_-}{2} \|\mathbf{u}_m^\pm\|_1^2 + \frac{1}{2\lambda_-} \|\mathbf{f}(t)\|_{-1}^2, \end{aligned}$$

which leads us to

$$\frac{d}{dt} \|\mathbf{u}_m^\pm(t)\|^2 + \lambda_- \|\mathbf{u}_m^\pm(t)\|_1^2 \leq \frac{1}{\lambda_-} \|\mathbf{f}(t)\|_{-1}^2. \quad (3.17)$$

Integrating (3.17) over $[0, s]$ and taking maximum over $[0, T]$ with (3.11), we have in particular

$$\max_{0 \leq s \leq T} \|\mathbf{u}_m^\pm(s)\|^2 \leq \|\mathbf{u}_{\text{IC},m}^\pm\|^2 + \frac{1}{\lambda_-} \int_0^T \|\mathbf{f}(t)\|_{-1}^2 dt \leq \|\mathbf{z}_{\text{IC}}^\pm\|^2 + \frac{1}{\lambda_-} \int_0^T \|\mathbf{f}(t)\|_{-1}^2 dt. \quad (3.18)$$

Again, integrating (3.17) from 0 to T yields

$$\|\mathbf{u}_m^\pm(T)\|^2 + \lambda_- \int_0^T \|\mathbf{u}_m^\pm(t)\|_1^2 dt \leq \|\mathbf{z}_{\text{IC}}^\pm\|^2 + \frac{1}{\lambda_-} \int_0^T \|\mathbf{f}(t)\|_{-1}^2 dt. \quad (3.19)$$

(ii) Due to (3.18) and (3.19), the sequence $\mathbf{u}_m^\pm$ is bounded in $L^2(0, T; \mathcal{V}) \cap L^\infty(0, T; \mathcal{H})$. Hence it assures the existence of an element $\mathbf{z}^\pm \in L^2(0, T; \mathcal{V}) \cap L^\infty(0, T; \mathcal{H})$ and a subsequence $\mathbf{u}_{m_l}^\pm$ such that

$$\mathbf{u}_{m_l}^\pm \rightharpoonup \mathbf{z}^\pm \text{ in } L^2(0, T; \mathcal{V}) \text{ weakly, and in } L^\infty(0, T; \mathcal{H}) \text{ weak-star, as } m_l \rightarrow \infty. \quad (3.20)$$

By the same manner used in [16], we have

$$\mathbf{u}_{m_l}^\pm \rightarrow \mathbf{z}^\pm \text{ in } L^2(0, T; \mathcal{H}) \text{ strongly.}$$

(iii) Consider a scalar function $\psi(t)$ continuously differentiable on $[0, T]$ such that $\psi(T) = 0$. Multiplying (3.9) by the function $\psi(t)$ and integrating with respect to t with integration by parts, it follows that

$$-\int_0^T \left(\mathbf{u}_m^\pm(t), \frac{d}{dt} \psi(t) \mathbf{w}_j^\pm \right) dt + \int_0^T (\mathbf{M} \nabla \mathbf{u}_m^\pm(t), \psi(t) \nabla \mathbf{w}_j^\pm) dt = (\mathbf{u}_{\text{IC},m}^\pm, \mathbf{w}_j^\pm) \psi(0) + \int_0^T (\mathbf{f}(t), \mathbf{w}_j^\pm) \psi(t) dt. \quad (3.21)$$

Now using the existence argument in the above, it follows that for $j = 1, 2, \dots, m$

$$-\int_0^T \left(\mathbf{z}^\pm(t), \frac{d}{dt} \psi(t) \mathbf{w}_j^\pm \right) dt + \int_0^T (\mathbf{M} \nabla \mathbf{z}^\pm(t), \psi(t) \nabla \mathbf{w}_j^\pm) dt = (\mathbf{z}_{\text{IC}}^\pm, \mathbf{w}_j^\pm) \psi(0) + \int_0^T (\mathbf{f}(t), \mathbf{w}_j^\pm) \psi(t) dt. \quad (3.22)$$

Therefore, for each $\mathbf{v}^\pm$ in the space spanned by $\{\mathbf{w}_j^\pm\}$ and for $j = 1, 2, \dots, m$,

$$-\int_0^T (\mathbf{z}^\pm(t), \mathbf{v}^\pm) \frac{d}{dt} \psi(t) dt + \int_0^T (\mathbf{M} \nabla \mathbf{z}^\pm(t), \nabla \mathbf{v}^\pm) \psi(t) dt = (\mathbf{z}_{\text{IC}}^\pm, \mathbf{v}^\pm) \psi(0) + \int_0^T (\mathbf{f}(t), \mathbf{v}^\pm) \psi(t) dt. \quad (3.23)$$

Hence, by continuity arguments, (3.23) holds for all $\mathbf{v}^\pm \in \mathcal{V}$. With a choice of ψ in $\mathcal{D}(0, T)$, applying integration by parts, we have that (3.23) becomes (3.9) in the distribution sense on $(0, T)$, that is,

$$\frac{d}{dt} (\mathbf{z}^\pm(t), \mathbf{v}^\pm) + (\mathbf{M} \nabla \mathbf{z}^\pm(t), \nabla \mathbf{v}^\pm) = (\mathbf{f}(t), \mathbf{v}^\pm), \quad \text{for all } \mathbf{v}^\pm \in \mathcal{V}. \quad (3.24)$$

Now it remains to check that $\mathbf{z}^\pm(0) = \mathbf{z}_{\text{IC}}^\pm$. Multiplying (3.24) by the same $\psi(t)$ above and integrating by parts, one gets

$$-\int_0^T (\mathbf{z}^\pm(t), \mathbf{v}^\pm) \frac{d}{dt} \psi(t) dt + \int_0^T (\mathbf{M} \nabla \mathbf{z}^\pm(t), \nabla \mathbf{v}^\pm) \psi(t) dt = (\mathbf{z}^\pm(0), \mathbf{v}^\pm) \psi(0) + \int_0^T (\mathbf{f}(t), \mathbf{v}^\pm) \psi(t) dt. \quad (3.25)$$

By comparison (3.25) with (3.23), we have

$$(\mathbf{z}_{\text{IC}}^\pm - \mathbf{z}^\pm(0), \mathbf{v}) \psi(0) = 0, \quad \text{for each } \mathbf{v} \in \mathcal{V}.$$

Once ψ is chosen as $\psi(0) \neq 0$, it follows that $\mathbf{z}_{\text{IC}}^\pm = \mathbf{z}^\pm(0)$.

(iv) (Uniqueness) Let $\mathbf{z}_1^\pm$ and $\mathbf{z}_2^\pm$ both satisfy (3.6)–(3.7). Then $\tilde{\mathbf{z}}^\pm := \mathbf{z}_1^\pm - \mathbf{z}_2^\pm$ satisfies

$$\frac{d}{dt}(\tilde{\mathbf{z}}^\pm, \mathbf{w}^\pm) + (M\nabla\tilde{\mathbf{z}}^\pm, \nabla\mathbf{w}^\pm) = \mathbf{0}, \quad \text{for all } \mathbf{w}^\pm \in \mathcal{V} \quad (3.26)$$

$$\tilde{\mathbf{z}}^\pm(0) = \mathbf{0}. \quad (3.27)$$

Set $\mathbf{w}^\pm = \tilde{\mathbf{z}}^\pm$ in (3.26), then we have

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{z}}^\pm(t)\|^2 + \lambda_- \|\nabla\mathbf{z}^\pm(t)\|^2 \leq \frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{z}}^\pm(t)\|^2 + (M\nabla\tilde{\mathbf{z}}^\pm(t), \nabla\tilde{\mathbf{z}}^\pm(t)) = 0.$$

Using (3.17) and (3.18), we have “ $\max_{0 \leq s \leq T} \|\tilde{\mathbf{z}}^\pm(s)\| = 0$ ”. Also (3.19) implies $\tilde{\mathbf{z}}^\pm = \mathbf{0}$ in $L^2(0, T; \mathcal{V})$. Hence the proof is completed.

In the above we showed the existence and uniqueness of solution to the weak formulations (3.6)–(3.7).

The following section introduces the Newton’s method to linearize Eq. (2.12) and suggests a proper initial guess for the Newton’s iteration using Theorem 3.2.

4. Newton algorithm based on time discretization

The Newton algorithm is one of the most popular tools to solve nonlinear partial differential equations using linearization process. Due to its iterative manner, choosing proper initial guess plays a very important role in finding convergent approximations to the solution. As we mentioned in the introduction, the goal of this paper is to find a valid initial guess in Newton iteration that leads to the converging sequence of approximations to the solution of the linearized MHD equations.

Define the following function spaces

$$\mathcal{X} := L^2(0, T; \mathcal{V}) \times L^2(0, T; [L_0^2(\Omega)]^2), \quad \mathcal{Y} = L^2(0, T; \mathcal{V}^*) \times \mathcal{H}.$$

Let $T : \mathbb{R}^2 \times \mathcal{Y} \rightarrow \mathcal{X}$ be the solution operator for the unsteady Stokes-like equations (3.1)–(3.4) and (3.5), that is, for $((v^+, v^-), (\mathbf{f}, \mathbf{z}_{IC}^\pm)) \in \mathbb{R}^2 \times \mathcal{Y}$, the solution $\mathbf{U} = (\mathbf{z}^\pm, \mathbf{q}) \in \mathcal{X}$ is defined as

$$\mathbf{U} = T_M(\mathbf{f}, \mathbf{z}_{IC}^\pm) \quad \text{for } ((v^+, v^-), (\mathbf{f}, \mathbf{z}_{IC}^\pm)) \in \mathbb{R}^2 \times \mathcal{Y}.$$

According to Theorem 3.2, the solution operator T is now well defined. Note that (2.12)–(2.15) can be rewritten as

$$\partial_t \mathbf{z}^\pm + \nabla \mathbf{q} - M\Delta \mathbf{z}^\pm = -K\mathbf{z}^\pm \quad \text{in } Q \quad (4.1)$$

$$\nabla \cdot \mathbf{z}^\pm = 0 \quad \text{in } Q \quad (4.2)$$

$$\mathbf{z}^\pm(\mathbf{x}, 0) = \mathbf{z}_{IC}^\pm(\mathbf{x}) \quad \text{in } \Omega \quad (4.3)$$

$$\mathbf{z}^\pm = \mathbf{0} \quad \text{on } \Gamma \times [0, T], \quad (4.4)$$

which can be written as

$$\mathbf{U} = T_M((v^+, v^-), (-K\mathbf{z}^\pm, \mathbf{z}_{IC}^\pm(\mathbf{x}))). \quad (4.5)$$

Motivated by (4.5), define a $\mathcal{C}^\infty$ operator G as $G : \mathcal{X} \rightarrow \mathbb{R}^2 \times \mathcal{Y}$ such that

$$G(\mathbf{U}) = ((v^+, v^-), (K\mathbf{z}^\pm, -\mathbf{z}_{IC}^\pm)), \quad \text{where } \mathbf{U} = (\mathbf{z}^\pm, \mathbf{q}).$$

Then, one may immediately check that the MHD equations (4.1)–(4.4) can be written as

$$F_M(\mathbf{U}) := \mathbf{U} + T_M(G(\mathbf{U})) = \mathbf{0}. \quad (4.6)$$

Note that

$$G(\mathbf{U}) = \left((v^+, v^-), \left(\begin{bmatrix} (\mathbf{z}^- \cdot \nabla) \mathbf{z}^+ \\ (\mathbf{z}^+ \cdot \nabla) \mathbf{z}^- \end{bmatrix}, -\mathbf{z}_{IC}^\pm \right) \right). \quad (4.7)$$

The conventional abstract theory on Newton’s method (see [12]) leads to the following iteration

$$\mathbf{U}_{n+1} = \mathbf{U}_n - [D_{\mathbf{U}} F_M(\mathbf{U}_n)]^{-1} \cdot F_M(\mathbf{U}_n), \quad n = 0, 1, 2, \dots, \quad (4.8)$$

where $D_{\mathbf{U}} F$ is the Fréchet derivative of F with respect to $\mathbf{U}$ acting on $\mathbf{U}_n := (\mathbf{z}_n^\pm, \mathbf{q}_n)$. This Newton’s algorithm becomes

$$[D_{\mathbf{U}} F_M(\mathbf{U}_n)](\mathbf{U}_{n+1} - \mathbf{U}_n) = -F_M(\mathbf{U}_n) = -\mathbf{U}_n - T_M(G(\mathbf{U}_n)) \quad (4.9)$$

in which

$$[D_{\mathbf{U}} F_M(\mathbf{U}_n)](\mathbf{U}_{n+1} - \mathbf{U}_n) = \mathbf{U}_{n+1} - \mathbf{U}_n + T_M \left((v^+, v^-), \left(\begin{bmatrix} (\mathbf{z}_n^- \cdot \nabla)(\mathbf{z}_{n+1}^+ - \mathbf{z}_n^+) + ((\mathbf{z}_{n+1}^- - \mathbf{z}_n^-) \cdot \nabla) \mathbf{z}_n^+ \\ (\mathbf{z}_n^+ \cdot \nabla)(\mathbf{z}_{n+1}^- - \mathbf{z}_n^-) + ((\mathbf{z}_{n+1}^+ - \mathbf{z}_n^+) \cdot \nabla) \mathbf{z}_n^- \end{bmatrix}, \mathbf{z}_{IC}^\pm \right) \right). \quad (4.10)$$

Thus replacing the left hand side of (4.9) with (4.10) yields

$$\begin{aligned} \mathbf{U}_{n+1} &= -T_M(G(\mathbf{U}_n)) - T_M\left((v^+, v^-), \left(\left[\begin{array}{c} (\mathbf{z}_n^- \cdot \nabla)(\mathbf{z}_{n+1}^+ - \mathbf{z}_n^+) + ((\mathbf{z}_{n+1}^- - \mathbf{z}_n^-) \cdot \nabla)\mathbf{z}_n^+ \\ (\mathbf{z}_n^+ \cdot \nabla)(\mathbf{z}_{n+1}^- - \mathbf{z}_n^-) + ((\mathbf{z}_{n+1}^+ - \mathbf{z}_n^+) \cdot \nabla)\mathbf{z}_n^- \end{array}\right], \mathbf{z}_{\text{IC}}^\pm\right)\right) \\ &= T_M\left((v^+, v^-), \left(\left[\begin{array}{c} -(\mathbf{z}_n^- \cdot \nabla)\mathbf{z}_n^+ \\ -(\mathbf{z}_n^+ \cdot \nabla)\mathbf{z}_n^- \end{array}\right], \mathbf{z}_{\text{IC}}^\pm\right)\right) \\ &\quad - T_M\left((v^+, v^-), \left(\left[\begin{array}{c} (\mathbf{z}_n^- \cdot \nabla)(\mathbf{z}_{n+1}^+ - \mathbf{z}_n^+) + ((\mathbf{z}_{n+1}^- - \mathbf{z}_n^-) \cdot \nabla)\mathbf{z}_n^+ \\ (\mathbf{z}_n^+ \cdot \nabla)(\mathbf{z}_{n+1}^- - \mathbf{z}_n^-) + ((\mathbf{z}_{n+1}^+ - \mathbf{z}_n^+) \cdot \nabla)\mathbf{z}_n^- \end{array}\right], \mathbf{z}_{\text{IC}}^\pm\right)\right) \\ &= T_M\left((v^+, v^-), \left(\left[\begin{array}{c} -(\mathbf{z}_n^- \cdot \nabla)(\mathbf{z}_{n+1}^+ - \mathbf{z}_n^+) - (\mathbf{z}_{n+1}^- \cdot \nabla)\mathbf{z}_n^+ \\ -(\mathbf{z}_n^+ \cdot \nabla)(\mathbf{z}_{n+1}^- - \mathbf{z}_n^-) - (\mathbf{z}_{n+1}^+ \cdot \nabla)\mathbf{z}_n^- \end{array}\right], \mathbf{0}\right)\right). \end{aligned} \quad (4.11)$$

Therefore, for a given initial guess $(\mathbf{z}_0^\pm, \mathbf{q}_0)$ the corresponding Newton iteration can be written as; for $n = 0, 1, 2, \dots$

$$\partial_t \mathbf{z}_{n+1}^\pm + \nabla \mathbf{q}_{n+1} + \left[\begin{array}{c} (\mathbf{z}_n^- \cdot \nabla)\mathbf{z}_{n+1}^+ + (\mathbf{z}_{n+1}^- \cdot \nabla)\mathbf{z}_n^+ \\ (\mathbf{z}_n^+ \cdot \nabla)\mathbf{z}_{n+1}^- + (\mathbf{z}_{n+1}^+ \cdot \nabla)\mathbf{z}_n^- \end{array}\right] - M \Delta \mathbf{z}_{n+1}^\pm = \left[\begin{array}{c} (\mathbf{z}_n^- \cdot \nabla)\mathbf{z}_n^+ \\ (\mathbf{z}_n^+ \cdot \nabla)\mathbf{z}_n^- \end{array}\right] \quad (4.12)$$

$$\nabla \cdot \mathbf{z}_{n+1}^\pm = 0 \quad \text{in } \Omega \quad (4.13)$$

$$\mathbf{z}_{n+1}^\pm(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega \quad (4.14)$$

$$\mathbf{z}_{n+1}^\pm = \mathbf{0} \quad \text{on } \Gamma. \quad (4.15)$$

Now, the question is whether one may choose the initial guess $(\mathbf{z}_0^\pm, \mathbf{q}_0)$ as the solution of the unsteady Stokes-like equations (3.1)–(3.4) and (3.5), that is,

$$\partial_t \mathbf{z}_0^\pm + \nabla \mathbf{q}_0 - M \Delta \mathbf{z}_0^\pm = \mathbf{0} \quad \text{in } Q \quad (4.16)$$

$$\nabla \cdot \mathbf{z}_0^\pm = \mathbf{0} \quad \text{in } Q \quad (4.17)$$

$$\mathbf{z}_0^\pm(\mathbf{x}, 0) = \mathbf{z}_{\text{IC}}^\pm(\mathbf{x}) \quad \text{in } \Omega \quad (4.18)$$

$$\mathbf{z}_0^\pm = \mathbf{0} \quad \text{on } \Gamma \times [0, T]. \quad (4.19)$$

Then, we verify that the initial guess $(\mathbf{z}_0^\pm, \mathbf{q}_0)$ as the solution of the unsteady Stokes-like equations (4.16)–(4.19) is the correct one for Newton iteration (4.12)–(4.15) with $\mathbf{f} = \mathbf{0}$.

Let

$$\mathbf{z}_n^\pm := \sum_{i=0}^n \mathbf{z}_{(i)}^\pm, \quad \mathbf{Q}_n := \sum_{i=0}^n \mathbf{q}_{(i)}, \quad \text{where } \mathbf{z}_{(0)}^\pm = \mathbf{z}_0^\pm, \quad \mathbf{q}_{(0)} = \mathbf{q}_0. \quad (4.20)$$

Consider the following iterations; find $(\mathbf{z}_{(j)}^\pm, \mathbf{q}_{(j)}) \in \mathcal{X}$ satisfying

$$\partial_t \mathbf{z}_{(j)}^\pm + \nabla \mathbf{q}_{(j)} + \left(\mathbf{P} \sum_{i=0}^{j-1} \mathbf{z}_{(i)}^\pm \cdot \nabla\right) \mathbf{z}_{(j)}^\pm + \left(\mathbf{P} \mathbf{z}_{(j)}^\pm \cdot \nabla\right) \sum_{i=0}^{j-1} \mathbf{z}_{(i)}^\pm - M \Delta \mathbf{z}_{(j)}^\pm = -\left(\mathbf{P} \mathbf{z}_{(j-1)}^\pm \cdot \nabla\right) \mathbf{z}_{(j-1)}^\pm \quad \text{in } \Omega \quad (4.21)$$

$$\nabla \cdot \mathbf{z}_{(j)}^\pm = 0 \quad \text{in } \Omega \quad (4.22)$$

$$\mathbf{z}_{(j)}^\pm = \mathbf{0} \quad \text{on } \Gamma. \quad (4.23)$$

Summing (4.21)–(4.23) from $j = 1$ up to $n + 1$, and then adding (4.16)–(4.19), it follows that

$$\partial_t \mathbf{z}_{n+1}^\pm + \nabla \mathbf{Q}_{n+1} + (\mathbf{P} \mathbf{z}_n^\pm \cdot \nabla) \mathbf{z}_{n+1}^\pm + (\mathbf{P} \mathbf{z}_{n+1}^\pm \cdot \nabla) \mathbf{z}_n^\pm - M \Delta \mathbf{z}_{n+1}^\pm = (\mathbf{P} \mathbf{z}_n^\pm \cdot \nabla) \mathbf{z}_n^\pm + \mathbf{b}(n), \quad \text{in } \Omega \quad (4.24)$$

$$\nabla \cdot \mathbf{z}_{n+1}^\pm = 0, \quad \text{in } \Omega \quad (4.25)$$

$$\mathbf{z}_{n+1}^\pm = \mathbf{0}, \quad \text{on } \Gamma, \quad (4.26)$$

where

$$\begin{aligned} \mathbf{b}(n) &= (\mathbf{P} \mathbf{z}_n^\pm \cdot \nabla) \mathbf{z}_{n+1}^\pm + (\mathbf{P} \mathbf{z}_{n+1}^\pm \cdot \nabla) \mathbf{z}_n^\pm - \sum_{j=1}^{n+1} \left[(\mathbf{P} \mathbf{z}_{j-1}^\pm \cdot \nabla) \mathbf{z}_{(j)}^\pm + (\mathbf{P} \mathbf{z}_{(j)}^\pm \cdot \nabla) \mathbf{z}_{j-1}^\pm \right] \\ &\quad - \sum_{j=1}^{n+1} \left(\mathbf{P} \mathbf{z}_{(j-1)}^\pm \cdot \nabla \right) \mathbf{z}_{(j-1)}^\pm - (\mathbf{P} \mathbf{z}_n^\pm \cdot \nabla) \mathbf{z}_n^\pm. \end{aligned}$$

Lemma 4.1. It follows that $\mathbf{b}(n) = \mathbf{0}$. Hence the system (4.24)–(4.26) is the same with (4.12)–(4.15).

Proof. We show all components of $\mathbf{b}(n) = \mathbf{0}$. We rewrite $\mathbf{b}(n)$ in terms of $\mathbf{z}_{(i)}^\pm$ using (4.20) as follows

$$\begin{aligned} \mathbf{b}(n) = & \sum_{i=0}^n (\mathbf{z}_{(i)}^\mp \cdot \nabla) \sum_{i=0}^{n+1} \mathbf{z}_{(i)}^\pm + \sum_{i=0}^{n+1} (\mathbf{z}_{(i)}^\mp \cdot \nabla) \sum_{i=0}^n \mathbf{z}_{(i)}^\pm - \sum_{j=1}^{n+1} \left[\sum_{i=0}^{j-1} (\mathbf{z}_{(i)}^\mp \cdot \nabla) \mathbf{z}_{(j)}^\pm + (\mathbf{z}_{(j)}^\mp \cdot \nabla) \sum_{i=0}^{j-1} \mathbf{z}_{(i)}^\pm \right] \\ & - \sum_{j=1}^{n+1} (\mathbf{z}_{(j-1)}^\mp \cdot \nabla) \mathbf{z}_{(j-1)}^\pm - \sum_{i=0}^n (\mathbf{z}_{(i)}^\mp \cdot \nabla) \sum_{i=0}^n \mathbf{z}_{(i)}^\pm. \end{aligned}$$

Here $\mathbf{b}(n)$ is composed of 6 components where the first 3 components are similar to the second 3 components (just flip over $+$ and $-$). So it is enough to show that the first 3 components are 0. We denote the first 3 components of $\mathbf{b}(n)$ as

$$\begin{aligned} [\mathbf{b}(n)_1, \mathbf{b}(n)_2, \mathbf{b}(n)_3]^T = & \sum_{i=0}^n (\mathbf{z}_{(i)}^- \cdot \nabla) \sum_{i=0}^{n+1} \mathbf{z}_{(i)}^+ + \sum_{i=0}^{n+1} (\mathbf{z}_{(i)}^- \cdot \nabla) \sum_{i=0}^n \mathbf{z}_{(i)}^+ - \sum_{j=1}^{n+1} \left[\sum_{i=0}^{j-1} (\mathbf{z}_{(i)}^- \cdot \nabla) \mathbf{z}_{(j)}^+ + (\mathbf{z}_{(j)}^- \cdot \nabla) \sum_{i=0}^{j-1} \mathbf{z}_{(i)}^+ \right] \\ & - m_{j=1}^{n+1} (\mathbf{z}_{(j-1)}^- \cdot \nabla) \mathbf{z}_{(j-1)}^+ - \sum_{i=0}^n (\mathbf{z}_{(i)}^- \cdot \nabla) \sum_{i=0}^n \mathbf{z}_{(i)}^+. \end{aligned}$$

For notational convenience, let $a_i := \mathbf{z}_{(i)}^- \cdot \nabla$ and let b_i be the k th ($k = 1, 2, 3$) component of $\mathbf{z}_{(i)}^+$. Then the k th component $\mathbf{b}(n)_k$ becomes

$$\mathbf{b}(n)_k = \sum_{i=0}^n a_i \sum_{i=0}^{n+1} b_i + \sum_{i=0}^{n+1} a_i \sum_{i=0}^n b_i - \sum_{j=1}^{n+1} \left[\sum_{i=0}^{j-1} a_i b_j + a_j \sum_{i=0}^{j-1} b_i \right] - \sum_{j=1}^{n+1} a_{j-1} b_{j-1} - \sum_{i=0}^n a_i \sum_{i=0}^n b_i.$$

Simple calculation using the following inequality implies $\mathbf{b}(n)_k = 0$ for $k = 1, 2, 3$

$$\left(\sum_{i=0}^n a_i \right) \left(\sum_{i=0}^{n+1} b_i \right) = \sum_{i=1}^{n+1} a_{i-1} b_{i-1} + \sum_{i=1}^{n+1} \left(\sum_{j=0}^{i-1} a_j \right) b_i + \sum_{i=1}^n a_i \left(\sum_{j=0}^{i-1} b_j \right).$$

These arguments complete the proof.

The above can be summarized step by step as follows; if one uses the solution of (4.16)–(4.19) ($\mathbf{z}_0^\pm, \mathbf{q}_0$), as the initial guess in the iteration solving (4.21)–(4.23), then $(\mathbf{z}_{n+1}^\pm, \mathbf{Q}_{n+1})$, which is defined in (4.20)

$$\mathbf{z}_{n+1}^\pm = \mathbf{z}_0^\pm + (\mathbf{z}_{(1)}^\pm + \cdots + \mathbf{z}_{(n+1)}^\pm), \quad \mathbf{Q}_{n+1} = \mathbf{q}_0 + (\mathbf{q}_{(1)} + \cdots + \mathbf{q}_{(n+1)})$$

with $(\mathbf{z}_{(i)}^\pm, \mathbf{q}_{(i)})$ the solution to (4.21)–(4.23) for $1 \leq i \leq n+1$, satisfies (4.12)–(4.15) by Lemma 4.1. That is, $(\mathbf{z}_{n+1}^\pm, \mathbf{Q}_{n+1})$ is the solution of (4.12)–(4.15) obtained by using initial guess $(\mathbf{z}_0^\pm, \mathbf{q}_0)$. The following theorem restates these arguments.

Theorem 4.2. Let Ω be an open, bounded and connected domain with Lipschitz boundary Γ . Assume that $\mathbf{f} \in H^{-1}(\Omega)$ and $q \in L_0^2(\Omega)$. Then the solution of (4.21)–(4.23) is a valid initial guess of Newton's iteration (4.12)–(4.15)

5. Numerical tests

In this section, we present two dimensional numerical results that solve (1.1) in steady state. All the theories developed in the previous sections for three dimension can be naturally applied to the two dimensional case. In two dimension, the MHD equation (1.1) is rewritten as

$$\begin{aligned} (\mathbf{u} \cdot \nabla) \mathbf{u} - \lambda (\mathbf{b} \cdot \nabla) \mathbf{b} + \nabla p - \nu \Delta \mathbf{u} &= \mathbf{f}_1, & (\mathbf{b} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{b} + \eta \Delta \mathbf{b} &= \mathbf{f}_2, \\ \nabla \cdot \mathbf{u} &= 0, & \nabla \cdot \mathbf{b} &= 0. \end{aligned}$$

Recall the definitions of new variables $\mathbf{z}^\pm := \mathbf{u} \pm \lambda^{\frac{1}{2}} \mathbf{b}$ and $v^\pm := (v \pm \eta)/2$ to transform the above equations into the following equivalent system

$$\begin{aligned} (\mathbf{z}^- \cdot \nabla) \mathbf{z}^+ + \nabla p - \Delta(v^+ \mathbf{z}^+ + v^- \mathbf{z}^-) &= \tilde{\mathbf{f}}_1, & (\mathbf{z}^+ \cdot \nabla) \mathbf{z}^- + \nabla p - \Delta(v^- \mathbf{z}^+ + v^+ \mathbf{z}^-) &= \tilde{\mathbf{f}}_2, \\ \nabla \cdot \mathbf{z}^\pm &= 0, \end{aligned}$$

where $\tilde{\mathbf{f}}_1 = \mathbf{f}_1 + \lambda^{\frac{1}{2}} \mathbf{f}_2$ and $\tilde{\mathbf{f}}_2 = \mathbf{f}_1 - \lambda^{\frac{1}{2}} \mathbf{f}_2$. Let $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ and let $\mathcal{T}_h$ denote its regular triangulation with mesh size h . The finite element method utilizing $P2 - P1$ Taylor–Hood elements is used to construct the linear system and GMRES is used to solve that linear system in each Newton iteration. The exact solution is chosen as

$$\mathbf{u}(x, y) = \begin{bmatrix} (\cos(2\pi x) - 1) \sin(2\pi y) \\ \sin(2\pi x)(1 - \cos(2\pi y)) \end{bmatrix}, \quad \mathbf{b}(x, y) = \begin{bmatrix} (x^4 - 2x^3 + x^2)(4y^3 - 6y^2 + 2y) \\ -(4x^3 - 6x^2 + 2x)(y^4 - 2y^3 + y^2) \end{bmatrix}$$

Table 1

Errors and convergence rates for the case: error = $\|\mathbf{u} - \mathbf{u}^h\|_1 + \|\mathbf{b} - \mathbf{b}^h\|_1 + \|p - p^h\|$, rate = $\log((\text{error with } h_1)/(\text{error with } h_2)) / \log(h_1/h_2)$, # = number of Newton iterations.

h	$\nu = 0.05, \eta = 0.01$			$\nu = 0.1, \eta = 0.05$			$\nu = 1.1, \eta = 1$		
	Error	Rate	#	Error	Rate	#	Error	Rate	#
1/4	4.885×10^{-1}		4	2.700×10^{-1}		16	1.087×10^{-1}		3
1/8	8.668×10^{-2}	2.5	4	8.668×10^{-2}	1.7	3	2.846×10^{-2}	1.9	2
1/10	4.805×10^{-2}	2.6	5	3.207×10^{-2}	4.4	3	1.834×10^{-2}	1.9	2
1/15	1.692×10^{-2}	2.6	5	1.218×10^{-2}	2.4	3	8.150×10^{-3}	2.0	2
1/20	1.386×10^{-2}	0.7	6	6.274×10^{-3}	2.3	8	4.574×10^{-3}	2.0	2
1/30	1.205×10^{-2}	0.3	6	2.572×10^{-3}	2.2	3	2.058×10^{-3}	2.0	2
1/40	8.995×10^{-3}	1.0	7	1.473×10^{-3}	1.9	4	1.209×10^{-3}	1.9	2
h	$\nu = \eta = \sqrt{10}$			$\nu = 5, \eta = 4$			$\nu = \eta = 10$		
	Error	Rate	#	Error	Rate	#	Error	Rate	#
1/4	1.691×10^{-1}		2	1.323×10^{-1}		2	2.782×10^{-1}		2
1/8	2.864×10^{-2}	2.6	2	2.996×10^{-2}	2.1	2	3.590×10^{-2}	3.0	2
1/10	1.827×10^{-2}	2.0	2	1.870×10^{-2}	2.1	2	2.087×10^{-2}	2.4	2
1/15	8.087×10^{-3}	2.0	2	8.178×10^{-3}	2.0	2	8.886×10^{-3}	2.1	2
1/20	4.534×10^{-3}	2.0	2	4.603×10^{-3}	2.0	2	5.305×10^{-3}	1.8	2
1/30	2.018×10^{-3}	2.0	2	2.123×10^{-3}	1.9	2	3.001×10^{-3}	1.4	2
1/40	1.156×10^{-3}	1.9	2	1.317×10^{-3}	1.7	2	2.280×10^{-3}	1.0	2

and $p(x, y) = \sin(2\pi x) \sin(2\pi y)$ with various $\lambda, \nu, \eta, \rho, \mu$ and σ . The right hand sides $\mathbf{f}_1$ and $\mathbf{f}_2$ are calculated using the exact solution. As we have claimed in the previous sections, we choose the steady-state solution of Eqs. (3.1)–(3.2) as the initial guess in Newton iteration. The Newton iteration stops when the following approximate relative error is small enough such as

$$\frac{\|\mathbf{z}_n^+ - \mathbf{z}_{n+1}^+\|_1^2 + \|\mathbf{z}_n^- - \mathbf{z}_{n+1}^-\|_1^2 + \|p_n - p_{n+1}\|^2}{\|\mathbf{z}_{n+1}^+\|_1^2 + \|\mathbf{z}_{n+1}^-\|_1^2 + \|p_{n+1}\|^2} < 10^{-8}.$$

Once an approximation, $(\mathbf{z}_{n+1}^\pm, p_{n+1})$, for $(\mathbf{z}^\pm, p)$ is obtained, the approximation for $(\mathbf{u}, \mathbf{b}, p)$ is set as

$$\mathbf{u}^h = (\mathbf{z}_{n+1}^+ + \mathbf{z}_{n+1}^-) / 2, \quad \mathbf{b}^h = (\mathbf{z}_{n+1}^+ - \mathbf{z}_{n+1}^-) / (2\sqrt{\lambda}), \quad p^h = p_{n+1}.$$

Table 1 presents the numerical results with various ν and η . Most of cases (except convection dominating case) show the optimal finite element convergence, $O(h)$, with proposed initial guess in Newton iterations. Since suggested exact solution is smooth, we have a super-convergence for convection–diffusion equally-balanced cases. One can observe that the convergence speed degrades when convection dominates (small ν and η). It is well known that once the convection gets dominating, the stability of the solution of original MHD equations themselves cannot be guaranteed. Those cases have been studied for a long time and many works had been done (definitely a lot more in Navier–Stokes equation itself).

6. Conclusion

The magnetohydrodynamic equations are a set of nonlinear partial differential equations and the Newton's algorithm is considered here to solve them. The Newton's algorithm is a popular approach to find an approximation to the nonlinear partial differential equations like Navier–Stokes equations. It is known that the initial guess has to be chosen carefully to guarantee the convergence of Newton's iteration and it is also well known that the solution of Stokes equations often can be used as the initial guess for Newton's iteration in solving Navier–Stokes equations. In this paper, the magnetohydrodynamic equations are rewritten as Navier–Stokes like equations as shown in (2.12)–(2.15) by rearranging the variables and introducing new operators in (2.1)–(2.3). Once we transformed the magnetohydrodynamic equations into the Navier–Stokes like equations, we have shown that the existence and uniqueness of the solution to the corresponding Stokes-like equations. Then it has proven that the solution of Stokes-like equations is a good initial guess for Newton's iteration for the targeted Eqs. (4.12)–(4.15) that are led from transformed magnetohydrodynamic Eqs. (4.1)–(4.4).

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