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On stability and convergence of semi-Lagrangian methods for the first-order time-dependent nonlinear partial differential equations in 1D

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Abstract

In this article, one-step semi-Lagrangian method is investigated for computing the numerical solutions of the first-order time-dependent nonlinear partial differential equations in 1D with initial and boundary conditions. This method is based on Lagrangian trajectory or the integration from the departure points to the arrival points (regular nodes) and Runge-Kutta method for ordinary differential equations. The departure points are traced back from the arrival points along the trajectory of the path. The convergence and stability are studied for the implicit and explicit methods. The numerical examples show that those methods work very efficient for the time-dependent nonlinear partial differential equations.

Keywords: Semi-Lagrangian methods, trajectory, Runge-Kutta method, time-dependent nonlinear partial differential equations, explicit and implicit methods
2000 MSC: 76D05, 74H15, 68Q25, 65Y20.

1. Introduction

It is well-known that mathematical models (differential equations) in a vast range of fields, from science and technology to sociology and business almost every field where measurements can be taken, describe how quantities change as the time or location change. It depends on how many variables

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are involved, this leads naturally to the formulations of ordinary differential equations (ODEs), or partial differential equations (PDEs) with different orders. For some of them, we can use the traditional techniques to provide an excellent explicit or implicit solutions. Most realistic mathematical models cannot be solved in this way; instead, they must be dealt with by computational methods that provide approximate solutions [4, 10, 11] in some sense.

In order to find the solutions, we must rely on numerical methods that produce approximations to the desired solutions. With the help of personal computers, a vast amount of effort has been expanded in designing, analyzing and applying computational techniques for differential equations [4, 11]. A great part of theoretical and practical developments have been made in this area, and new ideas continue to emerge.

For the well-posed initial-value problems of ODEs [4],

$$\frac{dy(t)}{dt} = f(t, y(t)), \quad a \le t \le b, \qquad y(a) = \alpha,$$

many numerical methods have been studied for different right side functions. The readers are referred to [4, 11]. Runge-Kutta methods obviously are a good choice to compute the approximations of ordinary differential equations. Runge-Kutta methods are one-step methods composed of a number of stages [11]. They have high-order local truncation error while eliminating the need to compute and evaluate the derivatives of $f(t, y)$. A weighted average of the slopes (f) of the solution computed at nearby points is used to determine the solution at t_{n+1} from that at $t = t_n$.

However, the partial differential equations or nonlinear partial differential equations are much more complicated. In this article, only time-dependent nonlinear partial differential equations are investigated. The numerical solution of time-dependent nonlinear partial differential equations is usually achieved by a proper combination of suitable spatial discretization and time integration processes. The method of lines [16] transforms initial-boundary value problems for time-dependent partial differential equations into initial-value problems for systems of ordinary differential equations. This is achieved by first discretization of the space variables by using finite difference, finite element or finite volume approximations. When a time-dependent partial differential equation involves terms of different types, it is a natural idea to employ different discretizations for them [7, 8]. Implicit-explicit time-discretization schemes [1] are an example of such a strategy. Fractional step Runge-Kutta methods [15] are a class of additive Runge-Kutta schemes that

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provide efficient time discretizations for evolutionary partial differential equations. Here the main idea is that the spatial discretization of PDEs leads to an initial-value problem of ODEs. Then, the traditional Runge-Kutta methods could be applied. Unfortunately, it is very difficult to apply those methods to nonlinear partial differential equations to obtain high-order methods.

Semi-Lagrangian methods have been introduced at the beginning of the eighties [20] and since then have been extensively used in the numerical simulation of models for weather forecast, oceanography, and for more general problems in fluid dynamics [2, 18, 19, 23]. The basic idea of the semi-Lagrangian method is to construct the method over a grid point along the characteristics, or the trajectory path. However, in this way, it is very difficult to obtain high order methods. Existing methods, like three-level or two-level Semi-Lagrangian methods, are order two. Despite the large number of papers of numerical solutions with semi-Lagrangian methods [23], only few theoretical studies were reported for each individual equations [9, 17]. The general study of numerical analysis for semi-Lagrangian methods is not reported.

In this article, the time-dependent nonlinear differential equations as follows are considered,

$$u_t + uu_x = f(t, x, u, u_x, u_{xx}), \quad t > 0, \quad x \in \Omega,$$

with initial-value and boundary conditions. Where Ω could be any dimensional. On the Lagrangian trajectory path, those equations could be considered as ODEs. In order to use Runge-Kutta methods, the arrival and departure points at t_{n+1} and $t = t_n$ are needed. One could be specified on regular nodes, but the other one needs to be computed. For example, if the arrival points are on the grid, then the departure points must be calculated. The approximate solution on departure points at $t = t_n$ must be approximated by some interpolating methods. Also Runge-Kutta methods could be applied to different types of partial differential equations [3, 5, 21, 25]. In the paper [14], the truncation error of the first-order semi-Lagrangian method was studied. The complete numerical analysis including stability and convergence is presented in this paper for the first-order and the second-order semi-Lagrangian methods. The numerical analysis and numerical example showed that it is possible to construct overall high order one-step difference method for the time-dependent partial differential equations. The main difficulty came from the calculation of the departure points.

This article is organized as follows: Section 2 shows the specified partial differential equations with given initial-value and boundary conditions to be

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studied; In Section 3, the departure point is computed; An one-step difference method is constructed and analyzed in Section 4; In Section 5, the semi-Lagrangian methods are investigated; The implicit semi-Lagrangian methods are shown in Section 6; The numerical examples are presented in Section 7; The final section is the summary and the consideration of future work.

2. Time-Dependent partial differential Equations

Many important partial differential equations (PDEs) have the format as

$$\frac{du(t,x)}{dt} = f(t,x,u,u_x,u_{xx}), \quad t>0, \quad x\in\Omega,$$

with initial-value condition

$$u(0,x) = u_0(x), \quad x\in\Omega,$$

where Ω is a domain in one, two, or three dimensions. The boundary condition is provided for $u(t,x)$ on $\partial\Omega$ if it is needed.

The full derivative reads

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\cdot\frac{\partial}{\partial x},$$

where u is the velocity field and x is the spatial variable. In this article, the dimensional one is considered for demonstration. It is more complicated for higher dimensions and it will be reported later.

For examples, the Burgers equation,

$$u_t + uu_x = \nu u_{xx},$$

with Dirichlet boundary conditions and initial conditions, and the Navier-Stokes equations,

$$\begin{cases} u_t + (u\cdot\nabla)u = f + \nu\Delta u - \nabla p, \\ \operatorname{div} u = 0, \end{cases}$$

with boundary conditions and the initial conditions. Even the reformulated Shallow-Water equations [12, 13] have the similar formate.

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In this article, in order to focus on the proposed method, we will study the general time-dependent differential equations as follows,

$$\begin{cases} \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(t,x,u), \quad 0 < t < T, a \le x \le b \\ u(0,x) = u_0(x), \quad a \le x \le b \end{cases} \tag{1}$$

with boundary conditions about x. Both implicit or explicit methods will be studied for the problem (1) for stability and convergence.

Theorem 1. *If $f(t,x,u)$ is a continuous function of t and x and satisfies the Lipschitz condition in u in the region $0 \le t \le T$, $a \le x \le b$, $-\infty < u < \infty$ and the first order derivative of $u_0(x)$ is continuous on $[a,b]$, then there exists at least a differentiable solution $u(t,x)$ of the initial-value boundary problem (1).*

This result can be found in many books on partial differential equations, such as [24] on page 282.

3. Computing the Departure Points

It is well-known that the exact solution of the problem (1) is hard to find. The approximations to u will be computed first at the given points, called grid points, in the domain $[0,T] \times [a,b]$. At other points in the domain, the approximate solutions are calculated from the interpolation of the approximate solutions on the grid points.

Now suppose that the grid points are equally distributed throughout the domain $[0,T] \times [a,b]$. Let N and M be two positive integer numbers. The grid points are formed by calculating

$$t_k = k\tau, \text{ for each } k = 0, 1, 2, \ldots, N,$$

$$x_i = a + ih, \text{ for each } i = 0, 1, 2, \ldots, M,$$

with the step sizes $\tau = T/N$ and $h = (b-a)/M$.

Assume that all solutions u on (t_k, x_i^k) for $i = 0, 1, \ldots, M$ are known, and we need to find the solutions u on (t_{k+1}, x_i^{k+1}) for $i = 1, 2, \ldots, M-1, M$. Let x_A^{k+1} be one of x_i at $t = t_{k+1}$ for $i = 1, 2, \ldots, M-1$, noted as arrival point. From the view of Lagrangian trajectory, the particles on arrival points at $t = t_{k+1}$ shall come from some points, called departure points at $t = t_k$ along

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the trajectory of path. In order to recover the departure point x_D^k at $t = t_k$ for the arrival point x_A^{k+1}, the following equation is considered,

$$\frac{dx}{dt} = u. \tag{2}$$

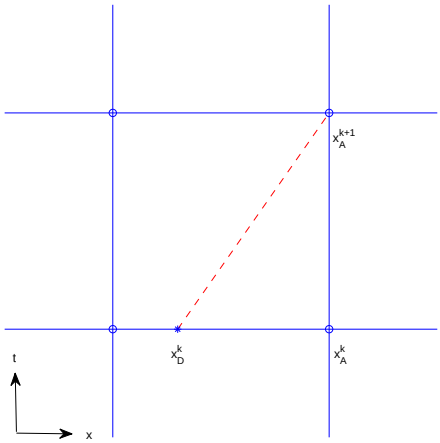


Figure 1: Departure and arrival points on grid.

Note that the departure point x_D^k may not be any of x_i, $i = 0, 1, \ldots, M$ at $t = t_k$ as shown in Figure 1. Then the departure point x_D^k is calculated by integrating the equation backwards along the trajectory from the arrival point x_A^{k+1} as

$$x_D^k = x_A^{k+1} - \int_{t_k}^{t_{k+1}} u \, dt.$$

It is obvious that Euler's method will provide the first order approximation as

$$x_D^k \approx x_A^{k+1} - \tau u(t_k, x_D^k).$$

This is an implicit equation about x_D^k and $u(t_k, x_D^k)$ is approximated by interpolation from $u(t_k, x_i)$ for $i = 0, 1, \ldots, M$. The results of numerical experiment showed that couple iterations are enough to obtain good approximation of x_D^k [22, 26].

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4. One-step difference method

For the problem (1), along the trajectory of the path, we propose a general one-step difference method in the form

$$\begin{cases} w_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0, 1, \cdots, M\}, \\ x_D^k = x_A^{k+1} - \tau\, w_D^k, \quad w_D^k \text{ Calculated from linear interpolation of } w_A^k, \\ w_A^{k+1} = w_D^k + \tau\phi(t_k, x_D^k, w_D^k, \tau, h), \quad \text{for each } k = 0, 1, \cdots, N-1, \end{cases} \tag{3}$$

where x_D^k is the departure point and most likely is not any of the regular grids; x_A^{k+1} is the arrival point and is one of regular grids; w_D^k and w_A^{k+1} are approximations of $u(t_k, x_D^k)$ and $u(t_{k+1}, x_A^{k+1})$ respectively for $k = 0, 1, \ldots, N-1$.

Remark: w_D^k is calculated from linear interpolation of w_A^k. If we consider a higher order difference method, a higher order interpolation is needed. From the experiment, those two orders are not required to be same.

Assuming that $\tau = O(h)$ (or $h = \alpha\tau$, $\alpha > 0$ a constant), and let $u(t, x)$ be the unique solution of the initial-value problem (1). We define the local truncation error for each time step as

Definition The local truncation error of the difference method (3) to the problem (1) is defined as

$$\begin{aligned} \varepsilon_{k+1}(\tau) &= \frac{u_A^{k+1} - [u_D^k + \tau\phi(t_k, x_D^k, u_D^k, \tau, h)]}{\tau} \\ &= \frac{u_A^{k+1} - u_D^k}{\tau} - \phi(t_k, x_D^k, u_D^k, \tau, h), \end{aligned}$$

where $u_D^k = u(t_k, x_D^k)$ and $u_A^{k+1} = u(t_{k+1}, x_A^{k+1})$ on the departure and arrival points respectively.

We need three lemmas. The proofs of first two can be found in any numerical analysis book, for example [4].

Lemma 1. *For all $t \geq -1$ and any positive m, we have $0 \leq (1+t)^m \leq e^{mt}$.*

Lemma 2. *If s and t are positive real numbers, $\{a_k\}_{k=0}^n$ is a sequence satisfying $a_0 \geq -t/s$, and*

$$a_{k+1} \leq (1+s)a_k + t, \quad \textit{for each } k = 0, 1, 2, \cdots, n-1,$$

then

$$a_{k+1} \leq e^{(k+1)s}(a_0 + \frac{t}{s}) - \frac{t}{s}.$$

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Lemma 3. *Suppose $x_i = a+ih$ for $i = 0, 1, \cdots, M$ and $h = (b-a)/M$. Then for each $x \in [a, b]$, if $p(x)$ and $q(x)$ are two piece-wise linear interpolations with $p(x_i) = u_i$ and $q(x_i) = v_i$, $i = 0, 1, \cdots, M$, then*

$$|p(x) - q(x)| \leq \max_{0\leq i\leq M} |u_i - v_i|.$$

Proof: For any $x \in [a, b]$, if $x = x_i$ for any $i \in \{0, 1, \cdots, M\}$, then the result is clearly true, since $p(x_i) = u_i$ and $q(x_i) = v_i$. Suppose that $x \in (x_i, x_{i+1})$ for some i, then

$$p(x) - q(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}}(u_i - v_i) + \frac{x - x_i}{x_{i+1} - x_i}(u_{i+1} - v_{i+1}).$$

Then we have

$$|p(x) - q(x)| \leq \frac{x_{i+1} - x}{x_{i+1} - x_i}|u_i - v_i| + \frac{x - x_i}{x_{i+1} - x_i}|u_{i+1} - v_{i+1}|$$

$$\leq (\frac{x_{i+1} - x}{x_{i+1} - x_i} + \frac{x - x_i}{x_{i+1} - x_i}) \max_{0\leq i\leq M} |u_i - v_i|$$

$$\leq \max_{0\leq i\leq M} |u_i - v_i|.$$

♯

Then, we have following theorem,

Theorem 2. *The initial-value boundary problem (1) is approximated by a one-step difference method (3). Suppose that there exists a number $\tau_0 > 0$ such that $\phi(t, x, w, \tau, h)$ is continuous and satisfies a Lipschitz condition in all variables with Lipschitz constant L on the set*

$$D = \{(t, x, w, \tau, h) | \, 0 \leq t \leq T, a \leq x \leq b, -\infty < w < \infty, 0 \leq \tau \leq \tau_0, 0 \leq h \leq \alpha\tau_0\}.$$

Then

(i) *The difference method (3) is stable;*

(ii) *The difference method (3) is convergent if and only if it is consistent, which is equivalent to*

$$\phi(t, x, w, 0, 0) = f(t, x, w), \text{ for all } 0 \leq t \leq T, a \leq x \leq b, -\infty < w < \infty;$$

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(iii) *if a function ε exists and, for each $k = 0, 1, \ldots, N$, the local truncation error $\varepsilon_k(\tau)$ satisfies $|\varepsilon_k(\tau)| \le \varepsilon(\tau)$ whenever $0 \le \tau \le \tau_0$, then*

$$\max_A |u(t_k, x_A^k) - w_A^k| \le e^{TL} \max_A |u_0(x_A) - w_A^0| + \frac{\varepsilon(\tau)}{L}(e^{TL} - 1),$$

$$\le \frac{\varepsilon(\tau)}{L}(e^{TL} - 1), \ \textit{for } k = 1, 2, \cdots, N.$$

Proof: (i) Suppose that $\{u_A^k\}_{k=1}^N$ and $\{v_A^k\}_{k=1}^N$ satisfy the difference equation (3) with the initial-value as u_0 and v_0 respectively. Let

$$E_A^k = u_A^k - v_A^k \text{ for the point } (t_k, x_A^k), \quad E_D^k = u_D^k - v_D^k \text{ for the point } (t_k, x_D^k)$$

and

$$E^k = \max |E_A^k| \text{ on the regular grid points (arrival points) at } t = t_k.$$

Then, by Lemma 3,

$$|E_D^k| \le \max |E_A^k| = E^k,$$

$$u_A^{k+1} - v_A^{k+1} = u_D^k - v_D^k + \tau[\phi(t_k, x_D^k, u_D^k, \tau, h) - \phi(t_k, x_D^k, v_D^k, \tau, h)],$$

and

$$|u_A^{k+1} - v_A^{k+1}| \le |u_D^k - v_D^k| + \tau L |u_D^k - v_D^k|$$
$$\le (1 + \tau L)|u_D^k - v_D^k|,$$

i.e.

$$|E_A^{k+1}| \le (1 + \tau L)|E_D^k| \le (1 + \tau L)E^k.$$

and

$$E^{k+1} \le (1 + \tau L)E^k.$$

Therefore from Lemma 1 and Lemma 2,

$$E^k \le (1 + \tau L)^k E^0 \le e^{k\tau L} E^0 \le e^{TL} E^0,$$

where $k \le N = T/\tau$.

So, the difference method (3) is stable.

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(ii) Let $\phi(t,x,w,0,0) = g(t,x,w)$. From the assumptions of ϕ, g satisfies the conditions of Theorem 1, then the differential equation

$$\begin{cases} \frac{dv}{dt} = g(t,x,v), \quad 0 < t < T, a \le x \le b, \\ v(0,x) = u_0(x), \quad a \le x \le b, \end{cases}$$

has a continuous differential solution $v(t,x)$. The numerical solution satisfies

$$\begin{cases} z_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0,1,\cdots,M\}, \\ x_D^k = x_A^{k+1} - \tau\, z_D^k, \quad z_D^k \text{ Calculated from linear interpolation of } z_A^k, \\ z_A^{k+1} = z_D^k + \tau\phi(t_k, x_D^k, z_D^k, \tau, h), \quad \text{for each } k = 0,1,\cdots,N-1, \end{cases} \tag{4}$$

By the mean value theorem,

$$v(t_{k+1}, x_A^{k+1}) = v(t_k, x_D^k) + \tau g(t_k + \xi\tau, x_D^k + \eta h, v(t_k + \xi\tau, x_D^k + \eta h)),$$

for some constants ξ and η $(0 \le \xi, \eta \le 1)$.

Similarly, define $E_A^k = z_A^k - v(t_k, x_A^k)$ and $E^k = \max|E_A^k|$ on the regular grid points (arrival points) at $t = t_k$. Subtract this from the numerical solution, we have

$$\begin{aligned} E_A^{k+1} = E_D^k + \tau[&\phi(t_k, x_D^k, z_D^k, \tau, h) - \phi(t_k, x_D^k, v(t_k, x_D^k), \tau, h) \\ &+\phi(t_k, x_D^k, v(t_k, x_D^k), \tau, h) - \phi(t_k, x_D^k, v(t_k, x_D^k), \tau, 0) \\ &+\phi(t_k, x_D^k, v(t_k, x_D^k), \tau, 0) - \phi(t_k, x_D^k, v(t_k, x_D^k), 0, 0) \\ &+\phi(t_k, x_D^k, v(t_k, x_D^k), 0, 0) - g(t_k + \xi\tau, x_D^k + \eta h, v(t_k + \xi\tau, x_D^k + \eta h))], \end{aligned}$$

$$|\phi(t_k, x_D^k, z_D^k, \tau, h) - \phi(t_k, x_D^k, v(t_k, x_D^k), \tau, h)| \le L|z_D^k - v(t_k, x_D^k)| \le L|E_D^k|,$$

$$\begin{aligned} |\phi(t_k, x_D^k, v(t_k, x_D^k), \tau, h) - \phi(t_k, x_D^k, v(t_k, x_D^k), \tau, 0)| \le Lh, \\ |\phi(t_k, x_D^k, v(t_k, x_D^k), \tau, 0) - \phi(t_k, x_D^k, v(t_k, x_D^k), 0, 0)| \le L\tau, \end{aligned}$$

and

$$|\phi(t_k, x_D^k, v(t_k, x_D^k), 0, 0) - g(t_k + \xi\tau, x_D^k + \eta h, v(t_k + \xi\tau, x_D^k + \eta h))|$$

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$$
\begin{aligned}
&\le |g(t_k, x_D^k, v(t_k, x_D^k)) - g(t_k + \xi\tau, x_D^k, v(t_k, x_D^k))| \\
&\quad + |g(t_k + \xi\tau, x_D^k, v(t_k, x_D^k)) - g(t_k + \xi\tau, x_D^k + \eta h, v(t_k, x_D^k))| \\
&\quad + |g(t_k + \xi\tau, x_D^k + \eta h, v(t_k, x_D^k)) - g(t_k + \xi\tau, x_D^k + \eta h, v(t_k + \xi\tau, x_D^k + \eta h))| \\
&\le L\xi\tau + L\eta h + L|v(t_k, x_D^k) - v(t_k + \xi\tau, x_D^k + \eta h))| \\
&\le L\xi\tau + L\eta h + LL_1\xi\tau + LL_1\eta h \\
&\le L\tau + Lh + LL_1\tau + LL_1 h \\
&\le L(1 + L_1)(\tau + h),
\end{aligned}
$$

where L_1 is a constant such that,

$$
|\frac{\partial v}{\partial t}| \le L_1 \qquad \text{and} \qquad |\frac{\partial v}{\partial x}| \le L_1,
$$

for all (t, x) in the region $0 \le t \le T, a \le x \le b$.

Then

$$
\begin{aligned}
|E_A^{k+1}| &\le |E_D^k| + L\tau|E_D^k| + L\tau h + L\tau^2 + L(1 + L_1)\tau(\tau + h) \\
&\le (1 + \tau L)E^k + L(2 + L_1)\tau(\tau + h),
\end{aligned}
$$

and

$$
E^{k+1} \le (1 + \tau L)E^k + L(2 + L_1)\tau(\tau + h).
$$

Thanks to Lemma 2 from which we have

$$
E^k \le e^{k\tau L}E^0 + (2 + L_1)(\tau + h)(e^{k\tau L} - 1),
$$

and

$$
E^k \le e^{TL}E^0 + (2 + L_1)(\tau + h)(e^{TL} - 1).
$$

This converges to zero as τ and h ($E^0 = 0$) go to zero. So the numerical solution converges to the solution of initial-value boundary problem (1) if $f = g$. Therefore the numerical solution from the difference method (3) is convergent.

On the other hand, if we have convergence, then u and v are the same. Suppose f and g differ at some point.Then we consider the initial-value boundary problem starting from that point. Two solutions should be different. This leads to a contradiction.

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(iii) Let E_A^k, E_D^k, and E^k are defined as in **(i)**. Then from the definition of the local truncation error, we have,

$$u(t_{k+1}, x_A^{k+1}) = u(t_k, x_D^k) + \tau\phi(t_k, x_D^k, u(t_k, x_D^k), \tau, h) + \tau\varepsilon_{k+1}(\tau).$$

Subtract this from the difference method, it reads

$$E_A^{k+1} = E_D^k + \tau[\phi(t_k, x_D^k, w_D^k, \tau, h) - \phi(t_k, x_D^k, u(t_k, x_D^k), \tau, h)] - \tau\varepsilon_{k+1}(\tau).$$

therefore,

$$\begin{aligned} |E_A^{k+1}| &\le |E_D^k| + \tau L|w_D^k - u(t_k, x_D^k)| + \tau|\varepsilon_{k+1}(\tau)|, \\ &\le (1+\tau L)|E_D^k| + \tau|\varepsilon_{k+1}(\tau)|, \end{aligned}$$

or

$$E^{k+1} \le (1+\tau L)E^k + \tau\varepsilon(\tau).$$

Then thanks Lemma 2, we have

$$E^k \le e^{k\tau L}(E^0 + \frac{\varepsilon(\tau)}{L}) - \frac{\varepsilon(\tau)}{L},$$

i.e.

$$E^k \le e^{TL}E^0 + \frac{\varepsilon(\tau)}{L}(e^{TL} - 1),$$

for $k = 1, 2, \cdots, N$. ♯

5. Semi-Lagrangian methods

In [14], the following semi-Lagrangian Euler method was studied for its stability and convergence,

$$\begin{cases} w_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0, 1, \cdots, M\}, \\ x_D^k = x_A^{k+1} - \tau\, w_D^k, \quad w_D^k \text{ Calculated from linear interpolation of } w_A^k, \\ w_A^{k+1} = w_D^k + \tau f(t_k, x_D^k, w_D^k), \quad \text{for each } k = 0, 1, \cdots, N-1. \end{cases} \tag{5}$$

Then Theorem 2 applies with $\phi(t, x, w, \tau, h) = f(t, x, w)$.

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Now we study the modified semi-Lagrangian Euler method for the initial-value boundary problem (1),

$$\begin{cases} w_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0, 1, \cdots, M\}, \\ x_D^k = x_A^{k+1} - \tau\, w_D^k, \quad w_D^k \text{ Calculated from linear interpolation of } w_A^k, \\ k_1 = f(t_k, x_D^k, w_D^k), \\ k_2 = f(t_{k+1}, x_A^{k+1}, w_D^k + \tau f(t_k, x_D^k, w_D^k)), \\ w_A^{k+1} = w_D^k + \frac{1}{2}\tau(k_1 + k_2), \end{cases} \tag{6}$$

for all of arrival points A and $k = 0, 1, 2, \cdots, N-1$.

Theorem 3. *If the function $f(t, x, u)$ in (1) is continuous and satisfies a Lipschitz condition in all variables with Lipschitz constant L on the set*

$$D = \{(t, x, w) |\, 0 \le t \le T, a \le x \le b, -\infty < u < \infty\}.$$

Then

(i) *The difference method (6) is stable;*

(ii) *The difference method (6) is convergent.*

The stability and convergence of (6) could be proved following the similar idea with Lipschitz continuity in Theorem 2. Actually, it follows for all semi-Lagrangian Runge-Kutta methods.

Theorem 4. *Suppose $h = O(\tau)$ and the function $f(t, x, u)$ in (1) is continuous and differentiable. If ε_k is the local truncation error of (6), then*

$$||\varepsilon_k(\tau)||_{L_\infty} = O(\tau^2).$$

Proof: Let $u(t, x)$ be the unique solution with up to order three continuous bounded partial derivatives on $[0, T] \times [a, b]$, so that for each $k = 0, 1, \ldots, N-1$,

$$f(t_{k+1}, x_A^{k+1}, w_D^k + \tau f(t_k, x_D^k, w_D^k)) = f(t_k, x_D^k, w_D^k) + \tau \frac{\partial f(t_k, x_D^k, w_D^k)}{\partial t}$$

$$+\tau w_D^k \frac{\partial f(t_k, x_D^k, w_D^k)}{\partial x} + \tau f(t_k, x_D^k, w_D^k) \frac{\partial f(t_k, x_D^k, w_D^k)}{\partial u} + O(\tau^2),$$

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and the last equation of (6) can be rewritten as,

$$w_A^{k+1} = w_D^k + \tau f(t_k, x_D^k, w_D^k) + \frac{1}{2}\tau^2[\frac{\partial f(t_k, x_D^k, w_D^k)}{\partial t} + w_D^k \frac{\partial f(t_k, x_D^k, w_D^k)}{\partial x} + f(t_k, x_D^k, w_D^k)\frac{\partial f(t_k, x_D^k, w_D^k)}{\partial u}] + O(\tau^3).$$

Since $u(t, x)$ satisfies the equation (1), then

$$\begin{aligned} u(t_{k+1}, x_A^{k+1}) &= u(t_k, x_D^k) + \tau \frac{du(t_k, x_D^k)}{dt} + \frac{1}{2}\tau^2 \frac{d^2u(t_k, x_D^k)}{dt^2} + O(\tau^3), \\ &= u(t_k, x_D^k) + \tau f(t_k, x_D^k, u_D^k) + \frac{1}{2}\tau^2[\frac{\partial f(t_k, x_D^k, u_D^k)}{\partial t} \\ &+ u_D^k \frac{\partial f(t_k, x_D^k, u_D^k)}{\partial x} + f(t_k, x_D^k, u_D^k)\frac{\partial f(t_k, x_D^k, u_D^k)}{\partial u}] + O(\tau^3). \end{aligned}$$

From the definition of local truncation error, we have

$$\begin{aligned} \varepsilon_{k+1}(\tau) = &\frac{u(t_{k+1}, x_A^{k+1}) - u(t_k, x_D^k)}{\tau} \\ &-\frac{1}{2}[f(t_k, x_D^k, u(t_k, x_D^k)) + f(t_{k+1}, x_A^{k+1}, u(t_{k+1}, x_A^{k+1}))] \end{aligned}$$

So it is easy to know that the local truncation error at the $(k+1)$th step for the method (6) is

$$||\varepsilon_{k+1}(\tau)||_{L_\infty} = O(\tau^2).$$

♯

Remark: Only the first order method is needed for the calculation of departure points and interpolation at the departure points. General speaking, for an overall nth order method the order of computing the departure points and the interpolation should be at least $(n-1)$.

Theorem 5. *If the function $f(t, x, u)$ in (1) is continuous and satisfies a Lipschitz condition in the variable u with Lipschitz constant L on the set*

$$D = \{(t, x, w)|\, 0 \le t \le T, a \le x \le b, -\infty < u < \infty\}.$$

Then, there exists a constant C such that $||\varepsilon_k(\tau)||_{L_\infty} \le C\tau^2$, and

$$\begin{aligned} \max_A |u(t_k, x_A^k) - w_A^k| &\le e^{TL} \max_A |u_0(x_A) - w_A^0| + \frac{C\tau^2}{L}(e^{TL} - 1), \\ &\le \frac{C\tau^2}{L}(e^{TL} - 1), \ \textit{for } k = 1, 2, \cdots, N. \end{aligned}$$

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Proof: It is obvious from Theorem 2 and 4. ♯

Remark 1: From Theorem 5, if we choose $w_A^0 = u_0(x_A)$, then w_A^k converges to $u(t,x)$ in order two.

Remark 2: For higher order semi-Lagrangian methods, a higher order method for the calculation of departure points is needed. For example, the semi-Lagrangian method of order three is constructed as follows

$$\begin{cases} w_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0, 1, \cdots, M\} \\ x_D^k = x_A^{k+1} - \tau\, w_D^k - \frac{1}{2}\tau^2 f(t_k, x_D^k, w_D^k) \\ \qquad\qquad w_D^k \text{ Calculated from interpolation of } w_A^k \\ k_1 = f(t_k, x_D^k, w_D^k) \\ k_2 = f(t_k + \frac{1}{2}\tau, x_D + \frac{1}{2}\tau w_D^k, w_D^k + \frac{1}{2}\tau k_1) \\ k_3 = f(t_{k+1}, x_A^{k+1}, w_D^k - \tau k_1 + 2\tau k_2) \\ w_A^{k+1} = w_D^k + \frac{1}{6}\tau(k_1 + 4k_2 + k_3). \end{cases} \tag{7}$$

The numerical analysis is much more complicated because of the need of higher order methods for computing of departure points and interpolation at the departure points.

6. Implicit semi-Lagrangian methods

The explicit semi-Lagrangian methods are straightforward to calculate the approximations, whereas the implicit semi-Lagrangian methods require that a simultaneous system of equations be solved at each time step. These are nonlinear unless f is a linear function of u. They can be solved by an iterative method which is guaranteed to converge for small enough step sizes τ and h.

In general it is hard or difficult to justify the additional work of the implicit methods by the increased accuracy attainable, but implicit methods are still used for partial differential equations.

The implicit semi-Lagrangian Euler method is given by

$$\begin{cases} w_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0, 1, \cdots, M\}, \\ x_D^k = x_A^{k+1} - \tau\, w_D^k, \quad w_D^k \text{ Calculated from linear interpolation of } w_A^k, \\ w_A^{k+1} = w_D^k + \tau f(t_{k+1}, x_A^{k+1}, w_A^{k+1}), \quad \text{for each } k = 0, 1, \cdots, N-1, \end{cases} \tag{8}$$

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and the implicit modified semi-Lagrangian Euler method reads,

$$\begin{cases} w_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0, 1, \cdots, M\}, \\ x_D^k = x_A^{k+1} - \tau\, w_D^k, \quad w_D^k \text{ Calculated from linear interpolation of } w_A^k, \\ k_1 = f(t_k, x_D^k, w_D^k), \\ k_2 = f(t_{k+1}, x_A^{k+1}, w_A^{k+1}), \\ w_A^{k+1} = w_D^k + \frac{1}{2}\tau(k_1 + k_2). \end{cases} \tag{9}$$

In order to study the stability and convergence of implicit semi-Lagrangian methods, let us consider the test equation as $f(t, x, u) = \lambda u$. Then, the initial boundary problem becomes

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \lambda u, \quad 0 < t < T, a \le x \le b, \\ u(0, x) = u_0(x), \quad a \le x \le b, \end{cases} \tag{10}$$

and the corresponding implicit modified semi-Lagrangian Euler method is

$$\begin{cases} w_A^0 = u_0(x_A^0), \quad x_A^0 \in \{x_i : i = 0, 1, \cdots, M\}, \\ x_D^k = x_A^{k+1} - \tau\, w_D^k, \quad w_D^k \text{ Calculated from linear interpolation of } w_A^k, \\ w_A^{k+1} = w_D^k + \frac{\lambda\tau}{2} w_D^k + \frac{\lambda\tau}{2} w_A^{k+1}. \end{cases} \tag{11}$$

Suppose that $\{u_A^k\}_{k=1}^N$ and $\{v_A^k\}_{k=1}^N$ satisfy the difference equation (11) with the initial-value as u_0 and v_0 respectively. Let

$$E_A^k = u_A^k - v_A^k \text{ for the point } (t_k, x_A^k)$$

and

$$E^k = \max |E_A^k| \text{ on the regular grid points (arrival points) at } t = t_k.$$

Then,

$$(1 - \frac{\lambda\tau}{2})(u_A^{k+1} - v_A^{k+1}) = (1 + \frac{\lambda\tau}{2})(u_D^k - v_D^k)$$

and

$$|u_A^{k+1} - v_A^{k+1}| = \left| \frac{1 + \frac{\lambda\tau}{2}}{1 - \frac{\lambda\tau}{2}} \right| |u_D^k - v_D^k|.$$

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i.e.

$$|E_A^{k+1}| \leq \left| \frac{1+\frac{\lambda\tau}{2}}{1-\frac{\lambda\tau}{2}} \right| |E_D^k|.$$

Using Lemma 4.5, we have

$$|E_A^{k+1}| \leq \left| \frac{1+\frac{\lambda\tau}{2}}{1-\frac{\lambda\tau}{2}} \right| E^k$$

and

$$E^{k+1} \leq \left| \frac{1+\frac{\lambda\tau}{2}}{1-\frac{\lambda\tau}{2}} \right| E^k.$$

Let $\lambda\tau/2 = a + bi$, where a and b are real numbers, we have

$$E^{k+1} \leq \left(\frac{1+a^2+b^2+2a}{1+a^2+b^2-2a} \right)^{1/2} E^k.$$

Then, if $a < 0$, the difference method (11) is stable, or it is A-stable as Dahlquist [6] defined.

It is similar to the proof of Theorem 4.5, we can prove that the difference method (11) is convergent.

7. Numerical Experiments and Results

The main computational effort in applying the semi-Lagrangian methods is the evaluation of f and the computation of the departure points. In the second-order method, the local truncation error is $O(\tau^2)$, and the cost is two functional evaluations per step and the computation of the departure points. The semi-Lagrangian method of order three requires four evaluations per step and the computation of the departure points, and the local truncation error is $O(\tau^3)$. The computation of the departure points depends on the interpolation method. In the practice, we can choose the order of interpolation method lower than the order of semi-Lagrangian method, or we can use one interpolation method for the variable x to get an interpolation function and use this function to evaluate the values at the departure points. In the following example, the linear interpolation is used for the calculation of the solution on departure points.

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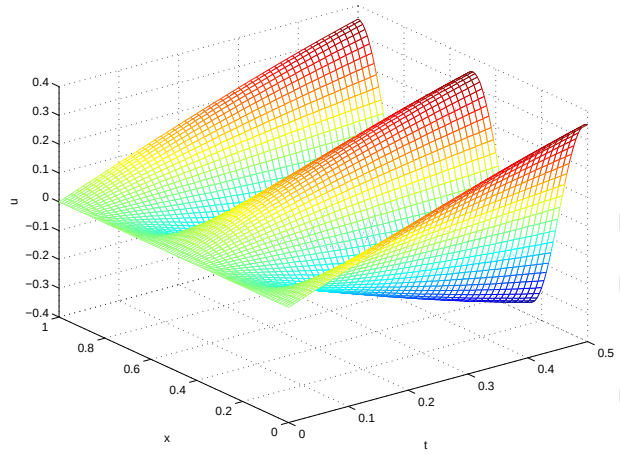


Figure 2: Approximated Solution of the initial-value problem.

Example: Consider the problem

$$\begin{cases} u_t + u\,u_x = \frac{1}{4}\pi\cos(\frac{1}{2}\pi t)\cos(4\pi x) - 2\pi u\,\sin(\frac{1}{2}\pi t)\sin(4\pi x), \\ u(0,x) = 0, \qquad 0 < t \leq 1,\ 0 < x < 1, \end{cases} \tag{12}$$

with periodic boundary condition for x.

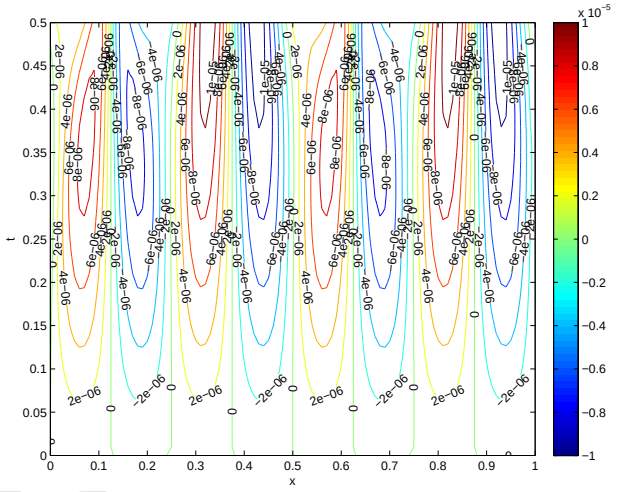


Figure 3: Contour lines of the difference between the solution and its approximation.

The exact solution for this problem is $u(t,x) = \frac{1}{2}\sin(\frac{1}{2}\pi t)\cos(4\pi x)$. The Figure 2 shows the graph of the approximate solution of $u(t,x)$ by using the

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semi-Lagrangian method of order three with $\tau = 0.01$ and $h = 0.01$. In order to focus on the truncation error from the proposed semi-Lagrangian methods, the periodic boundary condition is supplied. For other boundary conditions, the treatment near boundary is more complicated when the trajectories of the departure points hit the boundary.

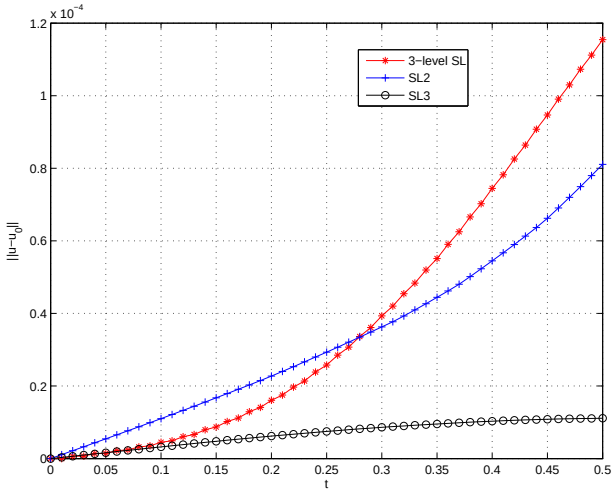


Figure 4: Truncation errors: $-*-$ three-level Semi-Lagrangian method (SL1); $-+-$ second-order method (SL2); $-\circ-$ third-order method (SL3).

In the Figure 3, it shows the contour lines of the difference between the exact solution and the approximation from the semi-Lagrangian method of order three at the common grid points. The truncation error is increasing as t goes from 0 to 0.5. The contour lines are matching the changes of the function. The larger errors occur when the function has the bigger changes. The maximum error reached at about $t = 0.5$ for the certain points of x as we expected.

Three-level Semi-Lagrangian method, the modified semi-Lagrangian Euler method, and the semi-Lagrangian method of order three are compared with $\tau = 0.01$ and $h = 0.01$ in the Figure 4. The maximum truncation error presented at the each step of t for the three-level semi-Lagrangian method, the modified semi-Lagrangian Euler method, and the semi-Lagrangian method of order three. From the graph, the semi-Lagrangian method of order three is better than the rest of two methods, even though we have not proven the convergence and its order theoretically.

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τ	0.1	0.05	0.02	0.01
3-Level SL ($\frac{\|\|u-u_0\|\|_\infty}{\tau^2}$)	1.9623	1.9664	1.9676	1.9688
SL2 ($\frac{\|\|u-u_0\|\|_\infty}{\tau^2}$)	1.8390	1.9595	2.0305	2.0456
SL3 ($\frac{\|\|u-u_0\|\|_\infty}{\tau^3}$)	2.4580	2.3869	2.5196	2.4771

Table 1: Numerical orders of the three-level Semi-Lagrangian method, SL2, and SL3 as applied to the problem (12)

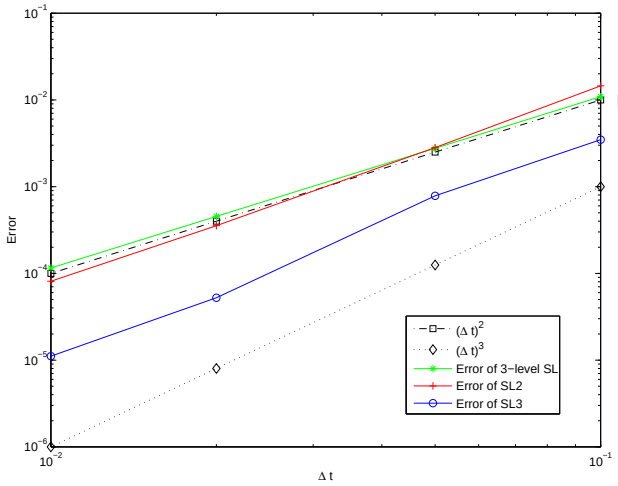


Figure 5: Scaling of the error of Semi-Lagrangian methods of order two and three to the problem (12)

In the table 1 and the Figure 5, we studied the numerical orders of convergence with $h = \tau$ and the comparison with three-level Semi-Lagrangian method [23]. It is clear that the modified semi-Lagrangian Euler method has a second order just like three-level Semi-Lagrangian method. However, the three-level Semi-Lagrangian method needs two interpolations on the previous two levels. The Semi-Lagrangian method of order three is not exactly order three, but is close to 2.5. The main difficulty for Semi-Lagrangian method of order three is the calculation of the departure points.

8. Conclusions

In this article, one-step semi-Lagrangian method was proposed and studied for the first-order time-dependent nonlinear partial differential equations in 1D. The modified semi-Lagrangian Euler method was proved to be consis-

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tence and stable. It was a second order method if $\tau = O(h)$, where τ and h were the step sizes of time and spatial variables respectively. When τ goes to zero, the numerical solution converges to the solution of the corresponding initial-boundary problem.

The theorem in the section 4 and the examples in the section 7 are shown that the semi-Lagrangian methods work for time-dependent partial differential equations. They are very convenient for implementation. The only disadvantage is that at each time step, a linear interpolation is needed to find the solutions on the departure points. However, the interpolation software can be found everywhere.

In the example, only one spatial variable was used. However, we believe that semi-Lagrangian methods should work for more spatial variables, but more complicated. We will report the similar results for the systems with more spatial variables. Also more terms involving higher order derivative only in spatial variables will be added to the right side function later.

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