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Identification of time-dependent source terms and control parameters in parabolic equations from overspecified boundary data

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Abstract

This paper presents a semigroup approach for inverse source problems for the abstract heat equation, when the measured output data is given in subject to the integral overspecification over the spatial domain. The existence of a solution to the inverse source problem is shown in appropriate function spaces and a representation formula for the solution is proposed. Such representation permits the derivation of sufficient conditions for the uniqueness of the solution. Also an approximation method based on the optimal homotopy analysis method (OHAM) is designed, and the error estimates are discussed using graphical analysis. Moreover, we conjecture that our approach can be applied for the determination of a control parameter in an inverse problem with integral overspecialization data. The proposed algorithm is examined through various numerical examples for the reconstruction of continuous sources and the determination of a control parameter in parabolic equations. The accuracy and stability of the method are discussed and compared with several finite-difference techniques. Computational results show efficiency and high accuracy of the proposed algorithm.

Key words: Inverse Source Problem; Identification Problem; Semigroup Theory; Homoyopy Analysis Method.

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1 Introduction

Let us consider a Banach space X, a linear operator A in X, $z \in X$, $g \in C^1([0,\tau];\mathbb{R})$, and a linear functional $\phi \in X^*$. We study the inverse problem of finding a pair of functions $u \in C^1([0,\tau];X)$ and $p \in C([0,\tau];\mathbb{R})$ from the set of relations

$$u'(t) = Au(t) + p(t)z, \qquad 0 \le t \le \tau, \tag{1}$$
$$u(0) = u_0, \tag{2}$$
$$\phi[u(t)] = g(t), \qquad 0 \le t \le \tau. \tag{3}$$

Mathematical models related to inverse problems of this type arise in various physical and engineering settings such as, the identification of water sources and air pollution in the environment, or the determination of heat sources in heat conduction.

Heat source identification problems are the most commonly encountered inverse problems in heat conduction. These problems have been studied for several decades due to their significance in a variety of scientific and engineering applications (see [4–6,10,13,15–20]). In many heat conduction and diffusion problems, the source terms are unknown and usually are not easy to be detected directly. Hence, only one of the following typical measured output data is available and feasible from experiments:

$$\begin{cases} \displaystyle\int_0^l u(x,t)k(x)\,dx = g(t), \\ u(x_0,t) = g(t). \end{cases}$$

These data are defined to be overspecified boundary (measured) data, according to inverse problems terminology.

The first attempt to study source identification problems for the time independent source $p(t)z \in X$, with the final overdetermination $u_T(x) := u(x,T)$, by the semigroup approach has been given in [33], where it is proved that when the elliptic operator $-A$ is positive definite and self-adjoint, the solution (u,p) of the source identification problem exists and is unique. A general representation formula for a solution of the source identification problem for the abstract parabolic equation $u_t(t) = Au(t) + F(t)$, was proposed in [18]. Note that an inverse source problem with final overdetermination for the one dimensional heat equation has first been considered by Tikhonov [35] in the study of geophysical problems. A semigroup approach for inverse source problems for the abstract heat equation $u_t = Au + F$, when the measured output data is given in the form the final overdetermination $u_T(x) := u(x,T)$ has been proposed in [20]. In this work abstract parabolic equation with overspecified boundary data is studied in half-plane and the uniqueness of the solution is proved. For parabolic equations in a bounded domain, various aspects of inverse source problems were studied in [1,2,21], etc.

Numerical methods for solving inverse problems for parabolic equations are considered in many works. Backward Euler approximation method has been introduced in [11] for the inverse problem of identifying a time dependent unknown coefficient in a parabolic problem subject to initial and non-local boundary conditions along with an overspecified condition defined at a specific point in the spatial domain. The idea in [34] is to change the problem of identifying an unknown time-dependent source term in an inverse problem of parabolic type with nonlocal boundary conditions to a system of Volterra integral equations and then to solve the system by means of a collocation method. Inverse problem of reconstructing the coefficient q in the parabolic equation $u_t - \Delta u + q(x)u = 0$ from the final measurement $u(x,T)$ has been solved using the optimization method combined with the finite element method in [14]. Recently, a numerical method depends on the Fourier regularization method for solving ill-posed problems of heat equation has been proposed in [12].

In comparison with previous studies on the subject of time-dependent heat source identification, we consider the case when A is a generator of a c_0-semigroup which is more general than [22,23] who considered only the case where A is a bounded linear operator and [32] who considered A as a generator of a c_0-semigroup with ϕA bounded. Our result in this case will be presented in Theorem 2 where much of detailed proof is completely different from the last mentioned studies. Moreover, we shall also consider the problem of finding a control parameter $p(t)$ in the following form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(t)u + g(x,t), \qquad 0 \le x \le 1,\ 0 < t \le T, \tag{4}$$

with the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{5}$$

and boundary conditions

$$u(0,t) = g_1(t), \qquad 0 < t \le T, \tag{6}$$
$$u(1,t) = g_2(t), \qquad 0 < t \le T, \tag{7}$$

with an additional condition which describes the overspecification over a portion of the spatial domain

$$\int_0^1 k(x)u(x,t)dx = E(t), \qquad 0 < t \le T, \tag{8}$$

which has never been investigated analytically and it represents one of the main contributions of the present study. Some of the ideas from the proof of Theorem 2 are combined to form a numerical method based on the OHAM of the underlying inverse problems.

The paper is organized as follows. Section 2 gives the mathematical analysis of the inverse problem and recalls some previous perturbation results for linear operators in Banach spaces. Moreover, it introduces a quasisolution of the inverse source problems (1)-(3) based on the solution to the corresponding direct problem. Sections 3 and 4 describe the optimal homotopy analysis method for solving analytically the inverse source and control problems with overspecified boundary data observations; it also discusses the numerical results. Section 5 gives the conclusion of the paper and possible future work.

2 Mathematical analysis

To solve the problem (1)-(3) when the operator A is the infinitesimal generator of a c_0-semigroup on X, we propose a method coupled with the perturbation theory of linear operators. The proposed method, requiring no conditions such as ϕA is bounded, can eliminate the restrictions of the traditional methods.

Let $(X, \|\cdot\|)$ be a Banach space and let A be the infinitesimal generator of a c_0-semigroup on X. It is well known that A is closed and hence its domain $D(A)$ equipped with the graph norm

$$\|x\|_A = \|x\| + \|Ax\|$$

becomes a Banach space, which we shall denote by X_A. Let us now present the following theorem which is the main result in [8,9].

Theorem 1 *Let X be a Banach space and let A be the infinitesimal generator of a c_0-semigroup $T(t)$ on X. If $B : X_A \longrightarrow X_A$ is a continuous linear operator, then $A + B$ is the infinitesimal generator of a c_0-semigroup on X.*

2.1 Strong solution in Banach spaces

The Main result of this work is given by the following theorem.

Theorem 2 *If A is the infinitesimal generator of a c_0-semigroup on X, $\phi[z] \neq 0$, $g \in C^1([0,\tau];\mathbb{R})$ and $z \in X$,then a solution of the inverse problem (1)-(3) exists and it is unique in the class of functions*

$$u \in C^1([0,\tau]; X), \qquad p \in C([0,\tau];\mathbb{R}).$$

PROOF. Applying the linear functional ϕ to both sides of (1) and using (3) we have

$$g'(t) - \phi[Au(t)] = p(t)\phi[z],$$

and we get that

$$p(t) = \frac{1}{\phi[z]}(g'(t) - \phi[Au(t)]). \tag{9}$$

Substituting (9) in (1), we get

$$u'(t) = Au(t) + \frac{1}{\phi[z]}(g'(t) - \phi[Au(t)])z. \tag{10}$$

Clearly (10) implies

$$u'(t) - (\ Au(t) + \frac{-1}{\phi[z]}(\phi[Au(t)])z\ ) = \frac{1}{\phi[z]}g'(t)z. \tag{11}$$

By imposing the operator

$$Bx = \frac{-1}{\phi[z]}(\phi[Ax])z, \tag{12}$$

equation (11) becomes

$$u'(t) - (A+B)u(t) = \frac{1}{\phi[z]}g'(t)z. \tag{13}$$

For the boundedness of B in X_A,

$$\begin{aligned}
\|B\|_A &= \sup_{\|x\|_A=1} \|Bx\| \\
&= \sup_{\|x\|_A=1} \left\| \frac{-1}{\phi[z]}(\phi[A(x)])z \right\| \\
&\le \sup_{\|x\|_A=1} \frac{1}{|\phi[z]|} \|z\| \, \|\phi\| \, \|Ax\| \\
&\le \frac{1}{|\phi[z]|} \|z\| \, \|\phi\| \, .
\end{aligned}$$

This implies that B is a bounded linear operator on X_A. Theorem 1 now implies that $A+B$ is the infinitesimal generator of a semigroup $S(t)$, $t \ge 0$. The Cauchy problem (1)-(2) has a unique solution $u(t)$ given by

$$u(t) = S(t)u_0 + \frac{1}{\phi[z]}\int_0^t S(t-s)g'(s)zds, \tag{14}$$

and, by (9) and (14), $p(t)$ is uniquely determined. Therefore, the problem is completely determined.

2.2 Identification of an unknown time-dependent heat source term from overspecified boundary data

We study the inverse problem of determining the temperature $u(x, t)$ and the heat source $f(t)$ in the parabolic heat equation

$$u_t(x,t) = u_{xx}(x,t) + F(x,t), \qquad (x,t) \in (0,l) \times (0,T) \tag{15}$$

where h is a given function, and l and T are given positive constants. Here l represents the length of the finite heat conductor and the subscripts t and x in Eq. (15) denote the partial derivatives with respect to t and x respectively. Equation (15) has to be solved subject to the initial temperature condition

$$u(x,0) = u_0(x), \qquad 0 \leq x \leq l, \tag{16}$$

the boundary conditions

$$u(0,t) = u(l,t) = 0, \qquad 0 < t \leq T, \tag{17}$$

and the overspecified boundary data

$$\int_0^l u(x,t)k(x)\ dx = g(t), \qquad 0 \leq t \leq T. \tag{18}$$

In the case of an arbitrary source term $F(x, t)$, difficulties related to the identifiability and uniqueness are discussed in some studies [21–23,32,33]. Evidently, one can only hope for a well-formulated inverse source problem only in the case if some a priori information is derived from the corresponding physical model. When the source is assumed to only depend on the spatial variable, the problem of determining the spacewise unknown source $F(x)$ in the inverse source problem governed by Eq. (1) and the measured final data $u_T(x) := u(x, T_f)$ is one of the most studied problems (see, for example, [4,16] and references therein). In the separable sources case of the form $F(x, t) = p(t)f(x)$, where $p(t)$ is known, the inverse problem of determining the spacewise unknown source has first been studied in [15].

In what follows we will discuss the inverse source problem with separable source, $F(x, t) = p(t)f(x)$. More precisely, the time-dependent source term $p(t)$ needs to be recovered from the overspecified boundary data defined by (18), assuming that the function $f(x)$ is known.

The symbol $u(t)$ will refer to the same function $u(x, t)$ but being viewed as an abstract function of the variable t with values in the Banach space

$$X = \{u : u \text{ is a continuous real-valued function with } u(0) = u(l)\}.$$

The symbol u_0 will be used in treating the function $u_0(x)$ as the element of the space X. Consider the linear functional $\varphi \in X^*$, defined by

$$\varphi[u(t)] = \int_0^l u(t)k(x)\, dx,$$

and a linear operator A in X defined by

$$D(A) = \{u : u, u', u'' \in X\},$$

and for $u \in D(A)$,

$$Au = u''.$$

It is not difficult to show that A generates an analytic c_0-semigroup on X. With these ingredients, the system (15)-(18) reduces to the inverse problem (1)-(3), which is, due to Theorem 2, has a unique solution.

3 Homotopy analysis solution

We are exploring the inverse problem of finding a pair of functions (u, p) from the set of relations

$$u_t(x,t) = u_{xx}(x,t) + p(t)F(x,t), \qquad (x,t) \in (0,l) \times (0,T), \tag{19}$$

$$u(x,0) = f(x), \qquad 0 \le x \le l, \tag{20}$$

$$\begin{aligned} u(0,t) &= h_1(t), \qquad 0 < t \le T, \\ u(l,t) &= h_2(t), \qquad 0 < t \le T, \end{aligned} \tag{21}$$

with an additional condition that describes the overspecification over a portion of the spatial domain

$$\int_0^l k(x)u(x,t)\, dx = g(t), \qquad 0 \le t \le T. \tag{22}$$

where F, f, h_1, h_2, k, and g are known functions.

Without loss of generality we solve (19)-(22) for $h_1(t) = h_1(t) = 0$. Note that we cannot apply the existing numerical techniques to solve this problem because it contains two unknowns. The inclusion of (9) has created a new scenario and thus required special considerations on its solution; it is this consideration that constitutes our focus point in this paper. In what follows, we shall show that under some reasonable assumptions there exists a unique solution pair (u, p) to (19)-(22).

3.1 Approach based on the Optimal HAM

Now we proceed the approximation of the solution pair (u, p) by the Optimal HAM algorithm (OHAM). The temperature distribution $u(x, t)$ and the heat source $p(t)$ can be expressed by the set of the base functions

$$\{x^n t^m | n \geq 0, \ m \geq 0\} \tag{23}$$

in the form

$$u(x,t) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} a_{1,n}^m x^n t^m, \tag{24}$$

$$p(t) = \sum_{n=0}^{+\infty} a_{2,n} t^n, \tag{25}$$

where $a_{1,n}^m$ and $a_{2,n}$ are coefficients to be determined. We choose carefully an initial approximation $u_0(x, t)$ of $u(x, t)$ that obeys to the *Rule of the Solution Expression* denoted by (23) and satisfies the initial and boundary conditions (20) and (21). Making use of (9), it is natural to choose an initial approximation of $p(t)$ in the form

$$p_0(t) = \left(g'(t) - \int_0^l k(x) \frac{\partial^2 u_0(x,t)}{\partial x^2} dx \right) / E(t), \tag{26}$$

where

$$E(t) = \int_0^l k(x) F(x,t) dx.$$

Besides that we select

$$\mathcal{L}_u[w(x,t)] = \frac{\partial w(x,t)}{\partial t},$$
$$\mathcal{L}_p[w(t)] = w(t),$$

as our auxiliary linear operators satisfying the following properties

$$\mathcal{L}_u[\eta(x)] = 0, \quad \mathcal{L}_p[0] = 0.$$

If $q \in [0, 1]$ is the embedding parameter and $\hbar_i$ $(i = 1, 2)$ are the auxiliary parameters, then according to the underlying principle of the HAM [25,27,29], the *zeroth-order deformation* problems can be constructed as follows:

$$(1-q)\mathcal{L}_u[\hat{u}(x,t;q) - u_0(x,t)] = q\hbar_1 \mathcal{N}_u[\hat{u}(x,t;q), \hat{p}(t;q)], \tag{27}$$

$$(1-q)\mathcal{L}_r[\hat{p}(t;q) - p_0(t)] = q\hbar_2 \mathcal{N}_p[\hat{u}(x,t;q), \hat{p}(t;q)], \tag{28}$$

subject to the boundary conditions

$$\hat{u}(0,t;q) = 0, \quad \hat{u}(l,t;q) = 0, \tag{29}$$

in which the non-linear operators $\mathcal{N}_u$ and $\mathcal{N}_p$ are defined by

$$\mathcal{N}_u[\hat{u}(x,t;q), \hat{p}(t;q)] = \frac{\partial \hat{u}(x,t;q)}{\partial t} - \frac{\partial^2 \hat{u}(x,t;q)}{\partial x^2} - \hat{p}(t;q)F(x,t),$$
$$\mathcal{N}_p[\hat{u}(x,t;q), \hat{p}(t;q)] = \hat{p}(t;q) - \left(g'(t) - \int_0^l k(x)\frac{\partial^2 \hat{u}(x,t;q)}{\partial x^2}dx\right)/E(t).$$

For $q = 0$ and $q = 1$, the above *zeroth-order deformation* equations (27)-(28) have the solutions

$$\hat{u}(x,t;0) = u_0(x,t), \quad \hat{p}(t;0) = r_0(t), \tag{30}$$

and

$$\hat{u}(x,t;1) = u(x,t), \quad \hat{p}(t;1) = r(t). \tag{31}$$

When q increases from 0 to 1, then $\hat{u}(x,t;q)$ and $\hat{p}(t;q)$ vary from the initial guesses $u_0(x,t)$ and $p_0(t)$ to the exact solutions $u(x,t)$ and $p(t)$, respectively.

Setting

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \hat{u}(x,t;q)}{\partial q^m}\bigg|_{q=0},$$
$$p_m(t) = \frac{1}{m!}\frac{\partial^m \hat{p}(t;q)}{\partial q^m}\bigg|_{q=0},$$

and expanding $\hat{u}$ and $\hat{p}$ into the Taylor series expansion with respect to the embedding parameter q, we have

$$\hat{u}(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{32}$$

$$\hat{p}(t;q) = p_0(t) + \sum_{m=1}^{\infty} p_m(t)q^m. \tag{33}$$

The convergence of the series in Eqs. (32) and (33) depends on $\hbar_1$ and $\hbar_2$. Assuming that $\hbar_1$ and $\hbar_2$ are selected in such a way that the series in Eqs. (32) and (33) are convergent at $q = 1$, then due to Eqs. (30) and (31) we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{34}$$

$$p(t) = p_0(t) + \sum_{m=1}^{\infty} p_m(t). \tag{35}$$

Differentiating the zeroth-order deformation Eqs. (27)-(28) m times with respect to q, then setting $q = 0$, and finally dividing by $m!$, the mth-order deformation problems can be expressed as

$$\mathcal{L}_u[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar_1 \mathcal{R}_m^u(x,t), \tag{36}$$
$$\mathcal{L}_p[p_m(t) - \chi_m p_{m-1}(t)] = \hbar_2 \mathcal{R}_m^p(t), \tag{37}$$
$$u_m(x,0) = 0,\ u_m(0,t) = 0,\ u_m(l,t) = 0 \tag{38}$$

where

$$\mathcal{R}_m^u(x,t) = \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - p_{m-1}(t)F(x,t),$$
$$\mathcal{R}_m^p(x,t) = p_{m-1}(t) - \left((1-\chi_m)g'(t) - \int_0^l k(x) \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} dx \right) / E(t)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

The general solutions of the higher-order deformation Eqs. (36)-(38) are

$$u_m(x,t) = (\chi_m + \hbar_1)\, u_{m-1}(x,t) - \hbar_1 \int_0^t \left(\frac{\partial^2 u_{m-1}(x,\tau)}{\partial x^2} + p_{m-1}(\tau)F(x,\tau) \right) d\tau, \tag{39}$$

$$p_m(t) = (\chi_m + \hbar_2)\, p_{m-1}(t) - \frac{\hbar_2}{E(t)} \left((1-\chi_m)g'(t) - \int_0^l k(x) \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} dx \right). \tag{40}$$

Therefore it is easy to solve the linear non-homogeneous Eqs. (39) and (40) one after one by using Mathematica in the following order $m = 1, 2, 3, \ldots$.

If we are not able to determine the sum of series in (24) and (25) then, we can accept the partial sum of these series

$$\hat{u}_n(x,t) \simeq \sum_{m=0}^{n} u_m(x,t), \tag{41}$$
$$\hat{p}_n(t) \simeq \sum_{m=0}^{n} p_m(t), \tag{42}$$

as the approximate solution of the considered equation.

3.2 Performance analysis of HAM algorithm: an optimal choice of the control parameters

We note that Eqs. (39) and (40) consist of the auxiliary parameters $\hbar_1$ and $\hbar_2$. It has been shown by Liao in his book [25] that the convergence and rate of approximation of such series depend on the values of $\hbar_1$ and $\hbar_2$. For this purpose, we let

$$E_n^u(\hbar_1,\hbar_2) = \int_{\Omega_1} \left\{ \mathcal{N}_u \left[\sum_{m=1}^{n} u_m(x,t), \sum_{m=1}^{n} p_m(t) \right] \right\}^2 d\Omega_1,$$
$$E_n^p(\hbar_1,\hbar_2) = \int_{\Omega_2} \left\{ \mathcal{N}_p \left[\sum_{m=1}^{n} u_m(x,t), \sum_{m=1}^{n} p_m(t) \right] \right\}^2 d\Omega_2,$$

denote the squared residual of the nth-order approximations of the governing equations integrated in the whole domain. The optimal value of $(\hbar_1,\hbar_2)$ is the value that minimizes E_n^u and E_n^p [28,29]. To our knowledge, the optimal value of the auxiliary parameter $\hbar$ in the HAM method always remains within the range $-3 \le \hbar \le 3$. By the Extreme Value Theorem, $E_n^u(\hbar_1,\hbar_2)$ attains its absolute minimum in the rectangle $D = \{(\hbar_1,\hbar_2) : -3 \le \hbar_1, \hbar_2 \le 3\}$. To do so, we will start by finding all the critical points that lie inside the given rectangle by solving simultaneously the system

$$\frac{\partial E_n^u(\hbar_1,\hbar_2)}{\partial \hbar_1} = 0 \quad \text{and} \quad \frac{\partial E_n^u(\hbar_1,\hbar_2)}{\partial \hbar_2} = 0.$$

Next, we need to find the absolute extrema of the function $E_n^u(\hbar_1,\hbar_2)$ along the boundary of the rectangle D. The same argument can be used with $E_n^p(\hbar_1,\hbar_2)$.

3.3 Numerical Results and Discussion

Let us use the investigated method for exploring the inverse problem of finding a pair of functions (u, f) from the set of relations

$$u_t(x,t) = u_{xx}(x,t) + p(t)\sin x, \qquad (x,t) \in (0,\pi)\times(0,T), \tag{43}$$

$$u(x,0) = 0, \qquad 0 \le x \le \pi, \tag{44}$$

$$u(0,t) = u(\pi,t) = 0, \qquad 0 < t \le T, \tag{45}$$

$$\int_0^\pi u(x,t)\sin x \, dx = t, \qquad 0 \le t \le T. \tag{46}$$

Solution of the above equation is given by the pair of functions $u_e(x,t) = \frac{2}{\pi} t \sin x$ and $p_e(t) = \frac{2}{\pi}(t+1)$.

By taking the initial approximation $u_0(x) = 0$, we get by (26) that $p_0(t) = 2/\pi$. Thus, we get successively

$$u_1(x,t) = -\frac{2}{\pi}\hbar_1 t \sin x,$$

$$u_2(x,t) = -\frac{\hbar_1^2}{\pi}t^2 \sin x - \frac{2}{\pi}\hbar_1 (\hbar_1 + 1) t \sin x,$$

$$u_3(x,t) = -(\hbar_1 + 1)\left(\frac{\hbar_1^2}{\pi}t^2 \sin x + \frac{2}{\pi}\hbar_1 (\hbar_1 + 1) t \sin x\right) - \frac{1}{3\pi}\left(\hbar_1^2 t^2 (3\hbar_1 + 3\hbar_2 + \hbar_1 t + 3) \sin x\right),$$

$$\vdots$$

and

$$p_1(t) = 0,$$

$$p_2(t) = \frac{2}{\pi}\hbar_1\hbar_2 t,$$

$$p_3(t) = \frac{1}{\pi}\hbar_1\hbar_2 (2 + 2\hbar_1 + \hbar_1 t) t + \frac{2}{\pi}\hbar_1\hbar_2 (1 + \hbar_2) t,$$

$$\vdots$$

In Fig. 1 the plots of squared residual E_n^u for $n = 5, 10, 15$ are presented. By minimizing the squared residual of governing equations, optimal values of the convergence control parameters were numerically determined

$$\hbar_1 = \hbar_2 = -0.88.$$

Assuming that $\hbar_1 = \hbar_2 = \hbar$, Figure 2 presents the $\hbar$-curve of $u(1,2)$. Table 1 compiles the percentage relative errors of the exact solution reconstruction for various values of the convergence control parameter $\hbar$. As revealed by the above results, together with increase of the components number in sums (32) and (33) the errors quickly decrease. The fastest error decrease can be observed for optimal value $\hbar = -0.88$. For this value the approximate solution $\hat{u}_{10}(x,t)$ provides the approximation of the sought function with the error not higher than $7.894 \times 10^{-2}\%$, while the approximate solution $\hat{u}_{18}(x,t)$ gives the error not higher than $7.83 \times 10^{-5}\%$. Obtained results indicate that the method is very rapidly convergent and the calculation of only the few first terms of the series ensures a very good approximation of the exact solution.

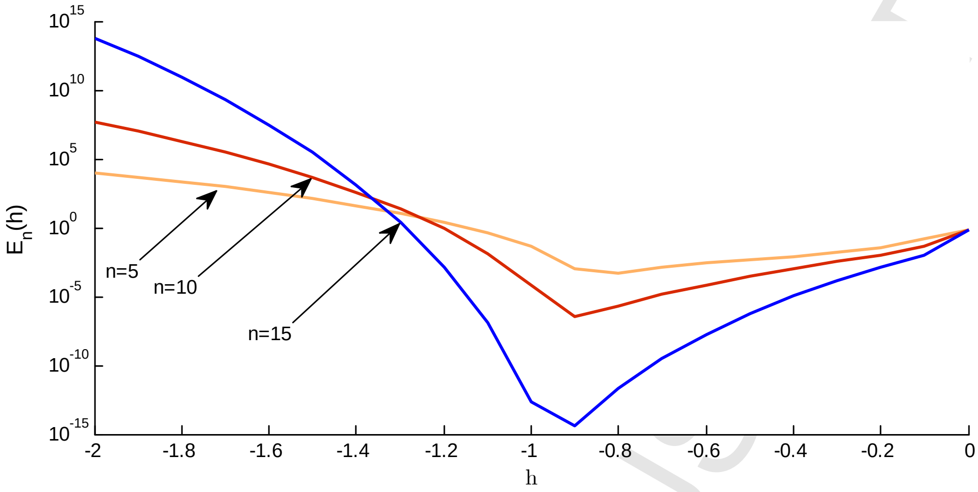

Fig. 1. Squared residual E_n^u for $n = 5, 10, 15$

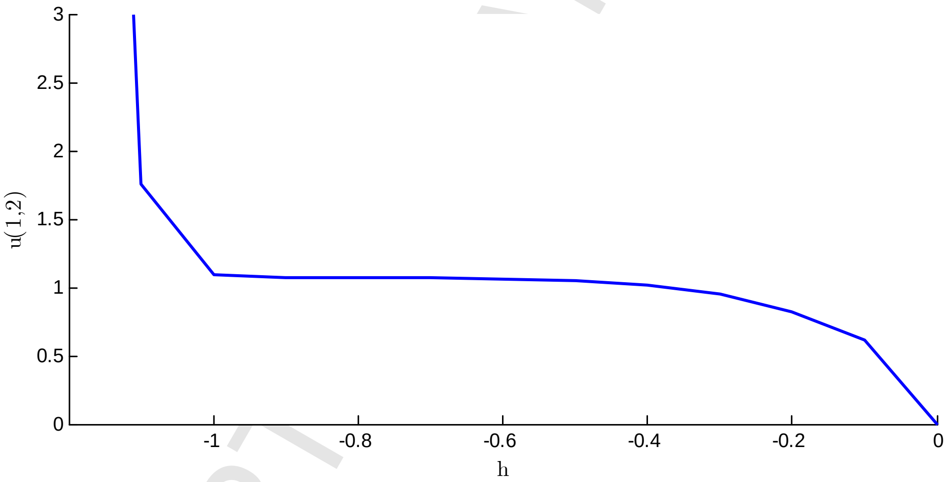

Fig. 2. The $\hbar$-curve of $u(1, 2)$

4 Determination of a control parameter in a parabolic partial differential equation

The purpose of this section is to discuss a numerical approach for solving the inverse problem with temperature overspecification (4)-(8). Our approach begins with the utilization of the following transformations [3]

$$w(x,t) = u(x,t)r(t),$$
$$r(t) = \exp\left(-\int_0^t p(\tau)d\tau\right).$$

n	$\hbar = -1.2$	$\hbar = -1$	$\hbar = -0.88$	$\hbar = -0.7$	$\hbar = -0.5$
4	83.68	20.83	10.40	12.18	22.13
6	0.7995	6.80	2.10	4.74	12.20
8	0.6369	1.76	0.40	1.74	6.56
10	0.4383	0.37	7.894×10^{-2}	0.61	3.46
12	0.2680	6.748×10^{-2}	1.513×10^{-2}	0.21	1.79
14	0.1486	1.041×10^{-2}	2.762×10^{-3}	6.898×10^{-2}	0.92
16	0.0757	1.409×10^{-3}	4.773×10^{-4}	2.201×10^{-2}	0.46
18	0.0359	1.695×10^{-4}	7.83×10^{-5}	6.833×10^{-3}	0.23
20	0.0159	1.836×10^{-5}	1.225×10^{-5}	2.069×10^{-3}	0.11

Table 1
Values of the percentage relative errors in reconstruction of the exact solution $u(2,1)$ with $\hbar_1 = \hbar_2 = \hbar$

So, we have

$$u(x,t) = \frac{w(x,t)}{r(t)}, \tag{47}$$

$$p(t) = -\frac{r'(t)}{r(t)}. \tag{48}$$

With this transformation, $p(t)$ will disappear and its role is represented implicitly by $r(t)$. So we overcome the difficulty in handling $p(t)$ and we obtain the following new non-classic parabolic partial differential equation which is equivalent to the original inverse problem, providing that some compatibility conditions are satisfied

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + r(t)g(x,t), \qquad 0 \leq x \leq 1,\ 0 < t \leq T, \tag{49}$$

with the initial condition

$$w(x,0) = f(x), \tag{50}$$

and the boundary conditions

$$w(0,t) = g_1(t)r(t), \qquad 0 < t \leq T, \tag{51}$$

$$w(1,t) = g_2(t)r(t), \qquad 0 < t \leq T, \tag{52}$$

subject to

$$\int_0^1 k(x)w(x,t)dx = r(t)E(t), \qquad 0 \leq t \leq T. \tag{53}$$

Obviously, if we obtain (w, r) from (49)-(53) then (u, p) can be found as

$$u(x,t) = \frac{w(x,t)}{r(t)}, \tag{54}$$

$$p(t) = -\frac{r'(t)}{r(t)}. \tag{55}$$

Defining the auxiliary linear operator

$$\mathcal{L}_w[v] = \frac{\partial v}{\partial t}$$

and assuming $\mathcal{L}_w$ to be invertible, we get

$$w(x,t) - w(x,0) = \mathcal{L}_w^{-1}\mathcal{L}_w[w]$$

or what amounts to the same as

$$w(x,t) = f(x) + \int_0^t \left(\frac{\partial^2 w(x,\tau)}{\partial x^2} + r(\tau)g(x,\tau) \right) d\tau. \tag{56}$$

System (49)-(53) is now a system similar to that in (19)-(22). Using the same procedure used in the previous section, we write the solution in the form

$$w(x,t) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t),$$

$$r(t) = r_0(t) + \sum_{m=1}^{\infty} r_m(t). \tag{57}$$

We choose initial approximations of $w(x,t)$ and $p(t)$ in the form

$$w_0(x,t) = f(x), \quad r_0(t) = \left(\int_0^1 k(x) w_0(x,t) dx \right) / E(t). \tag{58}$$

In this case, by applying the homotopy analysis method, we get the following formula for functions w_m:

$$w_m = \chi_m w_{m-1}(x,t) + \hbar_1 \mathcal{R}_m^w(x,t),$$

where $\mathcal{R}_m^w$ is defined by the relation

$$\mathcal{R}_m^w(x,t) = w_{m-1}(x,t) - \int\limits_0^t \left(\frac{\partial^2 w_{m-1}(x,\tau)}{\partial x^2} + p_{m-1}(\tau) g(x,\tau) \right) d\tau - (1-\chi_m) f(x).$$

By using definitions of the respective operators we obtain

$$w_1 = \hbar_1 \int_0^t \left(\frac{\partial^2 w_0(x,\tau)}{\partial x^2} + r_0(\tau) g(x,\tau) \right) d\tau$$

and for $m \geq 2$:

$$w_m = (1+\hbar_1)\, w_{m-1} - \hbar_1 \int_0^t \left(\frac{\partial^2 w_{m-1}(x,\tau)}{\partial x^2} + r_{m-1}(\tau) g(x,\tau) \right) d\tau.$$

Also, we get the following formula for r_m:

$$r_m = \chi_m r_{m-1}(t) + \hbar_2 \mathcal{R}_m^r(t),$$

where

$$\mathcal{R}_m^r(x,t) = r_{m-1}(t) - \frac{1}{E(t)} \int_0^1 k(x) w_{m-1}(x,t) dx,$$

$$r_1(t) = \hbar_2 \left(r_0(t) - \int_0^1 k(x) w_0(x,t) dx / E(t) \right)$$

and for $m \geq 2$,

$$r_m(t) = (1+\hbar_2)\, r_{m-1} - \frac{\hbar_2}{E(t)} \int_0^1 k(x) w_{m-1}(x,t) dx.$$

If we are not able to determine the sum of series in (57) then, we can accept the partial sum of these series

$$\begin{aligned} \hat{w}_n(x,t) &\simeq \sum_{m=0}^{n} w_m(x,t), \\ \hat{r}_n(t) &\simeq \sum_{m=0}^{n} r_m(t), \end{aligned} \tag{59}$$

as the approximate solution of the considered equation.

Example 3 *Consider the inverse problem (4)-(8) with*

$$\begin{aligned} f(x) &= x + \cos(\pi x), \quad g_1(t) = \exp(t), \quad g_1(t) = 0, \\ g(x,t) &= exp(t)[x + \cos(\pi x) + \pi^2 \cos(\pi x)] - \exp(t)(1+t^2)[x + \cos(\pi x)], \\ E(t) &= \exp(t) \left(\frac{3}{4} - \frac{2}{\pi^2} \right), \quad k(x) = 1 + x^2. \end{aligned}$$

Exact solution of this problem has the form

$$u_e(x,t) = exp(t)\,(x + \cos(\pi x))\,, \quad p_e(t) = 1 + t^2.$$

The $\hbar_1 - \hbar_2$ surface of $r(0.5)$ is plotted in Fig.3 to determine the valid regions of $\hbar_1$ and $\hbar_1$. It is found that the series of $r(t)$ converges in the region of

$-0.5 < \hbar_1 < 0$ and $-1 < \hbar_2 < -0.5$. Numerically determined, by minimizing the squared residual of governing equations, optimal values of the convergence control parameters were $\hbar_1 = -0.0018$ and $\hbar_2 = -0.97$. We can see that when $\hbar_1 = \hbar_2 = -1$, the solution (57) is exactly the same as that given by the Adomian decomposition method in [36]. However, the results given by the Adomian decomposition method converge to the corresponding numerical solutions in a rather small region, as shown in Fig. 4. But, the proposed method provides us with a simple way to adjust and control the convergence region of solution series by choosing an optimal value for the auxiliary parameters $\hbar_1$ and $\hbar_2$.

Differences $|r_e(t) - r_n(t)|$ for $n = 3, 5, 8$ are displayed in Fig. 4. As indicated by the example, with the properly chosen values of convergence control parameters $\hbar_1$ and $\hbar_2$, if it is impossible to predict a general form of functions w_m and r_m or calculate the sum of series in (57), it is sufficient to use the sum of several first functions w_m and r_m to obtain a very good approximation of the sought solutions.

Employing the pair of transformations (47) and (48), we can write (4)-(8) as follows,

$$w_t = w_{xx} + A(x,t)\int_0^1 k(x)w(x,t)dx, \qquad 0 \le x \le 1,\ 0 < t \le T, \tag{60}$$

$$w(x,0) = f(x), \qquad 0 \le x \le 1, \tag{61}$$

$$w(0,t) = B(t)\int_0^1 k(x)w(x,t)dx, \qquad 0 < t \le T, \tag{62}$$

$$w(1,t) = C(t)\int_0^1 k(x)w(x,t)dx, \qquad 0 < t \le T, \tag{63}$$

where

$$A(x,t) = \frac{g(x,t)}{E(t)}, \qquad B(t) = \frac{g_1(x,t)}{E(t)}, \qquad C(t) = \frac{g_2(x,t)}{E(t)}$$

and

$$r(t) = \frac{\int_0^1 k(x)w(x,t)dx}{E(t)}.$$

This means that a predicting-correcting mechanism can be constructed easily to solve (60)-(63) numerically. Once w is known numerically, the unknown (u,p) can be calculated through the inverse transformations (47) and (48) via numerical differentiation.

The domain $[0,1] \times [0,T]$ is divided into an $M \times N$ mesh with the spatial step size $h = 1/M$ in x direction and the time step size $l = T/N$, respectively. Grid points (x_i, t_n) are defined by

$$x_i = ih, \qquad i = 0, 1, 2, \ldots, M,$$
$$t_n = nh, \qquad n = 0, 1, 2, \ldots, N.$$

The one-dimensional forward time centred space (FTCS) finite-difference scheme leads to the following difference equation for (60),

$$w_i^{n+1} = sw_{i-1}^n + (1-2s)w_i^n + sw_{i+1}^n + (lh)\, z_i^n \sum_{j=0}^{M} c_j k_j w_j^n,$$

where

$$s = \frac{l}{h^2}, \qquad w_i^0 = f_i,$$
$$w_0^{n+1} = hB^n \sum_{j=0}^{M} c_j k_j w_j^n, \qquad n \geq 1,$$
$$w_M^{n+1} = hC^n \sum_{j=0}^{M} c_j k_j w_j^n, \qquad n \geq 1.$$

The notations w_i^n, p^n, z_i^n, r^n, B^n, C" and k_i are used for the finite-difference approximations of $w(ih, nl)$, $p(nl)$, $z(ih, nl)$, $r(nl)$, $B(nl)$, $C(nl)$ and $k(ih)$, respectively. Here $c_j = 1/2$, if $j = 0$ or M and unity, otherwise.

The Crank-Nicolson formula [7,30], for each $i = 1, 2, \ldots, M-1$, reads

$$-sw_{i-1}^{n+1} + 2(1+s)w_i^{n+1} - sw_{i+1}^{n+1} = sw_{i-1}^n + 2(1-s)w_i^n + sw_{i+1}^n + (lh)\, z_i^n \sum_{j=0}^{M} c_j k_j w_j^n.$$

Saulyev's finite-difference schemes are unconditionally stable and are explicit in nature and provide a useful and interesting alternative approach to the existing finite difference method [7,30]. The Saulyev's formula is given by,

$$w_i^{n+1} = \frac{1}{1+s}\left[sw_{i-1}^{n+1} + (1-s)w_i^n + sw_{i+1}^n + (lh)\, z_i^n \sum_{j=0}^{M} c_j k_j w_j^n\right],$$

for $i = 1, 2, \ldots, M-1$.

After we obtained the value of r^n, we can convert it into the corresponding value of p^n through the inverse transformation (48). This can be done by numerical differentiation. The finite-difference form of (48) is

$$p^n = -\frac{(r')^n}{r^n},$$

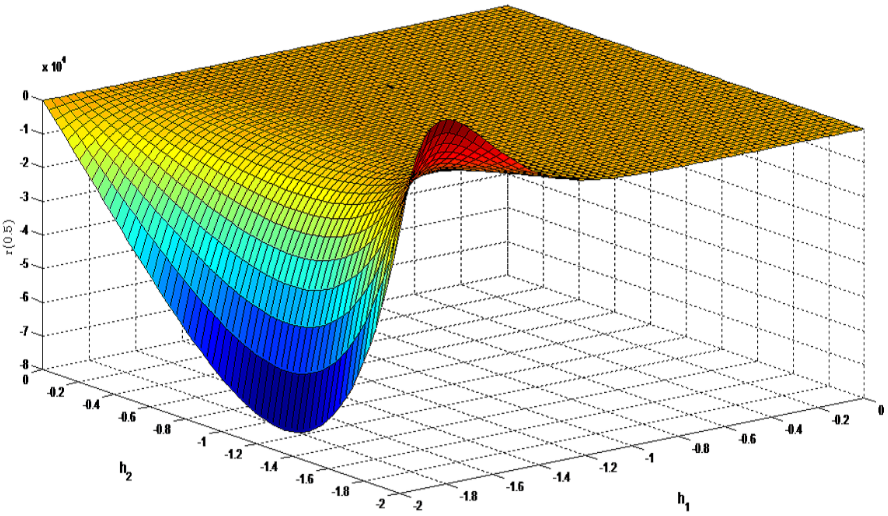

Fig. 3. The fifth-order approximation of $r(t)$ versus $\hbar_1$ and $\hbar_2$

and for the direct numerical differentiation, the following formula is used,

$$
\begin{aligned}
(r')^n &= \frac{r^{n+1} - r^{n-1}}{2l}, \qquad n = 2, 3, \ldots, N-1, \\
(r')^1 &= \frac{-3r^1 + 4r^2 - r^3}{2l}, \\
(r')^N &= \frac{3r^N - 4r^{N-1} + r^{N-2}}{2l}.
\end{aligned}
$$

For this example, we compared the proposed method (OHAM) with the FTCS, Crank-Nicolson and Saulyev methods. The absolute errors result from the numerical solutions to (4)-(8) are shown in Table 2. We observe that as soon as the number of terms in (59) is sufficiently large, the errors decrease rapidly. Although all the methods performed quite well, but we found that the proposed method performed the best.

5 Conclusion and future work

We have presented an elegant and efficient method, based on the optimal homotopy analysis method, for the numerical solution of inverse problems of identifying the time-dependent sources and control parameters from supplementary overspecified boundary measurements. Perturbation method for linear operators are employed to derive explicit formulas for the corresponding solutions, then the OHAM algorithm based on these explicit formulas is

t	FTCS	Crank-Nicolson	Saulyev	OHAM $\hat{p}_{10}(t)$	OHAM $\hat{p}_{30}(t)$
0.05	9.7×10^{-3}	7.2×10^{-3}	9.9×10^{-3}	8.0×10^{-4}	6.2×10^{-6}
0.10	9.8×10^{-3}	7.4×10^{-3}	9.8×10^{-3}	1.2×10^{-3}	3.1×10^{-6}
0.15	9.7×10^{-3}	7.5×10^{-3}	9.9×10^{-3}	1.3×10^{-3}	7.2×10^{-6}
0.20	9.7×10^{-3}	7.7×10^{-3}	1.1×10^{-2}	7.1×10^{-4}	6.8×10^{-6}
0.25	9.8×10^{-3}	7.8×10^{-3}	1.0×10^{-2}	5.8×10^{-4}	8.0×10^{-6}
0.30	9.5×10^{-3}	7.6×10^{-3}	9.8×10^{-3}	2.6×10^{-3}	7.4×10^{-5}
0.35	9.6×10^{-3}	7.7×10^{-3}	9.9×10^{-3}	5.5×10^{-3}	8.0×10^{-5}
0.40	9.9×10^{-3}	7.9×10^{-3}	9.9×10^{-3}	9.2×10^{-3}	1.3×10^{-5}
0.45	9.9×10^{-3}	7.8×10^{-3}	9.8×10^{-3}	1.3×10^{-2}	4.6×10^{-5}
0.50	9.8×10^{-3}	7.7×10^{-3}	9.7×10^{-3}	1.9×10^{-2}	8.1×10^{-5}
0.55	9.8×10^{-3}	7.6×10^{-3}	9.6×10^{-3}	2.5×10^{-2}	1.2×10^{-5}
0.60	9.7×10^{-3}	7.7×10^{-3}	9.8×10^{-3}	3.1×10^{-2}	5.6×10^{-5}
0.65	9.7×10^{-3}	7.6×10^{-3}	9.7×10^{-3}	3.9×10^{-2}	4.2×10^{-5}
0.70	9.6×10^{-3}	7.6×10^{-3}	9.5×10^{-3}	4.6×10^{-2}	2.5×10^{-4}
0.75	9.5×10^{-3}	7.5×10^{-3}	9.5×10^{-3}	5.4×10^{-2}	1.9×10^{-4}
0.80	9.5×10^{-3}	7.3×10^{-3}	9.4×10^{-3}	6.2×10^{-2}	7.1×10^{-4}
0.85	9.4×10^{-3}	7.4×10^{-3}	9.3×10^{-3}	7.1×10^{-2}	8.8×10^{-4}
0.90	9.5×10^{-3}	7.3×10^{-3}	9.3×10^{-3}	7.9×10^{-2}	8.0×10^{-4}
0.95	9.6×10^{-3}	7.2×10^{-3}	9.1×10^{-3}	8.7×10^{-2}	2.6×10^{-4}

Table 2
Results for the absolute error in the approximations to p, with $h = 0.02$, $s = 0.4$, $T = 1$, $\hbar_1 = -0.0018$ and $\hbar_2 = -0.97$.

proposed. For the sake of comparison, we have also suggested and tested further methods, namely, the FTCS, Crank-Nicolson and Saulyev methods. The resulting method is in most cases as good as the finite difference methods to a much easier implementation. The results presented for the most used classes of inverse problems show that the proposed algorithm is a very fast and effective reconstruction algorithm, if the optimal values of the control parameters are properly chosen.

The proposed method gives rapid convergence with a particularly good initial approximation as (58). It is not quite as efficient as Neural Network methods; it is slightly increasing in running time as the tolerance is increased as compared to the Neural Network methods. However, it has the added advantage of being

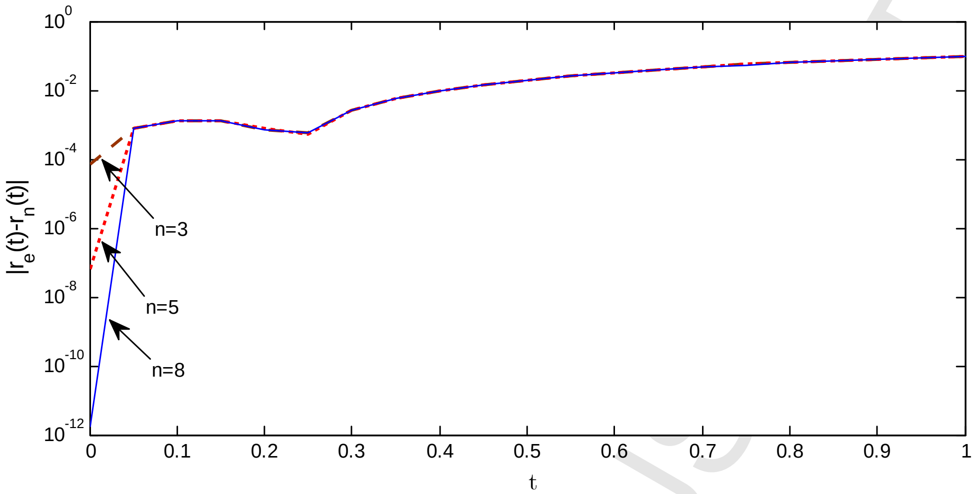

Fig. 4. Distribution of error of the exact solution approximation for $r(t)$ when $n = 3, 5, 8$

able to approximate the solution analytically.

Although this paper has considered, for simplicity, the inverse source problem of the first-order, extensions to higher-order mathematical models related to inverse problems arising in various physical and engineering settings such as, the identification of water sources, air pollution in the environment, or the determination of heat sources in heat conduction are also possible. Looking into such extensions will be our future work.

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