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Journal of Computational and Applied Mathematics 163 (2004) 189–197

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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# Fairing geometric modeling based on 4-point interpolatory subdivision scheme☆

Xiaonan Luo, Ning Liu, Chengying Gao*

*Computer Application Institute, Sun Yat-sen University, Guangzhou 510275, China*

Received 3 September 2002; received in revised form 26 May 2003

**Abstract**

A 4-point interpolatory subdivision scheme with a tension parameter is analyzed, and the local property of 4-point interpolatory subdivision scheme and a kind of $G^1$-continuity sufficient condition between surfaces as well as between curves are discussed. An efficient method of generating natural boundary points of 4-point interpolatory curve is presented, as well as a surface modeling method with the entire fairing property by combining energy optimization with subdivision scheme. The method has been applied in modeling 3D virtual garment surface.


*Keywords:* Subdivision; Fairing; Interpolatory; Natural boundary point; Surface modeling

## 1. Introduction

Subdivision schemes are important and efficient tools for generating curves and surfaces iteratively from a finite set of points. They have been widely applied in CAD and CG. A uniform 4-point interpolatory subdivision scheme for curve design was proposed by Dyn and Levin [3]. The method is different from the traditional interpolation methods. The traditional interpolatory methods generate a continuous curve using a finite set of points, then they discretize the curve, and display it on a computer screen. The process can be summarized as discrete–continuous–discrete. While the 4-point scheme does not generate continuous curve, it interpolates a discrete curve directly, which greatly speeds up the calculating and displaying. When control points are non-uniform, it is difficult to satisfy the fairing and the preserving shape of curve only by adjusting tension parameter. It constrains 4-point

☆ This work is supported by the Chinese National Science Foundation:60273063.
* Corresponding author.
*E-mail address:* mcsgcy@zsu.edu.cn (C. Gao).

0377-0427/$ - see front matter 
doi:10.1016/j.cam.2003.08.064

interpolatory subdivision scheme's application in geometrical modeling tremendously. Cai [1] once proposed a non-uniform 4-point interpolatory scheme by developing Dyn's 4-point interpolatory scheme. The method improves the shape controlling of curve and surface greatly. But it cannot preferably satisfy the entire fairing of curves and surfaces yet [4].

In the present paper, we analyze the local support properties of 4-point interpolatory subdivision scheme, $G^1$-continuity conditions between surfaces and between curves, and propose an efficient method to generate natural boundary points. In order to realize the entire fairing property of subdivision surface, we employ energy optimization to subdivide the mesh. By constructing the object function based on subdivision mesh, we not only simplify complex integral of common energy modeling, but also preferably satisfy the entire fairing of subdivision surfaces. The fairing method presented here may model the deformable surface more efficiently than subdivision schemes do. It has been realized in the design of garment surface, and has achieved good effect.

## 2. Properties of 4-point interpolatory subdivision scheme

**Definition 2.1** (Cai [1]). Given $n+1$ points $\{P_i\}_{i=0}^n, P_i=(x_i,y_i)$ let $\Delta x_i = x_{i+1}-x_i, 0\leqslant i\leqslant n-1$. If $\Delta x_i = \Delta x_j, \forall 0\leqslant i,j\leqslant n-1, \{P_i\}_{i=0}^n$ are called uniform control points; otherwise, $\{P_i\}_{i=0}^n$ are called nonuniform control points.

**Definition 2.2** (Cai [1]). Given $n+1$ numbers $\{x_i\}_{i=0}^n$, if $x_i < x_{i+1}, 0\leqslant i\leqslant n-1$, let

$$K = \max_{0\leqslant i\leqslant n-2}\left(\frac{\Delta x_{i+1}}{\Delta x_i}, \frac{\Delta x_i}{\Delta x_{i+1}}\right)$$

$K$ is called the non-uniform degree of $\{x_i\}$, where $K$ satisfies $K\geqslant 1$ and $(1/K)\leqslant(\Delta x_{i+1}/\Delta x_i)\leqslant K$.

**Lemma 2.1** (Cai [1]). *Suppose that* $\{x_i\}_{i=-2}^{n+2}$ *satisfy* $x_{-2}<x_{-1}<\cdots<x_{n+1}<x_{n+2}$, *with*

$$0<\frac{1}{K}\leqslant\frac{\Delta x_{i+1}}{\Delta x_i}\leqslant K, -2\leqslant i\leqslant n,$$

*where* $\Delta x_i = x_{i+1}-x_i, K\geqslant 1$ *is a constant. Let* $\{x_i^k\}$ *be defined by*

$$\begin{aligned} &x_{2i}^{k+1} = x_i^k, \quad -1\leqslant i\leqslant 2^k n+1,\\ &x_{2i+1}^{k+1} = (\tfrac{1}{2}+w)(x_i^k+x_{i+1}^k)-\omega(x_{i-1}^k+x_{i+2}^k), \quad -1\leqslant \imath\leqslant 2^k n, \end{aligned} \tag{2.1}$$

*where* $x_i^0 = x_i, -2\leqslant i\leqslant n+2$, *Then for* $0\leqslant\omega\leqslant K/2(K+1)^2$, *we have*

$$\Delta x_i^k = \Delta x_{i+1}^k - x_i^k > 0,$$

$$0<\frac{1}{K}\leqslant\frac{\Delta x_{i+1}^k}{\Delta x_i^k}\leqslant K.$$

Suppose that the initial control points $\{P_i\}_{i=0}^n$ are strictly monotone increasing; the limit curve of Eq. (2.1) is a strictly monotone increasing function. It is easily proved by Lemma 2.1.

So we may have the form of non-uniform 4-point interpolatory curve:

Given control points $\{P_i\}_{i=-2}^{n+2}, P_i \in \mathbb{R}^d$, let us denote the control points at the $k$-level set by $\{P_i^k\}_{i=-2}^{2^k+2}$. Then the subdivision scheme defines the control points at the level $k+1$ by

$$\begin{aligned} P_{2i}^{k+1} &= P_i^k, \quad -1 \leqslant i \leqslant 2^k n+1, \\ P_{2i+1}^{k+1} &= (\tfrac{1}{2}+\omega)(P_i^k + P_{i+1}^k) - \omega(P_{i-1}^k + P_{i+2}^k), \quad -1 \leqslant i \leqslant 2^k n, \end{aligned} \tag{2.2}$$

where $P_i^0 = P_i$, $-2 \leqslant i \leqslant n+2$, $0 \leqslant \omega \leqslant K/(2(K+1)^2)$. By letting $k$ tend to infinity, the process defines an infinite set of points in $\mathbb{R}^d$, and these points lie on a continuous curve in $\mathbb{R}^d$.

The subdivision scheme (2.2) can be easily extended for surface designing, which passes through a set of control points of the form $\{P_{i,j}^0; i=-2,\ldots,n+2, j=-2,\ldots,m+2\}$. Firstly, we apply (2.2) to the index $i$, inserting points between $P_{i,j}^k$ and $P_{i+1,j}^k, i=-1,\ldots,2^k n, j=-2,\ldots,2^k m+1$, where $0 \leqslant \omega \leqslant K/(2(K+1)^2)$. Then (2.2) is applied to the index $j$. The overall step results in the $(k+1)$th set of points:

$$\begin{aligned} P_{2i,2j}^{k+1} &= P_{i,j}^k, & -1 \leqslant i \leqslant 2^k n+1, -1 \leqslant j \leqslant 2^k m+1, \\ P_{2i+1,2j}^{k+1} &= (\tfrac{1}{2}+\omega)(P_{i,j}^k + P_{i+1,j}^k) - \omega(P_{i-1,j}^k + P_{i+2,j}^k), & -1 \leqslant i \leqslant 2^k n, -1 \leqslant j \leqslant 2^k m+1, \\ P_{i,2j+1}^{k+1} &= (\tfrac{1}{2}+\omega)(P_{i,2j}^{k+1} + P_{i,2j+2}^{k+1}) - \omega(P_{i,2j-2}^{k+1} + P_{i,2j+4}^{k+1}), & -1 \leqslant i \leqslant 2^k n+1, -1 \leqslant j \leqslant 2^k m. \end{aligned} \tag{2.3}$$

### 2.1. Generation of natural boundary points

To define a curve passing through $P_0, P_1, \ldots, P_{n-1}, P_n$ by Eq. (2.2), one needs to supply additional points $P_{-2}, P_{-1}, P_{n+1}, P_{n+2}$ which will affect the behavior of the curve ends. These extra points can be used to control the slope of the curve at the end points. In the case of non-closed curve, we usually adopt natural boundary points to define the curve. A common generation method of natural boundary points will involve solving complex calculation. To avoid the problem, we propose an efficient method to generate natural boundary points.

**Algorithm 2.1.** In the first phase, we get line $l$. $l$ satisfies

1. $l$ is on the same surface of $P_0, P_1$ and $P_2$;
2. $l$ is vertical to $\overrightarrow{P_0P_1}$ and goes across the midpoint of $\overrightarrow{P_0P_1}$.

In the second phase, we find point $P$ that is symmetric to $P_2$ with respect to line $l$, then we set $P_{-1} = P$. The same method is applied to get $P_{-2}, P_{n+1}$ and $P_{n+2}$. We can define a curve passing through $\{P_i\}_{i=0}^n$.

The algorithm only deals with simple linear operation. So it is efficient in practical application.

### 2.2. Local property of the scheme

**Lemma 2.2.** *The curve segment* $p(t), t \in (i, i+1)$, *only depends on* $P_{i-2}, \ldots, P_{i+3}$.

**Theorem 2.1.** *Given the control points* $\{P_i\}_{i=0}^n$, *we move any point* $P_i$ to $P_i'$, *whose offset is denoted as* $\Delta P_i = P_i' - P_i$. *Offset of any other point in the control points* $\{P_i\}_{i=0}^n$ *will satisfy*

(i) $\mathrm{O}(\omega \Delta P_i) \sim \Delta P_j, \quad P_j \in [P_{i-2}, \ldots, P_{i+2}]$,

(ii) $\mathrm{O}(\omega^2 \Delta P_j) \sim \Delta P_j, \quad P_j \in [P_{i+2}, P_{i+3}]$,

(iii) 0, *other*.

**Proof.** *Case* i: Where $k = 1$, generation of $P_{(i-2)+1/2}, P_{(i-1)+1/2}, P_{i+1/2}$, and $P_{(i+1)+1/2}$ which depend on $P_i$:

$$\begin{aligned}
P_{(i-2)+1/2} &= (\tfrac{1}{2} + \omega)(P_{i-2} + P_{i-1}) - \omega(P_{i-3} + P_i),\\
P_{(i-1)+1/2} &= (\tfrac{1}{2} + \omega)(P_{i-1} + P_i) - \omega(P_{i-2} + P_{i+1}),\\
P_{i+1/2} &= (\tfrac{1}{2} + \omega)(P_i + P_{i+1}) - \omega(P_{i-1} + P_{i+2}),\\
P_{(i+1)+1/2} &= (\tfrac{1}{2} + \omega)(P_{i+1} + P_{i+2}) - \omega(P_i + P_{i+3}).
\end{aligned} \tag{2.4}$$

When $P_i$ changes to $P_i'$, the change of $P_i$ will have the same impact on the curve segment beside $P_i$. The indices of the points in one segment are bigger than $P_i$ and those in other are smaller than $P_i$. So we need only to discuss the impact on the curve segment composing of points that have indices bigger than $P_i$:

$$\begin{aligned}
P_{i+1/2} &= (\tfrac{1}{2} + \omega)(P_i + P_{i+1}) - \omega(P_{i-1} + P_{i+2}),\\
P_{(i+1)+1/2} &= (\tfrac{1}{2} + \omega)(P_{i+1} + P_{i+2}) - \omega(P_i + P_{i+3}),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
P'_{i+1/2} &= (\tfrac{1}{2} + \omega)(P_i' + P_{i+1}) - \omega(P_{i-1} + P_{i+2}),\\
P'_{(i+1)+1/2} &= (\tfrac{1}{2} + \omega)(P_{i+1} + P_{i+2}) - \omega(P_i' + P_{i+3}).
\end{aligned} \tag{2.6}$$

Combining (2.5) with (2.6) we have

$$\begin{aligned}
\Delta P_{i+1/2} &= (P'_{i+1/2} - P_{i+1/2}) = (\tfrac{1}{2} + \omega)\Delta P_i,\\
\Delta P_{(i+1)+1/2} &= (P'_{(i+1)+1/2} - P_{(i+1)+1/2}) = -\omega \Delta P_i,
\end{aligned} \tag{2.7}$$

and then we have $\mathrm{O}(\omega \Delta P_i) \sim \Delta P_j, P_j \in [P_i, \ldots, P_{i+2}]$.

As the impact on the segment $[P_{i-2}, P_i]$ is similar to the above, we have $\mathrm{O}(\omega \Delta P_i) \sim \Delta P_j, P_j \in (P_{i-2}, \ldots, P_{i+2})$.

*Case* ii: Now we mainly discuss how the change of $P_i$ influences the segment of $[P_{i+2}, P_{i+3}]$.

When $k = 2$, we have

$$\begin{aligned}
P_{(i+2)+1/4} &= (\tfrac{1}{2} + \omega)(P_{i+2} + P_{(i+2)+1/2}) - \omega(P_{(i+1)+1/2} + P_{i+3}),\\
P'_{(i+2)+1/4} &= (\tfrac{1}{2} + \omega)(P_{i+2} + P_{(i+2)+1/2}) - \omega(P'_{(i+1)+1/2} + P_{i+3}).
\end{aligned} \tag{2.8}$$

Combining Eq. (2.6), we have

$$\Delta P_{(i+2)+1/4} = P'_{(i+2)+1/4} - P_{(i+2)+1/4} = -\omega \Delta P_{(i+1)+1/2} = \omega^2 \Delta P_i. \tag{2.9}$$

So we have

$$\mathrm{O}(\omega^2 \Delta P_i) \sim \Delta P_j, P_j \in (P_{i+2}, P_{i+3}).$$

*Case* iii: By Lemma 2.2, we know that any point with bigger indices than $P_{i+3}$, with indices smaller than $P_{i-2}$ of the curve will not be affected by the change of $P_i$. □

4-point interpolatory subdivision surface has similar local property.

### 2.3. $G^1$-continuity of the scheme

Given curves $C_1, C_2$ which are, respectively, generated by control points $\{P_i\}_{i=0}^n$ and $\{Q_i\}_{i=0}^m$,where $P_n$ is equal to $Q_0, P_n$ or $Q_0$ denotes connective point of curve $C_1$ and $C_2$. In order to realize $G^1$-continuity between curve $C_1$, $C_2$ at point $Q_0$ or $P_n$, we need to supply the following boundary conditions:

(i) Boundary condition of curve $C_1$:
- boundary condition of $P_n : P_{n+1} = Q_1, P_{n+2} = Q_2$;
- boundary condition of $P_0 : P_{-2}, P_{-1}$ generated by Algorithm 2.1.

(ii) Boundary conditions of curve $C_2$:
- boundary condition of $Q_0 : Q_{-2} = P_{n-2}, Q_{-1} = P_{n-1}$;
- boundary condition of $Q_m : Q_{n+1}, Q_{n+2}$ generated by Algorithm 2.1;

$G^1$-continuity sufficient conditions of curve can be easily extended to surface.

Given surfaces $S_1, S_2$ which are, respectively, generated by control points $\{P_{i,j} : i = 0, \ldots, n;$ $j = 0, \ldots, m_1\}$ and $\{Q_{i,j} : i = 0, \ldots, n;\ j = 0, \ldots, m_2\}$, where $P_{i,m_1} = Q_{i,0} : i = 0, \ldots n$, in order to realize $G^1$ continuity of surface $S_1, S_2$ at the adjacent curve generated by $\{P_{i,m_1}\}_{i=0}^n$ or $\{Q_{i,0}\}_{i=0}^n$, we need to supply the following boundary conditions:

(i) Boundary conditions of surface $S_1$:
- boundary condition of the adjacent curve: $P_{i,m_1+1} = Q_{i,1}, P_{i,m_1+2} = Q_{i,2},\ i = -2, \ldots, n+2$;
- other boundary conditions: $P_{i,-2}, P_{i,-1},\ i = -2, \ldots, n+2$ generated by Algorithm 2.1.

(ii) Boundary conditions of surface $S_2$:
- boundary condition of the adjacent curve: $Q_{i,-2} = P_{i,m_1-2}, Q_{i,-1} = P_{i,m_1-1},\ i = -2, \ldots, n+2$;
- other boundary conditions: $Q_{i,-2}, Q_{i,-1}, i = -2, \ldots, n+2$ generated by Algorithm 2.1.

We employ the above boundary conditions to refine $S_1, S_2$. The result is shown as Fig. 1. Red mesh denotes $S_1$, blue mesh denotes $S_2$ and, green line denotes the adjacent curve.

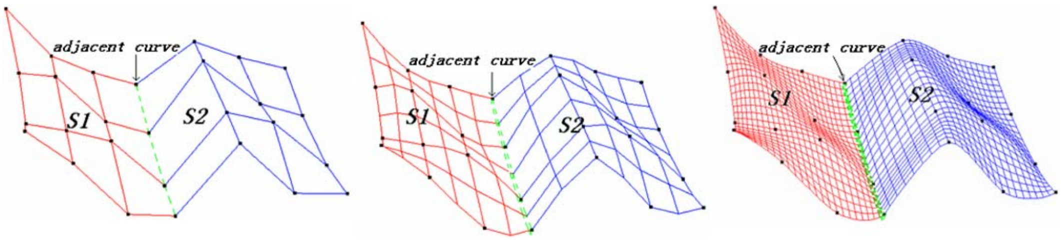


Fig. 1. $G^1$-continuity between surfaces.

## 3. Fairing modeling based on subdivision mesh

Surface modeling using subdivision scheme is hard to satisfy the entire fairing of surface [5]. Aiming at it, we introduce energy optimization method into subdivision surface. In order to employ energy optimization method, we must construct the object function and constrain equation efficiently. “Deformable energy function” proposed by Terzopoulo [2] is the most common object function at present. But it needs to deal with complex integral. We construct an object function, which is composed of energy functions in discrete form based on subdivision mesh. The method improves the entire fairing of subdivision surface efficiently.

### 3.1. Energy function based on geometrical constraint

#### 3.1.1. Stretching constraint

Stretching constraint can be expressed by restricting the distance between each adjacent point along longitudinal and latitudinal directions.

$$\sum_i \sum_j \left[ \frac{1}{2\omega_{\text{hstresh}}} (\|P_{i,j+1} - P_{i,j}\|_n - U_0)^2 + \frac{1}{2\omega_{\text{vstresh}}} (\|P_{i+1,j} - P_{i,j}\|_n - V_0)^2 \right], \tag{3.1}$$

where $\omega_{\text{hstresh}}$ and $\omega_{\text{vstresh}}$ are stretching constraint weights, respectively, in longitudinal and latitudinal directions. $U_0$ is the initial distance between adjacent grid points in the longitudinal direction, and $V_0$ is the initial distance between adjacent grid points in the latitudinal direction.

#### 3.1.2. Shearing constraint

Shearing constraint can be expressed by a deformation caused by a force applied in a direction not perpendicular on the surface. We restrict the diagonal distance between the control points of the mesh to be of some specified length:

$$\sum_i \sum_j \left[ \frac{1}{2\omega_{\text{shear}}} (\|P_{i+1,j+1} - P_{i,j}\| - L_0)^2 \right], \tag{3.2}$$

where $\omega_{\text{shear}}$ is the shearing constraint weight, $L_0$ is the initial distance between two points on the diagonal.

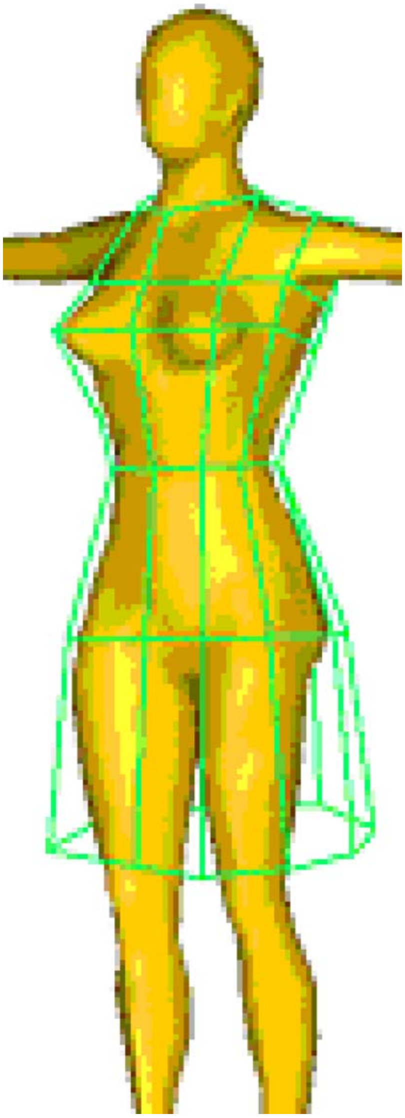

Fig. 2. Initial mesh.

*3.1.3. Bending constraint*

Bending constraint can be expressed by restricting the distance between every three neighboring points of the mesh in the longitudinal direction and latitudinal direction.

$$\sum_i \sum_j \left[ \frac{1}{2\omega_{\mathrm{hbend}}} (\|P_{i+2,j} - P_{i,j}\|_n - U_0)^2 + \frac{1}{2\omega_{\mathrm{vbend}}} (\|P_{i,j+2} - P_{i,j}\|_n - V_0)^2 \right], \tag{3.3}$$

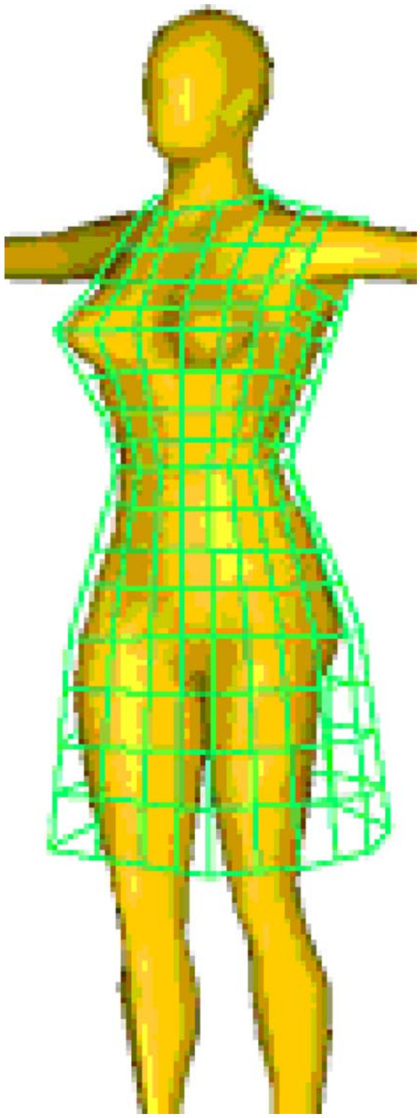

Fig. 3. Surface modeling by 4-point interpolatory subdivision surface.

where $\omega_{\text{hbend}}$ and $\omega_{\text{vbend}}$ are the bending constraint weights in the longitudinal direction and latitudinal direction. $u_0$ is initial distance between every second neighboring point of the mesh in the longitudinal direction. $v_0$ is initial distance between every second neighboring point of the mesh in latitudinal direction.

Combining these constraint function, an efficient object function can be constructed. With energy optimization method, an entire fairing surface can be modeled. The new method has been applied in the field of 3D virtual garment surface modeling. The result is shown as Figs. 2–4.

Fig. 4. Surface modeling by energy optimization.

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