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A bidimensional stability result for aluminium electrolytic cells

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Abstract

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This paper shows that the instabilities which are observed in cells cannot be explained by supposing that the mechanical and electromagnetic fields are independent of the space variable x_3 corresponding to the length of the cell. For a model which is infinitely long in the x_3-direction, we prove in particular that the kinetic energy due to the x_1- and x_2-components of the velocities tends to zero as the time tends to infinity.

Keywords: Magnetohydrodynamics, stability, aluminium production.

1. Introduction

To our knowledge, the specialists of the production of aluminium possess today only a rather limited understanding of the phenomena of instability which may arise in electrolytic cells. Among the different approaches which have been considered, let us mention the following ones: study of dispersion relations in Fourier analysis of linear models [5,6], shallow water models [8], perturbation methods [3], numerical simulation of the dynamic MHD equations.

In [7], the authors have analysed steady motions in a cell which is infinitely long in the x_3-horizontal direction by supposing that the fields are x_3-invariant. In this work, we consider the same situation for evolutionary flows; our main stability result is the following: as the time tends to infinity, the normal components of the velocity to the x_3-direction tend to zero. We notice as a main feature that the proof of this property requires no linearization of the MHD equations and no restriction concerning the variations of the interface between the electrolyte and the molten metal.

In Section 2, we define the problem by recalling the MHD equations and fixing the hypotheses. The main assumption, i.e., the invariance of the fields with respect to the x_3-variable, allows a decoupling of the equations into two sets. We study in Section 3 one of these sets by an energy method and deduce the stability results. Section 4 is devoted to the conclusions.

2. Description of the model

The geometry is defined schematically by Fig. 1. The cell is infinitely long in the ϵ_3-direction. The electrolyte and the liquid aluminium are contained in cylinders with generators parallel to ϵ_3 and sections $\Omega_1(t)$, $\Omega_2(t)$, respectively, in the x_1-, x_2-plane. t denotes the time. $\Gamma(t)$ is the section of the interface which separates the two fluids. We set $\Omega = \Omega_1(t) \cup \Omega_2(t)$ where $\Omega_1(t)$ and $\Omega_2(t)$ are considered as closed subsets of $\mathbb{R}^2$; Ω, which represents the section of the fluid part of the cell, is independent of t.

The fields with which we shall deal are listed as follows:

velocity,	$\boldsymbol{U} = \boldsymbol{u} + U\epsilon_3$;	(2.1)
magnetic induction,	$\boldsymbol{B} = \boldsymbol{b} + B\epsilon_3$;	(2.2)
electric field,	$\boldsymbol{E} = \boldsymbol{e} + E\epsilon_3$;	(2.3)
electric current,	$\boldsymbol{J} = \boldsymbol{j} + J\epsilon_3$;	(2.4)
pressure,	p;	(2.5)
density,	ρ;	(2.6)
viscosity,	η;	(2.7)
electric conductivity,	σ.	(2.8)

Let ω, $\omega_1(t)$, $\omega_2(t)$, $\gamma(t)$ be the cylinders with generators parallel to ϵ_3 and sections Ω, $\Omega_1(t)$, $\Omega_2(t)$, $\Gamma(t)$, respectively, with the x_1-, x_2-plane.

For $t \geqslant 0$, $\boldsymbol{U}$, $\boldsymbol{E}$, p, η and ρ are defined in ω, whereas $\boldsymbol{B}$, $\boldsymbol{J}$ and σ are considered in $\mathbb{R}^3$. Note that, for our purpose, it is not necessary to treat $\boldsymbol{E}$ on the entire of $\mathbb{R}^3$. In (2.1)–(2.4) the vector fields are decomposed in two parts; the first one contains the ϵ_1-, ϵ_2-components; the second one is the ϵ_3-component.

The two basic hypotheses are given as follows.

H1. $\boldsymbol{U}$, $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{J}$ and the gradient of p depend on x_1, x_2 and t but are independent of x_3.

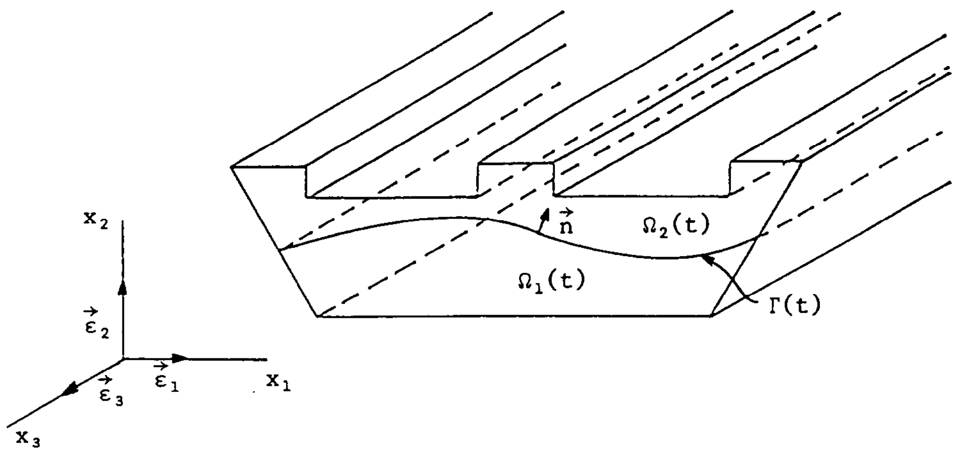

Fig. 1.

H2. The different fields and their partial derivatives may present discontinuities across $\gamma(t)$. $\gamma(t)$ and the restrictions of the fields to the subdomains $\omega_1(t)$, $\omega_2(t)$ or $\mathbb{R}^3 - \omega$ are sufficiently regular, with respect to the space and time variables, to insure the validity of the mathematical developments of Section 3.

In order to precise H1 and H2, we add the following hypothesis.

H3. $\boldsymbol{U}$ is continuous in ω. $\boldsymbol{B}$ is continuous in $\mathbb{R}^3$. The tangential component of $\boldsymbol{E}$ to $\gamma(t)$ is continuous across $\gamma(t)$. The jump of p across $\gamma(t)$ is independent of x_3. ρ, η and σ take positive constant values ρ_k, η_k, σ_k in $\omega_k(t)$, $k = 1, 2$, with $\rho_1 \neq \rho_2$.

We consider for $t \geqslant 0$ the following standard set of equations:

$$\rho\, \partial_t \boldsymbol{U} + \rho(\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = \eta\, \Delta \boldsymbol{U} + \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla P + \nabla \phi, \tag{2.9}$$

$$\operatorname{div} \boldsymbol{U} = 0, \tag{2.10}$$

$$\mathbf{curl}\, \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{2.11}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{2.12}$$

$$\partial_t \boldsymbol{B} + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0}, \tag{2.13}$$

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{U} \wedge \boldsymbol{B}). \tag{2.14}$$

Because of possible discontinuities across $\gamma(t)$, the Navier–Stokes equation (2.9) is only valid separately in $\omega_1(t)$ and $\omega_2(t)$, respectively. μ_0 is the magnetic permeability of the vacuum. P is the total pressure whereas $\nabla\phi$ represents the gravity forces; denoting by g the constant of gravitation, we have

$$P = p + \frac{1}{2\mu_0}|\boldsymbol{B}|^2, \qquad \phi = -g\rho_k x_2, \quad \text{in } \omega_k(t),\ k = 1, 2. \tag{2.15}$$

Equations (2.10), (2.13) and (2.14) are valid in $\omega_1(t)$ and $\omega_2(t)$ separately; recall that we do not consider $\boldsymbol{E}$ in $\mathbb{R}^3$ but only in ω. Equations (2.11), (2.12) are valid in $\omega_1(t)$, $\omega_2(t)$ or $\mathbb{R}^3 - \omega$, separately.

We complete the set of equations (2.9)–(2.14) by introducing boundary conditions. $\boldsymbol{n} = (n_1, n_2, 0)$ will denote the unit normal to $\gamma(t)$ pointing into $\omega_2(t)$. For any discontinuous field s, $\{s\}_\Gamma$ denotes the jump of s across $\gamma(t)$, i.e., the value of s in $\omega_2(t)$ minus its value in $\omega_1(t)$. For the fluids, we assume the no-slip condition on $\partial\omega$ and the force equilibrium on $\gamma(t)$, i.e.,

$$\boldsymbol{U} = \mathbf{0}, \quad \text{on } \partial\omega, \tag{2.16}$$

$$\sum_{j=1}^{3} \{\eta(\partial_i U_j + \partial_j U_i) - p\delta_{ij}\}_\Gamma n_j = 0, \quad i = 1, 2, 3, \tag{2.17}$$

where U_1, U_2, U_3 are the components of $\boldsymbol{U}$ and ∂_i is the partial derivative with respect to x_i. We remark that (2.17) takes into account the fact that the surface tension effects are negligible in cells.

We consider the set of relations formed by the $\boldsymbol{\epsilon}_1$-, $\boldsymbol{\epsilon}_2$-components of (2.9), (2.13), (2.16) and (2.17), the $\boldsymbol{\epsilon}_3$-components of (2.11), (2.14) and the scalar equations (2.10), (2.12). We ignore the other equations contained in (2.9)–(2.14), (2.16), (2.17) since we shall not use them in the following. By using hypothesis H1 and notations (2.1)–(2.4), we readily obtain:

$$\rho\,\partial_t \boldsymbol{u} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \eta\,\Delta\boldsymbol{u} + \frac{1}{\mu_0}(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \nabla P + \nabla\phi, \tag{2.18}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{2.19}$$

$$\operatorname{rot}\boldsymbol{b} = \mu_0 J, \tag{2.20}$$

$$\operatorname{div}\boldsymbol{b} = 0, \tag{2.21}$$

$$\partial_t\boldsymbol{b} + \mathbf{rot}\ E = 0, \tag{2.22}$$

$$J = \sigma(E + u_1 b_2 - u_2 b_1), \tag{2.23}$$

$$\boldsymbol{u} = 0, \quad \text{on } \partial\Omega, \tag{2.24}$$

$$\sum_{j=1}^{2}\{\eta\tau_{ij} - p\delta_{ij}\}_\Gamma n_j = 0, \quad \text{on } \Gamma(t),\ i = 1, 2, \tag{2.25}$$

$$\text{where } \tau_{ij} = \partial_i u_j + \partial_j u_i, \quad i, j = 1, 2. \tag{2.26}$$

We recall that in (2.18), P and ϕ are defined by (2.15). Because of H1 and of H3 (jump of p), we consider (2.18)–(2.25) in the sections $\Omega_1(t)$, $\Omega_2(t)$, $\mathbb{R}^2 - \Omega$, $\partial\Omega$, $\Gamma(t)$ of $\omega_1(t)$, $\omega_2(t)$, $\mathbb{R}^3 - \omega$, $\partial\omega$, $\gamma(t)$. In (2.20) and (2.22), we have used the notations $\operatorname{rot}\boldsymbol{b} = \partial_1 b_2 - \partial_2 b_1$, $\mathbf{rot}\ E = (\partial_2 E, -\partial_1 E)$.

The continuity of $\boldsymbol{b}$ (hypothesis H3), (2.20) and (2.21) imply the existence of a scalar potential A defined in $\mathbb{R}^2$ and depending on t such that

$$\boldsymbol{b} = \mathbf{rot}\ A, \quad -\Delta A = \mu_0 J. \tag{2.27}$$

To complete the modelling, we introduce a final hypothesis.

H4. (a) J vanishes outside a bounded region of $\mathbb{R}^2$.

(b) $\int_\Omega J = 0.$

(c) $$A(x, t) = -\frac{\mu_0}{2\pi}\int_{\mathbb{R}^2}\ln|x - \xi|\,J(\xi, t)\,\mathrm{d}\xi, \quad x\in\mathbb{R}^2,\ t \geqslant 0. \tag{2.28}$$

(d) J is time independent outside Ω.

Parts (a) and (b) of H4 are natural from an engineering point of view. Part (c) expresses nothing but the Biot–Savart law. Note that with (2.28) the second relation of (2.27) is satisfied. Part (d) is justified by the fact that in practice the bus bars have small sections with respect to the distance to the cells.

3. Study of the stability by an energy method

For initial conditions at $t = 0$ which need not be specified, we consider evolutionary fields which satisfy (2.18)–(2.25) and hypotheses H1–H4. We shall study their asymptotic behaviour as t tends to infinity with the help of an energy relation that we first establish.

We recall from a result of fluid mechanics (see, for example, [1]), which is valid for any scalar field $\psi(x, t)$, $x=(x_1, x_2)$ and for u satisfying (2.19):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_k(t)}\psi(x, t)\,\mathrm{d}x=\int_{\Omega_k(t)}\{\partial_t\psi(x, t)+\nabla\psi(x, t)\cdot u(x, t)\}\,\mathrm{d}x, \quad k=1, 2. \tag{3.1}$$

It is worth noticing that all "free boundary" features of our problem are implicitly contained in (3.1), in particular the immiscibility of the two fluids. We integrate on Ω the relation obtained by multiplying (2.18) by u:

$$\int_\Omega\rho(\partial_t u+(u\cdot\nabla)u)\cdot u=\int_\Omega(\eta\,\Delta u-\nabla P)\cdot u+\int_\Omega\nabla\phi\cdot u+\frac{1}{\mu_0}\int_\Omega((b\cdot\nabla)b)\cdot u. \tag{3.2}$$

Due to the presence of discontinuities on $\Gamma(t)$, each integral in (3.2) must be considered as the sum of integrals on $\Omega_1(t)$ and $\Omega_2(t)$ which have to be treated separately.

Applying (3.1) to the right-hand side member of the identity

$$(\partial_t u+(u\cdot\nabla)u)\cdot u=\tfrac{1}{2}\{\partial_t|u|^2+\nabla|u|^2\cdot u\},$$

we obtain for the left-hand side member of (3.2)

$$\int_\Omega\rho(\partial_t u+(u\cdot\nabla)u)\cdot u=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho|u|^2. \tag{3.3}$$

From now on, we shall use the Einstein sum convention. With the notation (2.26), we have by (2.19) that $u\cdot\Delta u=u_i\,\partial_j\tau_{ij}$. Taking into account the boundary condition (2.24), we obtain by integrations by parts

$$\int_\Omega\eta u\cdot\Delta u=-\int_\Omega\eta\tau_{ij}\,\partial_j u_i-\int_{\Gamma(t)}\{\eta u_i\tau_{ij}n_j\}_\Gamma, \tag{3.4}$$

where we recall that $n=(n_1, n_2)$ is the unit normal to $\Gamma(t)$ pointing into $\Omega_2(t)$ and where the jump $\{\cdot\}_\Gamma$ is the value in $\Omega_2(t)$ minus the value in $\Omega_1(t)$. From the definition (2.15) of P and the continuity of B (see H3), we get in the same way

$$-\int_\Omega\nabla P\cdot u=\int_{\Gamma(t)}\{p\}_\Gamma\delta_{ij}u_i n_j. \tag{3.5}$$

Combining (3.4), (3.5) and (2.25), we have:

$$\int_\Omega(\eta\,\Delta u-\nabla P)\cdot u=-\tfrac{1}{2}\int_\Omega\eta\tau_{ij}\tau_{ij}. \tag{3.6}$$

By (2.15), ϕ is independent of t so that we deduce from (3.1)

$$\int_\Omega\nabla\phi\cdot u=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\phi; \tag{3.7}$$

although ϕ is time independent, the integral of ϕ on Ω depends on t since the densities of the two fluids are different.

The treatment of the electromagnetic part of (3.2) is somewhat less conventional. We introduce the following fields:

$$A_0(x, t)=-\frac{\mu_0}{2\pi}\int_\Omega\ln|x-\xi|\,J(\xi, t)\,\mathrm{d}\xi, \quad x\in\mathbb{R}^2, \qquad b_0=\mathbf{rot}\,A_0=(\partial_2A_0, -\partial_1A_0). \tag{3.8}$$

By H4(c) A_0 is the contribution to the magnetic potential due to the ϵ_3-component of the current running in the cell Ω, whereas $\boldsymbol{b}_0$ is the magnetic field generated by the same current. As a consequence of H4(d), $A - A_0$ and $\boldsymbol{b} - \boldsymbol{b}_0$ are independent of t. As a consequence of H4(b), we have

$$A_0(\boldsymbol{x}, t) = \mathrm{O}\left(\frac{1}{|\boldsymbol{x}|}\right), \quad \boldsymbol{b}_0 = \mathrm{O}\left(\frac{1}{|\boldsymbol{x}|^2}\right), \qquad \text{as } |\boldsymbol{x}| \to \infty. \tag{3.9}$$

Consider the following identity:

$$((\boldsymbol{b}\cdot\nabla)\boldsymbol{b})\cdot\boldsymbol{u} = \operatorname{rot} \boldsymbol{b}(b_1u_2 - b_2u_1) + \tfrac{1}{2}\nabla|\boldsymbol{b}|^2\cdot\boldsymbol{u}. \tag{3.10}$$

We integrate (3.10) on Ω; recalling that $\boldsymbol{b}$ and $\boldsymbol{u}$ are continuous, we obtain by (2.19) and (2.24) that the integral of the second term of the right-hand side member vanishes. From (2.20) and (2.23) we deduce

$$\frac{1}{\mu_0}\int_\Omega((\boldsymbol{b}\cdot\nabla)\boldsymbol{b})\cdot\boldsymbol{u} = \int_\Omega\left(JE - \frac{1}{\sigma}J^2\right). \tag{3.11}$$

Comparing (2.22) and (2.27), the existence follows of a constant $c(t)$ depending only on t such that

$$E(\boldsymbol{x}, t) = -\partial_t A(\boldsymbol{x}, t) + c(t), \quad \boldsymbol{x}\in\Omega,\ t\geqslant 0. \tag{3.12}$$

By H4(d), $\partial_t A = \partial_t A_0$, where A_0 is defined by (3.8), so that by H4(b) we get

$$\int_\Omega JE = \int_\Omega J(-\partial_t A_0 + c) = \int_\Omega J(x, t)\int_\Omega \frac{\mu_0}{2\pi}\ln|\boldsymbol{x}-\boldsymbol{\xi}|\partial_t J(\boldsymbol{\xi}, t)\,\mathrm{d}\xi\,\mathrm{d}x,$$

$$\int_\Omega JE = \frac{\mu_0}{4\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_\Omega \ln|\boldsymbol{x}-\boldsymbol{\xi}|J(x, t)J(\boldsymbol{\xi}, t)\,\mathrm{d}\xi\,\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} J_0A_0, \tag{3.13}$$

where J_0 is the function defined in $\mathbb{R}^2$ which is equal to J in Ω and vanishes outside Ω. Equation (3.8) implies that

$$-\Delta A_0 = \mu_0 J_0, \quad \text{in } \mathbb{R}^2. \tag{3.14}$$

We multiply (3.14) by A_0, integrate over $\mathbb{R}^2$ and use Green's formula. Taking into account (3.9) and recalling the definition (3.8) of $\boldsymbol{b}_0$, we obtain

$$\int_{\mathbb{R}^2} J_0A_0 = \frac{1}{\mu_0}\int_{\mathbb{R}^2}|\nabla A_0|^2 = \frac{1}{\mu_0}\int_{\mathbb{R}^2}|\boldsymbol{b}_0|^2. \tag{3.15}$$

Combining (3.11), (3.13) and (3.15), we get

$$\frac{1}{\mu_0}\int_\Omega((\boldsymbol{b}\cdot\nabla)\boldsymbol{b})\cdot\boldsymbol{u} = \frac{-1}{2\mu_0}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\boldsymbol{b}_0|^2 - \int_\Omega\frac{1}{\sigma}J^2. \tag{3.16}$$

From (3.2), (3.3), (3.6), (3.7) and (3.16), we obtain the desired energy relation that we state explicitly in the following lemma.

Lemma 3.1. *Equations* (2.18)–(2.25) *and hypotheses* H1–H4 *imply the relation*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega \rho|\boldsymbol{u}|^2 + \frac{1}{\mu_0}\int_{\mathbb{R}^2}|\boldsymbol{b}_0|^2\right\} + \tfrac{1}{2}\int_\Omega \eta\tau_{ij}\tau_{ij} + \int_\Omega\frac{1}{\sigma}J^2 - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\phi = 0.$$

We need the following estimate.

Lemma 3.2. *There exists a positive constant* λ, *independent of* t, *such that*

$$\tfrac{1}{2}\int_\Omega \eta\tau_{ij}\tau_{ij} + \int_\Omega \frac{1}{\sigma}J^2 \geqslant \tfrac{1}{2}\lambda\left\{\int_\Omega \rho\,|\boldsymbol{u}|^2 + \frac{1}{\mu_0}\int_{\mathbb{R}^2}|\boldsymbol{b}_0|^2\right\}.$$

Proof. Since $\boldsymbol{u}$ is continuous and vanishes on $\partial\Omega$, we can apply the first Korn inequality (see, for example, [4]) and obtain the existence of a positive constant c_1 such that

$$\int_\Omega \tau_{ij}\tau_{ij} \geqslant c_1\int_\Omega |\boldsymbol{u}|^2. \tag{3.17}$$

We apply the Schwarz inequality to (3.15) by recalling that J_0 equals J in Ω and vanishes outside Ω; we get

$$\frac{1}{\mu_0}\int_{\mathbb{R}^2}|\boldsymbol{b}_0|^2 \leqslant \left(\int_\Omega J^2\right)^{1/2}\left(\int_\Omega A_0^2\right)^{1/2}. \tag{3.18}$$

By the Schwarz inequality again, we deduce from (3.8) the existence of a constant c_2, independent of t and of $\boldsymbol{x}\in\Omega$ such that

$$A_0^2(\boldsymbol{x}, t) \leqslant c_2\int_\Omega J^2(\xi, t)\,\mathrm{d}\xi, \quad \boldsymbol{x}\in\Omega,\ t\geqslant 0. \tag{3.19}$$

Integrating (3.19) on Ω and replacing in (3.18) proves the existence of a constant c_3, independent of t, such that

$$\frac{1}{\mu_0}\int_{\mathbb{R}^2}|\boldsymbol{b}_0|^2 \leqslant c_3\int_\Omega J^2. \tag{3.20}$$

Lemma 3.2 follows easily from (3.17) and (3.20). □

We are now in a position to state and prove the main result of this work.

Proposition 3.3. *Under hypotheses* H1–H4, *we have*

$$\lim_{t\to\infty}\tfrac{1}{2}\left\{\int_\Omega \rho\,|\boldsymbol{u}(\boldsymbol{x}, t)|^2\,\mathrm{d}x + \frac{1}{\mu_0}\int_{\mathbb{R}^2}|\boldsymbol{b}_0(\boldsymbol{x}, t)|^2\,\mathrm{d}x\right\} = 0.$$

Proof. With the notations

$$X(t) = \tfrac{1}{2}\left\{\int_\Omega \rho\,|\boldsymbol{u}|^2 + \frac{1}{\mu_0}\int_{\mathbb{R}^2}|\boldsymbol{b}_0(\boldsymbol{x}, t)|^2\right\}, \qquad Y(t) = \int_\Omega \phi, \tag{3.21}$$

we have by Lemmata 3.1 and 3.2:

$$X'(t) + \lambda X(t) - Y'(t) \leqslant 0, \quad t\geqslant 0. \tag{3.22}$$

By (2.15), there exists a constant d_1 such that

$$|Y(t)| \leqslant d_1, \quad t\geqslant 0. \tag{3.23}$$

Integrating (3.22), we get

$$X(t) + \lambda\int_0^t X(\tau)\,\mathrm{d}\tau \leqslant X(0) + Y(t) - Y(0), \quad t\geqslant 0. \tag{3.24}$$

Noticing that X is nonnegative and that λ is positive, (3.23), (3.24) show the existence of a constant d_2 such that

$$X(t) \leqslant d_2, \quad \int_0^t X(\tau)\,\mathrm{d}\tau \leqslant d_2, \qquad t \geqslant 0. \tag{3.25}$$

Applying the Schwarz inequality to (3.7), we obtain by (3.25) and (2.15) the existence of a constant d_3 such that

$$|Y'(t)| \leqslant d_3, \quad t \geqslant 0. \tag{3.26}$$

By (3.22), (3.25) and (3.26), there exists a constant M such that

$$X'(t) \leqslant M, \quad t \geqslant 0. \tag{3.27}$$

Suppose now that Proposition 3.3 is false, i.e., $X(t)$ does not converge to zero as t tends to infinity. Since $X(t) \geqslant 0$, there exist $a > 0$ and a sequence $0 \leqslant t_0 < t_1 < t_2 < \cdots$ such that

$$X(t_i) \geqslant a, \quad t_i \geqslant t_{i-1} + \frac{a}{M}, \; i = 1, 2, \ldots . \tag{3.28}$$

By (3.27), (3.28), setting $\alpha = a/M$, we have for $i \geqslant 1$:

$$X(t_i - \xi) = X(t_i) - \int_{t_i - \xi}^{t_i} X'(\tau)\,\mathrm{d}\tau \geqslant a - M\xi, \quad 0 \leqslant \xi \leqslant \alpha,$$

$$\int_{t_i - \alpha}^{t_i} X(\tau)\,\mathrm{d}\tau \geqslant \frac{a^2}{2M}, \qquad \int_0^{t_n} X(\tau)\,\mathrm{d}\tau \geqslant \frac{na^2}{2M}, \quad n = 1, 2 \ldots;$$

letting n tend to infinity in this last relation, we obtain a contradiction with (3.25). This ends the proof of Proposition 3.3. □

Remark 3.4. Although this is not a consequence of Proposition 3.3, assume that for some T, $X(t)$, defined by (3.21), vanishes for $t \geqslant T$, i.e., $\boldsymbol{u}(\boldsymbol{x}, t) = 0$, $\boldsymbol{x} \in \Omega$, $t \geqslant T$, and $\boldsymbol{b}_0(\boldsymbol{x}, t) = 0$, $\boldsymbol{x} \in \mathbb{R}^2$, $t \geqslant T$. By (3.8), (3.14), it follows that $J = 0$ in Ω and by H4(d) that $\boldsymbol{b}$ is time independent for $t \geqslant T$. Furthermore by (2.25), p is continuous along $\Gamma(t)$; by (2.15), since $\boldsymbol{B}$ is continuous, P is also continuous along $\Gamma(t)$; it easily follows (see [2]) from (2.18) that the jump of ϕ along $\Gamma(t)$ is constant; since by H3, $\rho_1 \neq \rho_2$, we conclude by (2.15) that $\Gamma(t)$ is a horizontal line, i.e., parallel to ϵ_1, which by mass conservation is time independent for $t \geqslant T$.

4. Conclusions

The purpose of this work was to show that it is not possible to explain the instabilities which can be observed in cells by supposing that the fields are invariant for translations with respect to the x_3-direction. In other words, the phenomena of instabilities must be studied in genuine three-dimensional situations.

Our main result which is contained in Proposition 3.3 states that the kinetic energy relative to the ϵ_1- and ϵ_2-components of the velocity and the magnetic energy generated by the ϵ_3-component of the current running into the cell tend to zero as t tends to infinity.

Even by supposing that the cell is " very long", we are conscious that our model contains some unrealistic aspects. For example, the x_3-independence of the currents implies that the bus bars are not connected to the cell. On another side the model is fully nonlinear; in particular, no restriction is imposed on the interface $\Gamma(t)$. The geometry of the section Ω of the cell is quite general. Note furthermore that little use has been made of the specific form (2.15) of ϕ; in fact Proposition 3.3 is valid by supposing that, in each subdomain, ϕ is a function only depending on x_1 and x_2. Finally we notice that, by assuming that Ω is not a disk, we can replace the no-slip boundary condition by the slip one without modifying the results.

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