Mixed finite elements for the Richards' equation: linearization procedure

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Abstract

We consider mixed finite element discretization for a class of degenerate parabolic problems including the Richards' equation. After regularization, time discretization is achieved by an Euler implicit scheme, while mixed finite elements are employed for the discretization in space. Based on the results obtained in (Radu et al. RANA Preprint 02-06, Eindhoven University of Technology, 2002), this paper considers a simple iterative scheme to solve the emerging nonlinear elliptic problems.
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1. Introduction

Water flow in porous media can be modelled by the Richards' equation [4], a nonlinear, possibly degenerate, parabolic differential equation. Taking the pressure as a primary unknown, Richards' equation becomes

$$\partial_t \Theta(\psi) - \nabla \cdot K(\Theta)\nabla(\psi + z) = 0, \tag{1}$$

where ψ is the pressure head, Θ the fluid saturation, K the conductivity and z the vertical height. Assuming a constant air pressure constant, in the fully saturated region we have $\psi \geqslant 0$, while $\psi < 0$ for partially saturated regions. Several retention curves are proposed in the literature to express

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relations between ψ, K and Θ. Here we are interested in both partially saturated and saturated flow, therefore we retain the pressure ψ as primary unknown.

As suggested in [1], saturation may be less regular than pressure. Therefore we expect to achieve better results by applying a Kirchhoff transform

$$\mathcal{K} : \mathbb{R} \to \mathbb{R}, \quad \psi \mapsto \int_0^\psi K(\Theta(s)) \, ds. \tag{2}$$

Due to the properties of K and Θ assumed below, $\mathcal{K}$ is invertible. With $u := \mathcal{K}(\psi)$ as primary unknown, after defining

$$b(u) := \Theta \circ \mathcal{K}^{-1}(u), \quad k(b(u)) := K \circ \Theta \circ \mathcal{K}^{-1}(u), \tag{3}$$

if $\vec{e}_z$ stands for the vertical unit vector, (1) becomes

$$\partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\vec{e}_z) = 0 \quad \text{in } (0, T) \times \Omega. \tag{4}$$

Qualitative properties for the Richards' equation are studied in several papers (see, e.g., [1,7]). Numerous articles investigate numerical methods that are appropriate for this equation. Adaptive time stepping is studied, e.g., in [20]. In case of implicit schemes, iterative methods are considered for solving the resulting nonlinear equations (see, e.g., [6,10,12]). For the spatial discretization, mixed finite elements or finite volumes provide a good approximation of the solution [8,11]. Hybrid mixed finite elements are studied from an algorithmic point of view in [20].

Convergence results are obtained in [24] (for mixed finite element discretization), [8] (for an implicit finite volume method), [10] (for a relaxation scheme) and [16] (for an implicit scheme in unsaturated regime). This paper is a continuation of [19], where error estimates are obtained for an Euler implicit mixed finite element scheme. It discusses an iterative scheme used for solving the nonlinear problems appearing as a result of the discretization procedure.

2. The numerical scheme

We are interested in solving Eq. (4) with initial and boundary conditions,

$$\partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\vec{e}_z) = 0 \quad \text{in } (0, T) \times \Omega,$$

$$u = u^0 \quad \text{in } 0 \times \Omega,$$

$$u = 0 \quad \text{on } J \times \Gamma. \tag{5}$$

Throughout this paper we make use of the following assumptions:

(A1) $\Omega \subset \mathbb{R}^d$ $(d \geqslant 1)$ is bounded with Lipschitz continuous boundary, $T < \infty$.
(A2) $b \in C^1$ is such that $0 \leqslant b'(u) \leqslant L_b$ for all reals u.
(A3) $k(b(z))$ is continuous and bounded in z and satisfies, for all $z_1, z_2 \in \mathbb{R}$,

$$|k(b(z_2)) - k(b(z_1))|^2 \leqslant C_k(b(z_2) - b(z_1))(z_2 - z_1).$$

(A4) $b(u_0)$ is essentially bounded (by 0 and 1) in Ω and $u_0 \in L^2(\Omega)$.

Assuming k bounded is reasonable since, for Richards' equation, k models the medium conductivity of the medium. The same is also valid for (A4), recalling that $b(u_0)$ stands for the initial saturation in the porous medium. The growth condition on $k(b(\cdot))$ (see also [16,18]) relaxes the more often assumed Lipschitz continuity of k (see, for example, [2] or [13]) and ensures uniqueness of a weak solution (see [1]). Source terms satisfying (A3) can also be considered here without any additional difficulty.

Because of the degenerate character, solutions of (5) should be understood in a weak sense. To give the corresponding definition we let $(\cdot,\cdot)$ stand for the inner product on $L^2(\Omega)$ or the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, $\|\cdot\|$ for the norm in $L^2(\Omega)$, $\|\cdot\|_1$ and $\|\cdot\|_{-1}$ for the norms in $H^1(\Omega)$, respectively $H^{-1}(\Omega)$. For vector valued functions the L^2 inner product and norm are defined by summing the correspondings for each component. We also make use of the space $H(\mathrm{div}, \Omega) = \{\vec{q} \in (L^2(\Omega))^d / \nabla \cdot \vec{q} \in L^2(\Omega)\}$, with the associated norm $\|\vec{q}\|_{\mathrm{div}}^2 := \|\nabla \cdot \vec{q}\|^2 + \|\vec{q}\|^2$.

Analogous notations are used for the inner product and corresponding norm on $L^2(0, T; \mathscr{H})$, with $\mathscr{H}$ being either $L^2(\Omega)$, $H^1(\Omega)$, or $H^{-1}(\Omega)$. In addition, we often write u or $u(t)$ instead of $u(t, x)$ and use C to denote a generic positive constant, not depending on the discretization or regularization parameters.

A weak solution for problem (5) by defined as (see [1]).

Definition 1. A function u is called a weak solution for Eq. (5) iff $b(u) \in H^1(0, T; H^{-1}(\Omega))$, $u \in L^2(0, T; H_0^1(\Omega))$, $u(0) = u^0$ (in H^{-1} sense) and for all $\varphi \in L^2(0, T; H_0^1(\Omega))$ it holds

$$\int_0^T (\partial_t b(u(t)), \varphi(t)) + (\nabla u(t) + k(b(u(t)))\vec{e}_z, \nabla \varphi(t)) \, dt = 0. \tag{6}$$

Existence, uniqueness and essential bounds for a weak solution of the above problem is studied in several papers (see, for example, [1,14] and the references therein).

Our numerical approach employs the lowest order Raviart–Thomas (RT_0) finite elements in space and Euler implicit in time. These schemes are applied after performing a regularization step. Specifically, for an $\varepsilon > 0$ we define b_ε as an approximation of the original nonlinearity b, satisfying

$$0 < \varepsilon \leqslant b_\varepsilon'(u) \leqslant C_1 < \infty, \quad |b(u) - b_\varepsilon(u)| \leqslant C_2 \varepsilon,$$

for any real u. Possible choices are

$$b_\varepsilon(u) = b(u) + \varepsilon u \quad \text{or} \quad b_\varepsilon(u) = \int_0^u \max\{\varepsilon, \min\{b'(v), 1/\varepsilon\}\} \, dv. \tag{7}$$

With $N > 0$ integer, set $\tau = T/N$ and let $\mathscr{T}_h$ being a decomposition of Ω into closed d-simplices; h stands for the mesh size.

For a rigorous formulation of the scheme we make use of the discrete subspaces $W_h \times V_h \subset L^2(\Omega) \times H(\mathrm{div}, \Omega)$ defined as

$$W_h := \{p \in L^2(\Omega) \mid p \text{ is constant on each element } T \in \mathscr{T}_h\},$$

$$V_h := \{\vec{q} \in H(\mathrm{div}, \Omega) \mid \vec{q}_{|T} = \vec{a} + b\vec{x} \text{ for all } T \in \mathscr{T}_h\}. \tag{8}$$

So W_h denotes the space of piecewise constant functions, while V_h is the RT_0 space (see [5]). Now the fully discrete mixed finite element approximation of problem (5) is defined in

Definition 2. Let $n \in \{1, \ldots, N\}$ and $p_h^{n-1} \in W_h$ be given. Find $(p_h^n, \vec{q}_h^n) \in W_h \times V_h$ such that

$$(b_\varepsilon(p_h^n), w_h) + \tau(\nabla \cdot \vec{q}_h^n, w_h) = (b_\varepsilon(p_h^{n-1}), w_h), \tag{9}$$

$$(\vec{q}_h^n, v_h) - (p_h^n, \nabla \cdot \vec{v}_h) + (k(b(p_h^n))\vec{e}_z, \vec{v}_h) = 0, \tag{10}$$

for all $w_h \in W_h$ and $\vec{v}_h \in V_h$.

Initially we take $p_h^0 = b_\varepsilon^{-1}(P_h b_\varepsilon(u^0))$, where P_h is the usual L^2 projector

$$P_h : L^2(\Omega) \to W_h, \quad ((P_h w - w), w_h) = 0, \quad \forall w_h \in W_h. \tag{11}$$

Since $P_h b_\varepsilon(u^0)$ is constant on any $T \in \mathcal{T}_h$, the same holds for $b_\varepsilon^{-1}(P_h b_\varepsilon(u^0))$, so $p_h^0 \in W_h$. Moreover, with this choice we obtain, for all $w_h \in W_h$,

$$(b_\varepsilon(p_h^0), w_h) = (b_\varepsilon(u^0), w_h).$$

Defining, for $0 \leqslant t \leqslant T$, the time-integrated flux as

$$\tilde{q}(t) = -\int_0^t (\nabla u(s) + k(b(u(s)))\vec{e}_z) \, ds,$$

if u is the solution of problem (5) in the sense of Definition 1, the following error estimates are proven in [19, Theorem 4.14, Corollary 4.15, and Remark 4.16]:

Theorem 3. *Assuming* (A1)–(A4), *we get*

$$\left\| \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (u(t) - p_h^n) \, dt \right\|^2 + \left\| \tilde{q}(T) - \tau \sum_{n=1}^N \vec{q}_h^n \right\|^2 \leqslant C(\tau + \varepsilon + h^2),$$

$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (b_\varepsilon(u(t)) - b_\varepsilon(p_h^n), u(t) - p_h^n) \, dt \leqslant C(\tau^{1/2} + \varepsilon^{1/2} + h^2/\tau^{1/2}). \tag{12}$$

3. Linearization procedure

The numerical scheme under consideration is implicit, leading at each time step to a nonlinear equation. To solve it we take $K \geqslant L_b$ being a real number and define the following linearization procedure.

Definition 4. Let $n = \overline{1, N}$, $i > 0$, and $p_h^{n-1} \in W_h$, $p_h^{n,i-1} \in W_h$, $\vec{q}_h^{n,i-1} \in V_h$ given. Find $(p_h^{n,i}, \vec{q}_h^{n,i}) \in W_h \times V_h$ such that

$$K(p_h^{n,i}, w_h) + \tau(\nabla \cdot \vec{q}_h^{n,i}, w_h) = (Kp_h^{n,i-1} + b_\varepsilon(p_h^{n-1}) - b_\varepsilon(p_h^{n,i-1}), w_h), \tag{13}$$

$$(\vec{q}_h^{n,i}, \vec{v}_h) - (p_h^{n,i}, \nabla \cdot \vec{v}_h) = -(k(b(p_h^{n,i-1}))\vec{e}_z, \vec{v}_h), \tag{14}$$

for all $w_h \in W_h$ and $\vec{v}_h \in V_h$, with $p_h^{n,0} = p_h^{n-1}$.

The scheme above is inspired from partial differential equations books (see, e.g., [22, p. 96]), where it is used for constructing sub- and super-solutions for nonlinear elliptic problems. In [25] (see also [15,17]) the same idea is used to construct iteration schemes for degenerate (slow diffusion) problems, and the proof is based on fixed point arguments in the H^1 space. Nevertheless, in case of slow diffusion problems the iterations are converging slowly, so such schemes are not of practical interest.

Closely related ideas are adopted in the succeeding paper [21], where the same approach is used for constructing an effective iteration scheme for fast diffusion problems. Convection is treated there explicitly, and formulation is conformal. Here we deal with mixed formulation and discretize the convection implicitly.

Remark 5. The setting of Definition 4 includes a spatial discretization based on RT_0 elements. Defining the iterative procedure for the spatially continuous case is straightforward, and all the results below apply without any change.

Remark 6. Convergence is shown below by contraction arguments and holds for any starting point $p_h^{n,0}$. However, the number of iterations required for obtaining a good approximation of p_h^n and $\vec{q}_h^n$ is influenced by this choice, therefore we take $p_h^{n,0} = p_h^{n-1}$ above, which seems reasonable.

To show convergence of the iteration scheme we use the following notations:

$$e_p^i := p_h^n - p_h^{n,i}, \quad e_q^i := \vec{q}_h^n - \vec{q}_h^{n,i}, \tag{15}$$

where p_h^n and $\vec{q}_h^n$ are from Definition 2. The following lemma is proven in [23].

Lemma 7. *If the domain Ω is convex, for any $f_h \in W_h$ a $\vec{v}_h \in V_h$ exists such that*

$$\nabla \cdot \vec{v}_h = f_h \quad and \quad \|\vec{v}_h\| \leqslant C_\Omega \|f_h\|, \tag{16}$$

where the constant $C_\Omega > 0$ does not depend on f_h.

To deal with convection, a mild restriction is imposed for the time step

$$2C_k \tau < 1, \tag{17}$$

with no additional assumption for the regularization parameter ($\varepsilon \geqslant 0$).

Theorem 8. *Assuming* (17), *the scheme in Definition* 4 *converges, and for any* $i > 0$ *we have*

$$\|e_p^i\| \leqslant \alpha^{i/2}\|e_p^0\| \quad and \quad \|e_q^i\| \leqslant \alpha^{i/2}\sqrt{\frac{(K + \tau/(2C_\Omega^2))}{\tau}}\,\|e_p^0\| \tag{18}$$

with $\alpha := [(K - \varepsilon(1 - 2\tau C_k))/(K + \tau/(2C_\Omega^2))] < 1$, *with* C_Ω *from Lemma* 7 *and* C_k *from* (A3).

Proof. We restrict ourselves to proving (18), convergence of $p_h^{n,i}$ to p_h^n, respectively, $\vec{q}_h^{n,i}$ to $\vec{q}_h^n$ (as $i \nearrow \infty$) following directly since $\alpha < 1$. Subtracting (13) from (9) and (14) from (10) gives

$$K(e_p^i - e_p^{i-1}, w_h) + \tau(\nabla \cdot e_q^i, w_h) + (b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1}), w_h) = 0, \tag{19}$$

$$(e_q^i, \vec{v}_h) - (e_p^i, \nabla \cdot \vec{v}_h) + ((k(b(p_h^n)) - k(b(p_h^{n,i-1})))\vec{e}_z, \vec{v}_h) = 0. \tag{20}$$

By Lemma 7, a $\vec{v}_h \in V_h$ exists such that $\nabla \cdot \vec{v}_h = e_p^i$ and $\|\vec{v}_h\| \leqslant C_\Omega\|\nabla \cdot \vec{v}_h\|$. Using this $\vec{v}_h$ into (20) together with the inequalities of Cauchy and of means $|ab| \leqslant \delta a^2 + b^2/(4\delta)$ (for all reals a and b and $\delta > 0$) gives

$$\|e_p^i\|^2 \leqslant 2C_\Omega^2\{\|e_q^i\|^2 + \|k(b(p_h^n)) - k(b(p_h^{n,i-1}))\|^2\}. \tag{21}$$

Taking now $w_h = e_p^i \in W_h$ into (19), respectively $\vec{v}_h = \tau e_q^i$ into (20), and adding the resulting, by the identity $(x - y, x) = [x^2 - y^2 + (x - y)^2]/2$ we get

$$\frac{K}{2}(\|e_p^i\|^2 - \|e_p^{i-1}\|^2 + \|e_p^i - e_p^{i-1}\|^2) + \tau\|e_q^i\|^2 + (b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1}), e_p^{i-1})$$

$$= -(b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1}), e_p^i - e_p^{i-1}) - \tau((k(b(p_h^n)) - k(b(p_h^{n,i-1})))\vec{e}_z, e_q^i).$$

Using again Cauchy's inequality and the inequality of means, we get

$$K\|e_p^i\|^2 - K\|e_p^{i-1}\|^2 + \tau\|e_q^i\|^2 + 2(b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1}), e_p^{i-1})$$

$$\leqslant \frac{1}{K}\|b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1})\|^2 + \tau\|k(b(p_h^n)) - k(b(p_h^{n,i-1}))\|^2. \tag{22}$$

Adding $\tau\|k(b(p_h^n)) - k(b(p_h^{n,i-1}))\|^2$ on both sides of (22), using (21), (A2) and (A3) gives

$$\left(K + \frac{\tau}{2C_\Omega^2}\right)\|e_p^i\|^2 \leqslant K\|e_p^{i-1}\|^2 + \left(\frac{L_b}{K} + 2\tau C_k - 2\right)(b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1}), e_p^{i-1}).$$

Since b_ε is monotone, the scalar product on the right is positive. In fact we have

$$\varepsilon\|e_p^{i-1}\|^2 \leqslant (b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1}), e_p^{i-1}) \leqslant L_b\|e_p^{i-1}\|^2.$$

By (17), because $K \geqslant L_b$ we have $L_b/K + 2\tau C_k - 2 \leqslant 2\tau C_k - 1 < 0$, so we end up with

$$\left(K + \frac{\tau}{2C_\Omega^2}\right)\|e_p^i\|^2 \leqslant (K - \varepsilon(1 - 2\tau C_k))\|e_p^{i-1}\|^2, \tag{23}$$

and the first part of (18) follows immediately.

For estimating the flux error we use (22) again and proceed as before to get

$$K\|e_p^i\|^2 + \tau\|e_q^i\|^2 \leqslant (K - \varepsilon(1 - 2\tau C_k))\|e_p^{i-1}\|^2.$$

Using now the estimates in the pressure gives the second part of (18). □

Remark 9. Theorems 10 and 8 show that the linearization given in Definition 4 is convergent in both pressure and flux. This is consistent with previous results, where H^1 estimates are obtained in the conformal formulation.

The convergence rate becomes squared if convection is 0, as follows from below. Moreover, as in [21] or [25], the scheme converges for any $\tau > 0$ and $\varepsilon \geqslant 0$.

Theorem 10. *In the absence of convection ($k = 0$), the scheme given in Definition 4 is convergent. With $\alpha := (K - \varepsilon)/(K + \tau/C_\Omega) < 1$, for any $i > 0$ we have*

$$\|e_p^i\| \leqslant \alpha^i\|e_p^0\| \quad \text{and} \quad \|e_q^i\| \leqslant \alpha^i[(K + \tau/C_\Omega)/(2K\tau^{1/2})]\|e_p^0\|. \tag{24}$$

Proof. We use similar arguments as for proving Theorem 8. Subtracting (13) from (9) and taking $w_h = e_p^i \in W_h$ yields

$$K\|e_p^i\|^2 + \tau(\nabla \cdot e_q^i, e_p^i) = (Ke_p^{i-1} - (b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1})), e_p^i). \tag{25}$$

Similarly, from (14) and (10) we get

$$(e_q^i, \vec{v}_h) - (e_p^i, \nabla \cdot \vec{v}_h) = 0, \tag{26}$$

for all $\vec{v}_h \in V_h$. Testing with $\vec{v}_h = \tau e_q^i \in V_h$ and adding the result to (25) gets

$$K\|e_p^i\|^2 + \tau\|e_q^i\|^2 = (Ke_p^{i-1} - (b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1})), e_p^i). \tag{27}$$

By Lemma 7, a $\vec{v}_h \in V_h$ exists such that $\nabla \cdot \vec{v}_h = e_p^i \in W_h$ and $\|\vec{v}_h\| \leqslant C_\Omega\|e_p^i\|$. Using this as a test function into (25) gives

$$\|e_p^i\|^2 = (e_p^i, \nabla \cdot \vec{v}_h) = (e_q^i, \vec{v}_h) \leqslant \|\vec{v}_h\|\|e_q^i\| \leqslant C_\Omega\|e_p^i\|\|e_q^i\|,$$

showing that

$$\|e_p^i\| \leqslant C_\Omega\|e_q^i\|. \tag{28}$$

Since $\varepsilon \leqslant b_\varepsilon' \leqslant L_b \leqslant K$, it follows that

$$|Ke_p^{i-1} - (b_\varepsilon(p_h^n) - b_\varepsilon(p_h^{n,i-1}))| \leqslant (K - \varepsilon)|e_p^{i-1}| \tag{29}$$

almost everywhere in Ω. By Cauchy's inequality, (28) and (29) imply

$$\left(K + \frac{\tau}{C_\Omega}\right)\|e_p^i\|^2 \leqslant (K - \varepsilon)\|e_p^{i-1}\|\|e_p^i\|,$$

so $\|e_p^i\| \leqslant \alpha\|e_p^{i-1}\|$ and the first part of (24) is proven. For the second estimate we use (27) again and the mean inequality to obtain

$$K\|e_p^i\|^2 + \tau\|e_q^i\|^2 \leqslant (K - \varepsilon)\|e_p^{i-1}\|\|e_p^i\| \leqslant K\|e_p^i\|^2 + \frac{(K - \varepsilon)^2}{4K^2}\|e_p^{i-1}\|^2,$$

and the rest of the proof is straightforward. □

Table 1
L^2 errors for the test problem in [9]

N	τ	h	Squared error	Convergence order
1	0.04	0.25	7.084860e-06	—
2	0.02	0.176	3.620121e-06	0.81
3	0.01	0.125	2.057357e-06	0.82
4	0.005	0.088	9.574648e-07	1.10
5	0.0025	0.0625	5.362180e-07	0.84
6	0.00125	0.044	2.431789e-07	1.14
7	0.000625	0.03125	1.355457e-07	0.84

Remark 11. The same result is obtained if convection is discretized explicitly (see also [21]).

The theoretical estimates are validated on a test problem proposed in [9]. After performing the Kirchhoff transformation we end up with (4), where $k \equiv 0$ and

$$b(u) = \begin{cases} \dfrac{\pi^2}{2} - \dfrac{u^2}{2} & \text{for } u \leqslant 0, \\ \dfrac{\pi^2}{2} & \text{for } u > 0. \end{cases}$$

With appropriate initial and (Dirichlet) boundary conditions, an exact solution can be given

$$u_{\text{ex}}(t, x, y) = \begin{cases} \dfrac{-2(e^s - 1)}{e^s + 1} & \text{for } s \geqslant 0, \\ -s & \text{for } s < 0, \end{cases}$$

where $s = x - y - t$. The equation is solved in the unit square $(0, 1)^2$, and the final time is $T = 1.0$.

We consider a hybrid implementation of the mixed finite element method. The emerging algebraic system of equations is solved by adding Lagrange multipliers. Within an iteration step we first eliminate the flux on each element and then continuity equation is solved locally for pressure. The algorithm is implemented in *UG* (version 3.8, see also [3]) on a SUN workstation.

Computations are performed on a uniform triangular mesh. Initially we take $h = 0.25$ and $\tau = 0.04$. Then τ and h^2 are successively halved, up to $\tau = 0.000625$ and $h = 0.03125$. Here we took $\varepsilon = 0$ and $K = 2$. Few iterations (5 to 8) were sufficient to obtain solutions differing by at most 10^{-10}. Similar results are provided by the Newton method developed in [20].

The order of convergence is estimated by the logarithm of the ratio between two squared errors, obtained for successive discretization parameters. Results are given in Table 1, confirming the theoretical estimates in Theorem 3.

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