



Rational approximations to $1/\sqrt{1-s^2}$, one-way wave equations and absorbing boundary conditions

R.A. Renaut^{*,1}, J.S. Parent

Department of Mathematics, Arizona State University, Tempe, AZ 85287-1804, USA

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Abstract

One-way wave equations (OWWEs), derived from rational approximations, $C(s)$ to $1/\sqrt{1-s^2}$, are considered. Absorbing boundary conditions obtained from these OWWEs are easily implemented, producing systems of differential equations at the boundary which are different from those produced by rational approximations, $r(s)$ to $\sqrt{1-s^2}$. Although these systems are different, a particular choice of difference approximation for the system yields numerical methods such that stability properties of both approaches are equivalent. In particular, for $C(s) = 1/r(s)$, the two systems possess equivalent stability properties. In other cases, numerical results are presented which demonstrate that, where $C(s)$ and $r(s)$ are not derived via interpolation, the $C(s)$ OWWEs can provide better absorption.

Keywords: Absorbing boundary conditions; One-way wave equations; Well-posedness; Stability; Discrete approximations.

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1. Introduction

In the modelling of seismic events the solution of the acoustic wave equation on an unbounded domain is often desired. One approach numerically is to cut the unbounded domain to a computationally reasonable size. The boundaries introduced by this procedure allow for nonphysical reflections back into the domain that should be minimized in order to generate physically meaningful solutions. A technique that has proven successful is the application of absorbing boundary conditions which have been derived from approximations to a one-way wave equation (OWWE) at the boundary [2, 5, 7, 13, 9]. In this paper we reconsider this approach and show that the methods presented by Lindman [7] and Clayton and Engquist [1] have equivalent numerical properties. In particular,

^{*} Corresponding author: e-mail: renaut@math.la.asu.edu.

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although the generated OWEs are different in each case their numerical implementations lead to methods with identical stability and accuracy properties.

Both approaches result from a consideration of the dispersion relation,

$$\omega^2 = c^2(\xi^2 + \eta^2), \quad (1)$$

of the constant-coefficient acoustic wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}). \quad (2)$$

The solutions of this equation are plane waves, $u = u(x, y, t) = e^{i(\omega t + \xi x + \eta y)}$, where ω is the frequency and ξ and η are the spatial wave numbers.

A wave with wave numbers ξ, η travels at the velocity $(c_x, c_y) = c(-\xi/\omega, -\eta/\omega) = c(-\cos \theta, -\sin \theta)$, where θ is the angle measured counterclockwise from the negative x -axis. Thus, waves with $|\theta| < 90^\circ$ travel to the left, $c_x < 0$, and waves with $|\theta| > 90^\circ$ travel to the right, $c_x > 0$. At an artificial boundary $x = 0$, waves should travel to the left and thus $c_x \leq 0$ is desirable. Equivalently, ξ and ω should have the same sign and these waves should satisfy

$$\xi = \frac{\omega}{c} \sqrt{1 - \left(\frac{\eta c}{\omega}\right)^2}, \quad (3)$$

which is the dispersion relation of the ideal one-way wave equation.

To translate (3) to a practical OWE, approximants to the square root are required. There are two formulations which can be considered. One finds an approximation to $\sqrt{1 - s^2}$, and the other to $1/\sqrt{1 - s^2}$, where $s = \eta c/\omega$. Denote the former approximation by $r(s) = P_m(s)/Q_n(s)$, and the latter by $C(s) = A_M(s)/B_N(s)$, where the subscript denotes the degree of the polynomial, as a polynomial in s , in the numerator or denominator. Then $r(s)$ and $C(s)$ lead to the partial differential equations (PDEs):

$$\sum_{j=0}^n q_j c^{j+1} u_{y^j t^K - jx} = \sum_{j=0}^m p_j c^j u_{y^j t^K - j+1}, \quad (4)$$

and

$$\sum_{j=0}^M a_j c^{j+1} u_{y^j t^K - jx} = \sum_{j=0}^N b_j c^j u_{y^j t^K - j+1}, \quad (5)$$

respectively. Here $K = \max\{n, m - 1\}$ in the first case and $K = \max\{M, N - 1\}$ in the second case.

The only significant difference between (4) and (5) is the manner in which the approximations $C(s)$ and $r(s)$ are derived. Several possible approaches were presented by Halpern and Trefethen [5]. Collino [3] also gave explicit formulae for the coefficients of the appropriate Padé approximants. Furthermore, for the approximations in [5], Renaut [9] investigated the stability of their associated absorbing boundary conditions. Here these techniques are extended to the absorbing boundary conditions associated with $C(s)$.

In Section 2 we present well-posed approximations to $1/\sqrt{1 - s^2}$. For these approximations some representative graphs of the reflection coefficient are also given. The boundary conditions can be

implemented using an approach of Lindman [7]. These discretizations are described in Section 3. A stability analysis demonstrates that the schemes are then completely analogous to those presented in [9]. Furthermore, this also demonstrates that restrictions on timestep are determined by the interior scheme rather than the boundary scheme. Hence, these OWEs can be used without compromising the grid size or time step one would normally use to stably solve the wave equation on a periodic domain.

In the final section some numerical results are presented which demonstrate that the methods presented here offer a viable alternative to those considered by the earlier authors. Furthermore, their implementation, which replaces a time derivative of u in the $r(s)$ form by a spatial derivative in the $C(s)$ form, has many potential uses in other situations. In particular this formulation is advantageous in both spectral methods [11] and in the solution of the elastic wave equation where evanescent waves are present [10].

2. The rational approximations

The material in this section generalizes the results of Halpern and Trefethen, [5] and [15], for the approximations $C(s)$.

From [15], a necessary condition for the initial value and initial boundary value problems, derived from $r(s)$ to be well-posed is $n \leq m \leq n + 2$. The equivalent result for $C(s)$ is that these problems are well-posed only if $M \leq N \leq M + 2$ [12].

To explain this similarity consider the rational approximation $r(s)$ which interpolates $\sqrt{1-s^2}$ at $m+n+1+\chi_{mn}$ points in $(-1, 1)$, where χ_{mn} is 0 if $m+n$ is odd and 1 if $m+n$ is even. At these points, s_k , $k = 1 : m+n+1+\chi_{mn}$,

$$r(s_k) = \frac{P_m(s_k)}{Q_n(s_k)} = \sqrt{1-s_k^2}, \quad (6)$$

$s_k \in (-1, 1)$. Therefore, $P_m(s_k) - \sqrt{1-s_k^2}Q_n(s_k) = 0$ and $C(s) = Q_n(s)/P_m(s)$ interpolates $1/\sqrt{1-s^2}$ at the same set of points s_k . Hence, we set $n=M$ and $m=N$ and $C(s) = 1/r(s)$ in [15]. The proofs then easily follow those in [15] but with adjustment to deal with interpolation to $1/\sqrt{1-s^2}$ instead of $\sqrt{1-s^2}$.

Because of the necessary condition for well-posedness it is only appropriate to consider approximations for which M and N are related by the given inequalities. In particular only approximants with $M=N$ and $M=N-2$, and, to minimize complexity, for which $K=1, \dots, 5$, where $K = \frac{1}{2}(M+N+2)$, are considered. Note that K defines M and N uniquely with $M=N$ and $M=N-2$, M and N even.

Coefficients of Chebyshev–Padé, least squares and minmax, (L^∞) , approximants on the subintervals $[-\sin^{-1}\theta, \sin^{-1}\theta]$, $\theta = 80^\circ, 85^\circ, 86^\circ \dots 89^\circ$ and α are given in Tables 1, 2 and 3, respectively. Here α depends on K according to $K = \{1 \dots 5\}$, $\alpha = \{10^\circ, 20^\circ, 45^\circ, 60^\circ, 75^\circ\}$. Descriptions of how to find these approximants are given in [5]. The families of approximants considered in [5], and not presented here, are based on interpolation and therefore for the same set of interpolatory points, $C(s) = 1/r(s)$. Further details are given in [12].

Table 1
Coefficients of the Chebyshev–Padé approximants

K	β	80°	85°	86°	87°	88°	89°	α
1	a_0	2.03848	2.43864	2.58637	2.76586	3.02115	3.46050	1.00767
2	a_0	0.48112	0.08474	−0.04959	−0.22583	−0.47815	−0.91516	0.99927
	b_2	0.47393	−55.79394	106.30977	26.49547	14.63672	9.56263	−0.06419
3	a_0	1.32773	1.65246	1.77514	1.94110	2.18486	2.61561	1.00326
	a_2	−1.19999	−5.16158	−6.54735	−8.38780	−11.05178	−15.70504	−0.86983
	b_2	1.85143	1.60506	1.56529	1.52708	1.48922	1.44890	−1.06339
4	a_0	0.82333	0.52407	0.40860	0.24999	0.01323	−0.41167	0.99269
	a_2	7.45035	14.08429	16.55323	19.89781	24.82813	33.57660	2.02578
	b_2	2.46480	3.02862	3.13570	3.23964	3.34183	3.44815	1.45623
	b_4	4.95133	7.19729	7.60742	7.99960	8.37882	8.76605	−0.22041
5	a_0	1.13706	1.36827	1.47078	1.61716	1.84295	2.25913	1.04358
	a_2	−1.81236	−11.10674	−15.05572	−20.64083	−29.19590	−44.88222	1.04865
	a_4	25.46457	62.25432	77.68815	99.44904	132.69472	193.52486	12.60893
	b_2	2.46773	2.49239	2.49013	2.48615	2.48056	2.47315	2.34586
	b_4	3.60843	3.64457	3.63389	3.61892	3.59989	3.57601	3.32524

Table 2
Least squares

K	β	80°	85°	86°	87°	88°	89°	α
1	a_0	1.41756	1.48887	1.50452	1.52019	1.53640	1.55304	1.00509
2	a_0	0.86418	0.75446	0.71406	0.65691	0.57793	0.45593	0.99984
	b_2	−0.82236	−0.89311	−0.90935	−0.92646	−0.94554	−0.96767	−0.51344
3	a_0	1.02821	1.06754	1.08367	1.10642	1.14133	1.25123	1.00023
	a_2	−0.60803	−0.75479	−0.80052	−0.85792	−0.93537	−1.08381	−0.31269
	b_2	−0.94705	−0.97487	−0.98042	−0.98585	−0.99109	−0.99479	−0.80574
4	a_0	0.99421	0.98073	0.97415	0.96387	0.94597	0.90630	0.99984
	a_2	−0.83578	−0.89302	−0.90248	−0.90899	−0.90865	−0.88740	−0.65756
	b_2	−1.39277	−1.52215	−1.55972	−1.60450	−1.66258	−1.74509	−1.16170
	b_4	0.40737	0.52721	0.56325	0.60717	0.66369	0.74543	0.22175
5	a_0	1.00123	1.00494	1.00822	1.01151	1.01894	1.03713	1.00036
	a_2	−1.22244	−1.36173	−1.41463	−1.46789	−1.54335	−1.66853	−1.12733
	a_4	0.27801	0.38189	0.42513	0.46866	0.53149	0.63410	0.21700
	b_2	−1.70429	−1.80970	−1.84022	−1.87094	−1.90420	−1.94293	−1.62020
	b_4	0.70841	0.81085	0.84099	0.87133	0.90436	0.94296	0.62846

The potential effectiveness of these rational approximants as absorbing boundary conditions is measured by the reflection coefficient

$$R(s) = \left| \frac{C(s) - 1/\sqrt{1-s^2}}{C(s) + 1/\sqrt{1-s^2}} \right|, \quad s \in [-1, 1]. \quad (7)$$

Table 3
Minmax: L^∞

K	β	80°	85°	86°	87°	88°	89°	α
1	a_0	3.37940	6.23685	7.66779	14.27001	14.8270	29.1352	1.00771
2	a_0	0.63699	0.35247	0.30080	0.25616	0.21751	0.18379	0.99751
	b_2	-0.90936	-0.97449	-0.98212	-0.98798	-0.99259	-0.99639	-0.51567
3	a_0	1.08467	1.28443	1.39632	1.57029	1.93346	2.44106	1.00038
	a_2	-0.73939	-1.07228	-1.21848	-1.42844	-1.83130	-2.37569	-0.32105
	b_2	-0.96429	-0.98781	-0.99164	-0.99484	-0.99742	-0.99907	-0.81162
4	a_0	0.98261	0.92330	0.88769	0.82246	0.68898	0.43125	0.99969
	a_2	-0.95629	-0.87530	-0.85387	-0.80219	-0.68041	-0.42973	-0.67321
	b_2	-1.46777	-1.65463	-1.71054	-1.78002	-1.86655	-1.95470	-1.17969
	b_4	0.46909	0.65679	0.71180	0.78060	0.86672	0.95471	0.23454
5	a_0	1.00360	1.02108	1.03383	1.05796	1.11608	1.30553	1.00090
	a_2	-1.28887	-1.50432	-1.58464	-1.69872	-1.90234	-2.39876	-1.17024
	a_4	0.32762	0.49645	0.55947	0.64565	0.78830	1.09365	0.24492
	b_2	-1.75257	-1.87364	-1.90159	-1.92999	-1.95899	-1.98467	-1.65735
	b_4	0.75532	0.87413	0.90186	0.93010	0.95902	0.98467	0.66395

Representative reflection coefficients for the approximations given in the tables, for $K = 3$ and 5 are plotted in Figs. 1–3. $K = 1$ gives very low order and we shall see in Section 3 that $K = 2$ and 4 require the same number of equations as $K = 3$ and 5, respectively, and hence it is preferable to choose $K = 3$ or $K = 5$ rather than $K = 2$ or $K = 4$. In those cases where no curve is drawn, the reflection coefficient becomes greater than one and thus these approximations are of no use practically.

Observe, also, that the Chebyshev–Padé approximants do not interpolate $(1/\sqrt{1-s^2}) M + N + 1 + \chi_{MN}$ times on $[-1, 1]$. This can be confirmed by evaluating the roots of $C(s) - 1/\sqrt{1-s^2}$ numerically. The well-posedness results prove that interpolation at $M + N + 1 + \chi_{MN}$ points on $[-1, 1]$ is necessary for well-posedness. But implementations with the Chebyshev–Padé indicate that the interpolation condition may also be sufficient.

It is clear in every case that as we take β nearer to 90° the approximation becomes less effective for small angles θ . Also, the L^2 minimization procedure produces approximations for which the reflection coefficient is smaller on a larger interval than for the equivalent L^∞ approximations.

3. Implementation of one-way wave equations

Renaut and Peterson [13] demonstrated that it is not sufficient to derive the PDE associated with a given approximation in order to have a satisfactory absorbing boundary condition. Rather, it is also important to determine a difference approximation of the OWWE which, in conjunction with the difference approximation of the wave equation on the domain, away from the boundaries, leads to stable solutions. For example, when $m=2$ and $n=0$, the difference approximation to (4) proposed by Clayton and Engquist [2] has a stability interval $\mu \in [0, \gamma]$, $\mu = c\Delta t/\Delta x$, where Δt is the timestep,

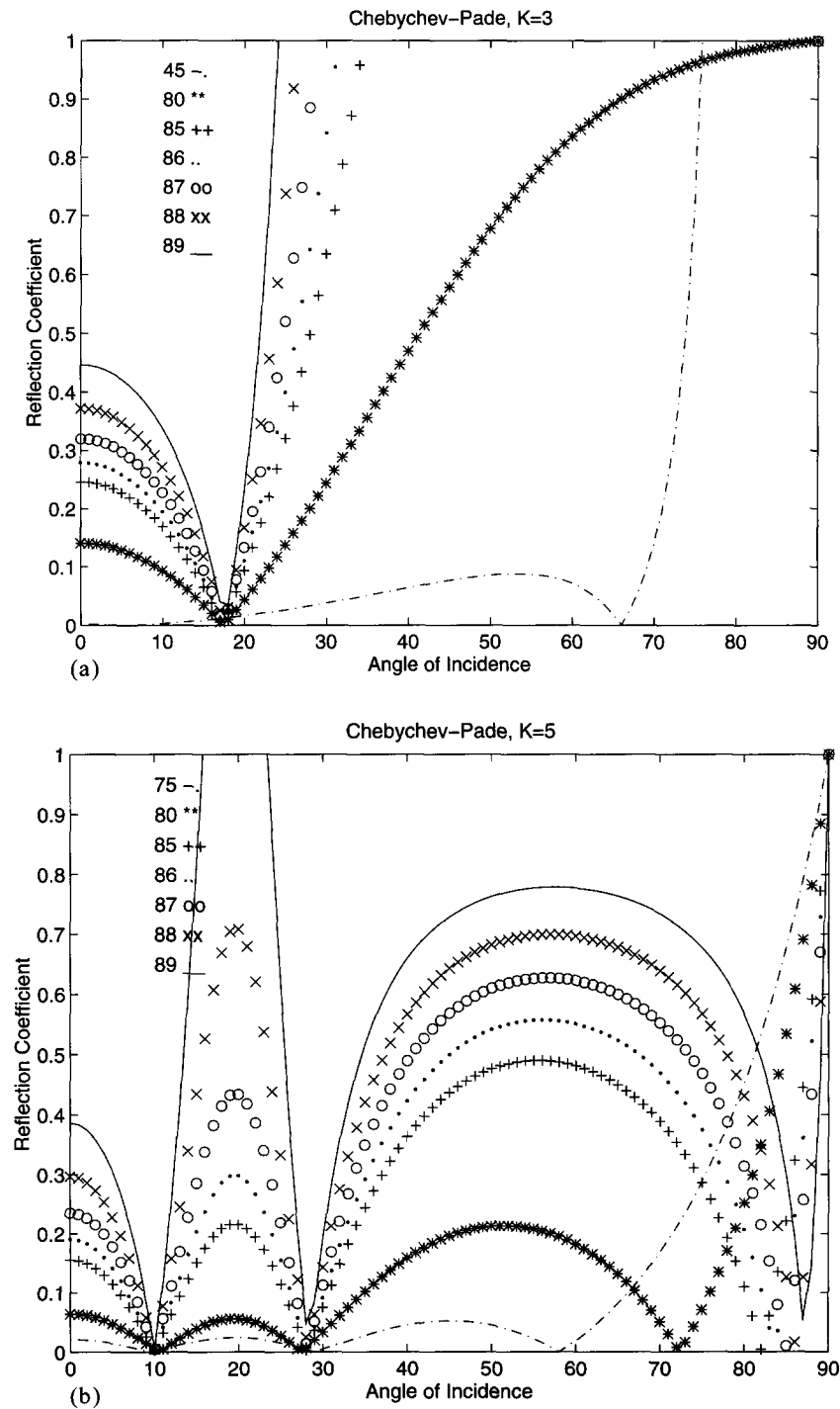


Fig. 1. Reflection coefficient for Chebyshev-Pade: (a) $K = 3$; and (b) $K = 5$.

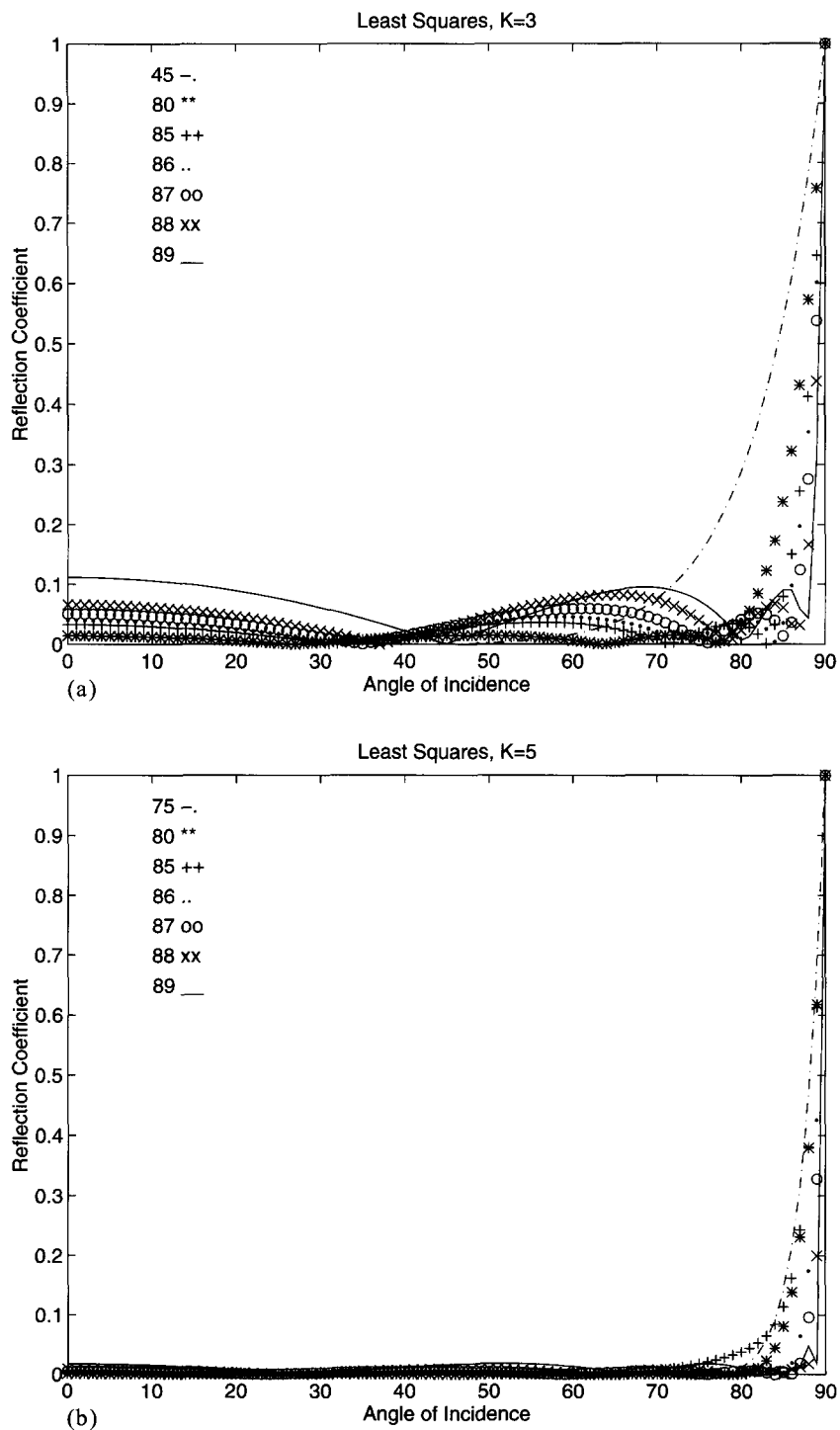


Fig. 2. Reflection coefficient for least squares: (a) $K = 3$; and (b) $K = 5$.

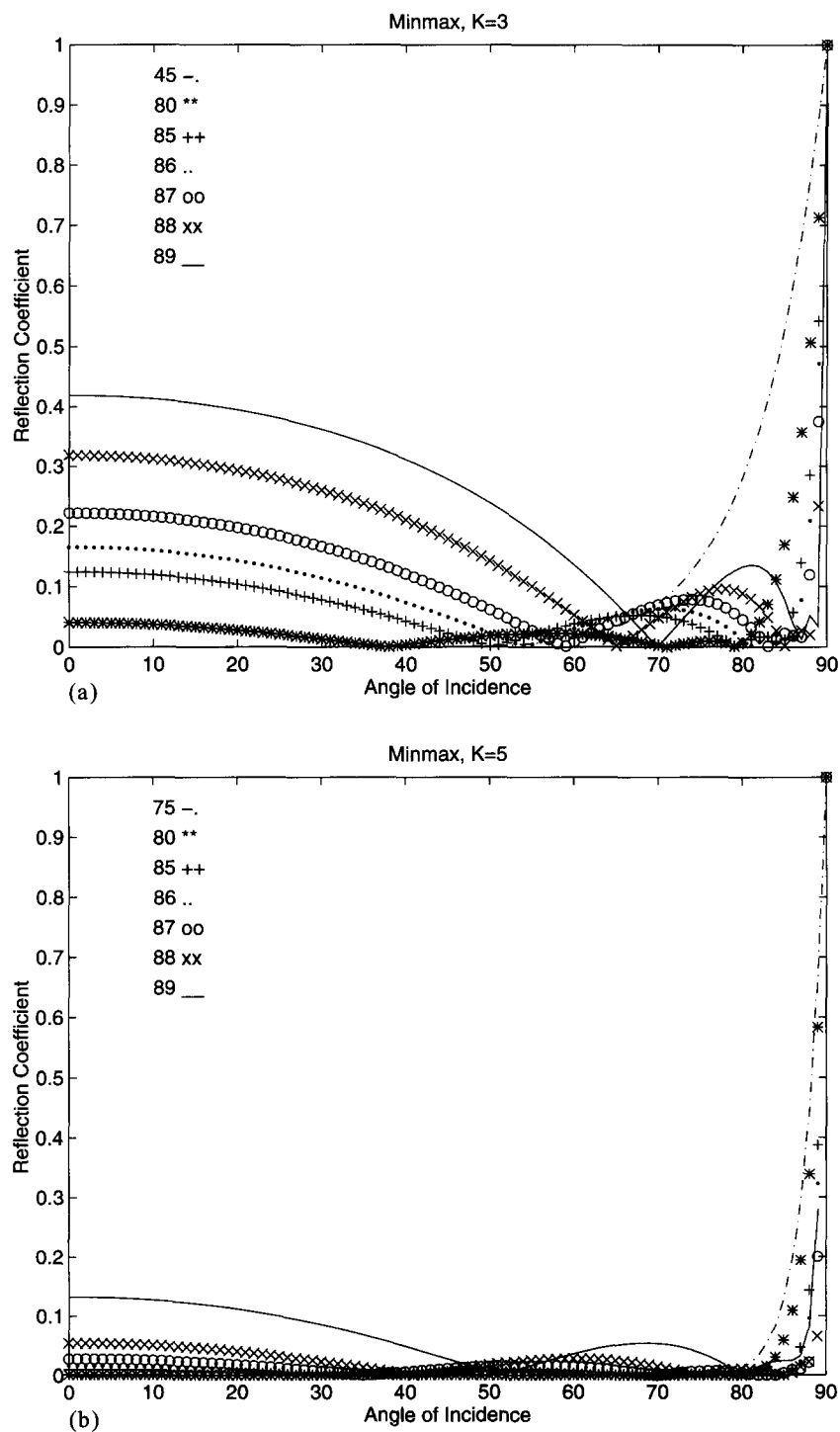


Fig. 3. Reflection coefficient for minimax scheme: (a) $K = 3$; and (b) $K = 5$.

Δx the grid spacing and c the wave speed. The number γ takes on values from 0.5 to $1/\sqrt{2}$ for the approximations $r(s)$ given in [5]. But note that the von Neumann stability condition applied to the usual second-order central differencing of the wave equation is $\mu \in [0, 1/\sqrt{2}]$. Therefore, $\gamma < 1/\sqrt{2}$ represents a reduced stability interval. Renaut [9] has shown how this problem is overcome if the approach of Lindman [7] is adopted and the PDE is replaced by a system of equations at the boundary, each of the same type.

We write $C(s)$ in the form

$$C(s) = a_0 \left(1 + \sum_{j=1}^L \frac{\alpha_j s^2}{1 - \beta_j s^2} \right), \quad (8)$$

where we do not insist $a_0 = 1$ and $2L = M = N$ if none of the coefficients are zero. Inserting (8) into the dispersion relation (3) as follows,

$$\xi a_0 \left(1 + \sum_{j=1}^L \frac{\alpha_j s^2}{1 - \beta_j s^2} \right) = \frac{\omega}{c}, \quad (9)$$

leads to the form

$$i(\omega - a_0 \xi c) = i a_0 \left(\xi c \sum_{j=1}^L \frac{\alpha_j s^2}{1 - \beta_j s^2} \right). \quad (10)$$

Associating $i\omega$, $i\xi$ and $i\eta$ with partial differentiation by t, x and y , respectively, gives the system of equations:

$$\frac{\partial u}{\partial t} - a_0 c \frac{\partial u}{\partial x} = a_0 c \sum_{j=1}^L h_j, \quad (11a)$$

$$\frac{\partial^2 h_j}{\partial t^2} - \beta_j c^2 \frac{\partial^2 h_j}{\partial y^2} = \alpha_j c^2 \frac{\partial^2}{\partial y^2} \frac{\partial u}{\partial x}, \quad j = 1, 2, \dots, L. \quad (11b)$$

Therefore, instead of solving (5) at the boundary, Lindman solves the system described by (11). The first of these equations corresponds to the absorbing boundary condition which annihilates plane waves at angle of incidence $\cos^{-1}(1/a_0)$ modified by the correction functions h_j . These correction functions are the solutions of the one-dimensional wave equations along the boundary.

Note that in (11) for a given N both $M = N$ and $M = N - 2$ lead to the same value of L , i.e., $2L = N$. Hence, the number of equations to be solved at the boundary is the same regardless of whether we choose $M = N$ or $M = N - 2$. Since the choice $M = N$ corresponds to the higher degree of interpolation, and thus smaller reflection coefficient for a wider range of angles, it is preferable to choose $M = N$. Note that when $M = N$, we have $K = \frac{1}{2}(M + N + 2) = N + 1$ and, therefore, the odd values of K , $K = 2L + 1$, (since N is even) can be used to give higher-order approximations for the same amount of work as for $K = 2L$.

The same technique can be applied to the approximation $r(s)$ to give the system of equations:

$$c \frac{\partial u}{\partial x} - p_0 \frac{\partial u}{\partial t} = p_0 \sum_{j=1}^L g_j, \quad (12a)$$

$$\frac{\partial^2 g_j}{\partial t^2} - \delta_j c^2 \frac{\partial^2 g_j}{\partial y^2} = c^2 \gamma_j \frac{\partial^3 u}{\partial y^2 \partial t}, \quad j = 1, 2, \dots, L, \quad (12b)$$

where $r(s) = p_0(1 + \sum_{j=1}^L [\gamma_j s^2 / (1 - \delta_j s^2)])$. The system described by (12) is similar to that described by (11) but differs on the right-hand side of (12b) where $\partial/\partial t$ is replaced by $\partial/\partial x$. Again the choice $m = n$, and hence K odd, leads to higher-order approximation for the same amount of work.

Consider now a simple example in which $m = 2$, $r(s) = p_0 + p_2 s^2$. In this case, $\gamma_1 = p_2/p_0$ and $\delta_1 = 0$. Thus, (12) reduces to

$$c \frac{\partial u}{\partial x} - p_0 \frac{\partial u}{\partial t} = p_0 g_1, \quad (13a)$$

$$\frac{\partial^2 g_1}{\partial t^2} = c^2 \frac{p_2}{p_0} \frac{\partial^3 u}{\partial y^2 \partial t}. \quad (13b)$$

The direct inverse of $r(s)$ is

$$C(s) = \frac{1}{p_0} \left(\frac{1}{1 + p_2/p_0 s^2} \right) \quad (14)$$

and $a_0 = 1/p_0$, $\alpha_1 = p_2/p_0$ and $\beta_1 = -p_2/p_0$. System (11) is then given by

$$\frac{\partial u}{\partial t} - \frac{1}{p_0} c \frac{\partial u}{\partial x} = \frac{1}{p_0} c h_1, \quad (15a)$$

$$\frac{\partial^2 h_1}{\partial t^2} + \frac{p_2}{p_0} c^2 \frac{\partial^2 h_1}{\partial y^2} = \frac{p_2}{p_0} c^2 \frac{\partial^2}{\partial y^2} \frac{\partial u}{\partial x}. \quad (15b)$$

Here (15a) is equivalent to (13a) if $g_1 = -(c/p_0)h_1$. But then (15b) becomes

$$\frac{\partial^2 h_1}{\partial t^2} + c^2 \frac{p_2}{p_0} \frac{\partial^2 h_1}{\partial y^2} = -c^2 \frac{p_2}{p_0} \frac{\partial^2}{\partial y^2} \frac{\partial u}{\partial x}, \quad (16)$$

which is not equivalent to (13b). Therefore, systems (11) and (12) are, indeed, different, even when $r(s)$ and $C(s)$ are inverses of one another.

Difference approximations to these systems are obtained, again using the ideas of Lindman [7], in which second-order derivatives are approximated by a second-order central difference and the first-order derivatives by forward differences averaged over two levels. Suppose that D_+^q , D_-^q and D_0^q are standard forward, backward and central difference operators and u_{nk}^j is an approximation to $u(j\Delta x, k\Delta y, n\Delta t)$, Δx , Δy are gridsizes in x and y directions and Δt is the timestep. Similarly,

h_{nk}^j and g_{nk}^j are approximations to $h(j\Delta x, k\Delta y, n\Delta t)$ and $g(j\Delta x, k\Delta y, n\Delta t)$, respectively. Then (11) is replaced by

$$D_+^t \left(\frac{u_{nk}^0 + u_{nk}^1}{2} \right) - a_0 c D_+^x \left(\frac{u_{n+1k}^0 + u_{nk}^0}{2} \right) = a_0 c \sum_{j=1}^L (h_j)_{nk}^0, \quad (17a)$$

$$(D_+^t D_-^t - \beta_i c^2 D_+^y D_-^y) (h_j)_{n-1k}^0 = \frac{\alpha_i c^2}{2} (D_+^y D_-^y) D_+^x (u_{nk}^0 + u_{n-1k}^0). \quad (17b)$$

Applying the same technique to (12) gives

$$c D_+^x \left(\frac{u_{nk}^0 + u_{n+1k}^0}{2} \right) - p_0 D_+^t \left(\frac{u_{nk}^0 + u_{nk}^1}{2} \right) = p_0 \sum_{j=1}^L (g_j)_{nk}^0, \quad (18a)$$

$$(D_+^t D_-^t - \delta_j c^2 D_+^y D_-^y) (g_j)_{n-1k}^0 = \frac{\gamma_j c^2}{2} (D_+^y D_-^y) D_+^t (u_{n-1k}^0 + u_{n-1k}^1). \quad (18b)$$

4. Stability

Stability analysis of (11) is identical to that for (12) and uses the theory of Schur transforms [6]. In particular, either of (11) or (12) can be expressed in an operator form,

$$D(E, Y, Z^{-1}) u_{n+1k}^0 = 0, \quad (19)$$

where D is a polynomial in the three variables E, Y and Z which denote the forward shift operators with respect to x, y and t , respectively. The GKS (Gustafsson, Kreiss and Sundstöm) [4] stability condition is then

$$D(\kappa, y, z^{-1}) \neq 0, \quad \text{whenever } |z| \geq 1, \quad |\kappa| \leq 1, \quad (20)$$

see [9]. To verify this condition the Schur analysis is applied to D as a polynomial in $W = z^{-1}$ and is used to derive conditions under which the zeros of D satisfy $|W| > 1$, and thus $|z| < 1$, whenever $|\kappa| < 1$.

Fortuitously, it is not necessary to repeat the complete analysis used in [9] because here the operators D are polynomials in κ and W of the same degree as those for (12), just with different polynomial coefficients. Hence after appropriate identification of these coefficients, the next theorem follows immediately from the stability theorem given in [9].

Theorem 1. *The difference approximation at the boundary is stable in conjunction with the five point difference stencil for (2) only if*

- (a) For $K = 2$:
 - (i) $a_0 > 0$,
 - (ii) $b_2 < 0$ and $\mu^2 < 1/|b_2|$,
- (b) For $K = 3$:
 - (i) $a_0 > 0$,
 - (ii) $b_2 a_0 - a_2 < 0$ and $\mu^2 < a_0/(a_2 - a_0 b_2)$,

- (iii) if $a_2 > 0$ then $b_2 < 0$ and $\mu^2 > 1/|b_2|$,
- (iv) if $a_2 < 0$ then, if $b_2 < 0$, $\mu^2 < 1/|b_2|$.
- (c) For $K = 4$:
 - (i) $a_0 > 0$,
 - (ii) $a_2 - b_2 a_0 > 0$ and $\mu^2 < a_0/(a_2 - b_2 a_0)$,
 - (iii) if $a_2^2 + b_4 a_0^2 - b_2 a_2 a_0 < 0$ then, if $b_4 a_0 + a_2 b_2 - b_2^2 a_0 < 0$,

$$\mu^2 < \frac{a_2 - b_2 a_0}{|b_4 a_0 + a_2 b_2 - b_2^2 a_0|},$$
 - (iv) if $a_2^2 + b_4 a_0^2 - b_2 a_2 a_0 > 0$, then $b_4 a_0 + a_2 b_2 - b_2^2 a_0 < 0$ and

$$\mu^2 > \frac{a_2 - b_2 a_0}{|b_4 a_0 + a_2 b_2 - b_2^2 a_0|},$$
 - (v) if $b_4 > 0$ then, if $b_2^2 - 4b_4 > 0$, $b_2 < 0$ and

$$\mu^2 < \left[\frac{-b_2 - \sqrt{b_2^2 - 4b_4}}{2b_4} \right],$$
 - (vi) if $b_4 < 0$ then $b_2^2 - 4b_4 > 0$ and

$$\mu^2 > \left[\frac{-b_2 - \sqrt{b_2^2 - 4b_4}}{2b_4} \right].$$

Observation. In each case above the operator D is equivalent to an operator B in [9] where the only difference is in the definition of the constants which occur in the proof. For example, for the $K = 2$ case, we have $a = 1/a_0 \mu$ and $b = b_2 \mu/a_0$ as compared to $a = p_0/\mu$ and $b = p_2 \mu$ in [13]. Thus, if we put $a_0 = 1/p_0$ and $b_2 = p_2/p_0$ in (a) we get $p_0 > 0$ and $\mu^2 < \sqrt{p_0/|p_2|}$ which are precisely the conditions of Theorem 5.1 in [13] for $K = 2$. Furthermore, if we consider the approximation $p_0 + p_2 s^2$ to $\sqrt{1-s^2}$ and its inverse $1/(p_0 + p_2 s^2) = (1/p_0)/(1 + p_2/p_0 s^2)$, we see that we obtain the approximation $a_0/(1 + b_2 s^2)$, where a_0 and b_2 are defined exactly as before, with $a_0 = 1/p_0$ and $b_2 = p_2/p_0$. Therefore, although (11) and (12) are not equivalent differential equations at the boundary their difference approximations are equivalent. Thus in terms of stability it makes no difference whether we solve (11) or (12) at the boundary!

It can also be shown that the same result applies for both the $K = 3$ and 4 boundary conditions [12]. Therefore, the results of Theorem 1 which are derived for approximations to $1/\sqrt{1-s^2}$ may immediately be applied to boundary conditions derived for approximations to $\sqrt{1-s^2}$ by making the appropriate substitutions. In addition, we can use (11) or (12) at the boundary, for $C(s)$ or $r(s)$, respectively, and in those cases described in Section 3 where the approximation to $\sqrt{1-s^2}$ is the exact inverse of the approximation to $1/\sqrt{1-s^2}$, $r(s) = 1/C(s)$, (11) and (12) have equivalent stability properties.

To complete the analysis of stability of the approximations presented in this paper we need to investigate the existence of generalized eigensolutions, i.e., those roots which have $|\kappa| = |z| = 1$. To do this a numerical search is used to look for generalized eigensolutions for those values of μ for which the method is potentially stable.

The numerical search proceeds by calculating the roots κ of $D(\kappa, y, z^{-1})=0$, where $|z|=|\kappa|=|y|=1$, as a function of η and θ , $z=e^{i\theta}$ and $y=e^{i\eta}$. For each root a test is then made to determine whether it satisfies the dispersion relation for the interior stencil. Due to rounding error we cannot expect to find roots which satisfy the interior operator exactly, instead roots which satisfy the interior operator to some specified error are found. Note also that as the numerical search is discrete it cannot be taken to be conclusive, but, insofar as it supports numerical experiments which would show instability, it is an accurate indicator of potential instability due to eigensolutions.

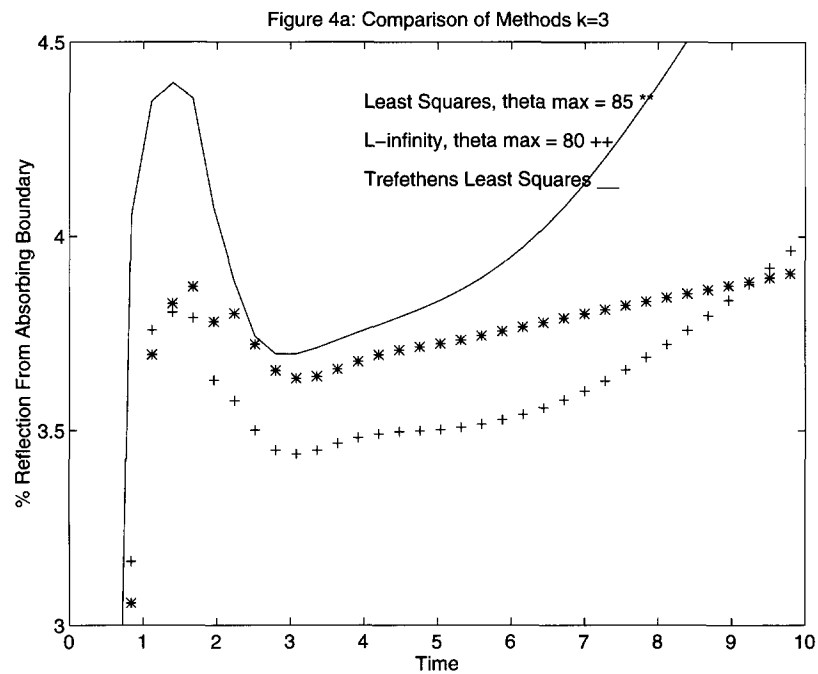
For the Chebyshev–Padé $K=2$ schemes the reflection coefficient is greater than one for some angles of incidence and, therefore, analysis of stability is not significant. For the least-squares schemes no eigensolutions were found and for the L^∞ , the 88° and 89° schemes showed eigensolutions for $\mu=0.1$. The $K=3$ Chebyshev–Padé schemes are all unstable for $\mu < 1/\sqrt{2}$ except for $\theta=45^\circ$ which showed eigensolutions for $\mu=0.1, 0.2$ and 0.3 . The least-squares schemes and L^∞ schemes showed eigensolutions only for $\mu=0.1$ and 88° and 89° , and 86° and 87° , respectively. For $K=4$ there were no generalized eigensolutions at all when an error tolerance less than 10^{-4} was used.

We see that only for small values of μ have generalized eigensolutions been found. Further, we observe that the bounds on the Courant number given by Theorem 1 impose no restriction on the least squares or L^∞ approximations. In each case, the calculated upper bounds on μ are greater than $1/\sqrt{2}$ which is the maximum allowable μ for stability of the interior scheme. The Chebyshev–Padé approximations are unstable except for the $K=3$ case on $[0, 45^\circ]$ and the $K=2$ cases calculated on the intervals $[0, 20^\circ]$, $[0, 80^\circ]$ and $[0, 85^\circ]$. The maximum values of μ are 0.39 and 0.13 in the latter two cases and thus neither of these approximate one-way wave equations are practical. The existence of generalized eigensolutions for small μ in the least squares and L^∞ cases does not pose a problem since for practical simulations it is preferred to run near the largest allowable μ , which is $1/\sqrt{2}$ in all cases of interest.

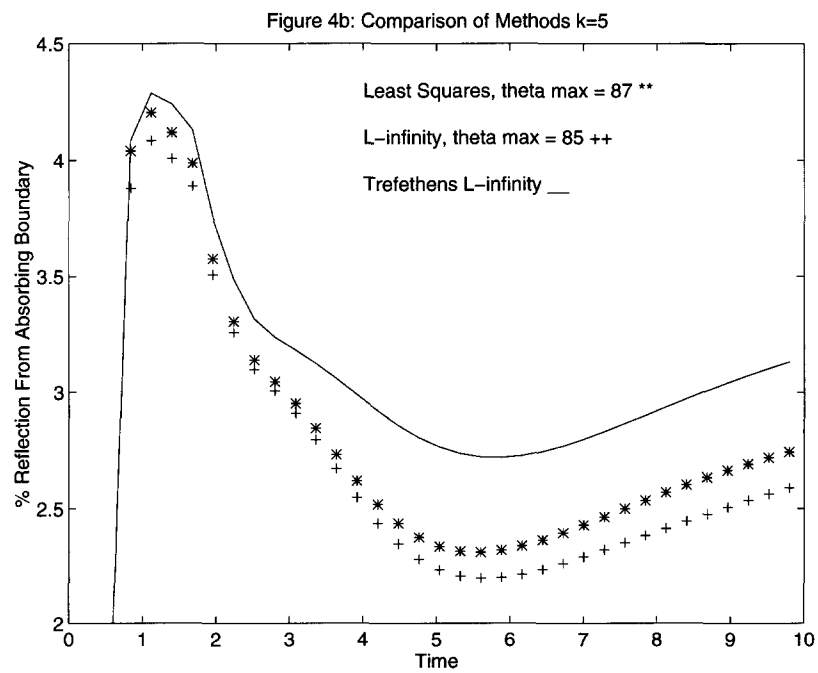
5. Numerical results

The OWWEs derived here were tested for both satisfaction of the stability bounds derived in the last section and their effectiveness at absorbing waves incident for a wide range of angles on the absorbing boundaries. Identical tests were carried out for the OWWEs derived in [5].

The results shown in Figs. 4(a) and (b) compare the optimum results from each of the following 3 groups: (1) the L^2 family of methods presented in Table 2; (2) the L^∞ family of methods presented in Table 3; and (3) the family of methods presented in [5]. In the latter case the least-squares case was best for $K=3$ and the L^∞ for $K=5$. We see from both Figs. 4(a) and (b) that the minimization of $\|C(s) - 1/\sqrt{1-s^2}\|$ with respect to the max norm leads to the best performance at the boundary. Furthermore, for the same amount of work, both the L^2 and L^∞ methods outperform the rest of the OWWEs derived from $r(s)$. These criteria were tested by simulating the propagation of a pulse down a long deep well. The pulse was propagated long enough to allow incidence of the pulse on the boundary for a wide range of angles from normal to glancing incidence. In addition, the long-time interval allowed for the development of instabilities in the solution due to the boundary condition. A complete set of results for these tests can be found in [8].



(a)



(b)

Fig. 4. Comparison of methods: (a) $K = 3$; and (b) $K = 5$.

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