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# A regularization framework for mildly ill-posed problems connected with pseudo-differential operator

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## Abstract

Recently filter-based regularization methods have been well investigated for ill-posed problems when the forward operators are compact. There are many ill-posed problems connected with pseudo-differential operators. But there is no uniform method for this kind of problems. The work on generalization of filter-based regularization methods to pseudo-differential operator is necessary. In this paper, we present a regularization framework for solving the mildly ill-posed problems involved pseudo-differential operators. A general regularization method for this kind of problems is given. The order-optimal error estimates are derived under the usual source conditions. As an example, a new fractional Tikhonov regularization method could be cast into the general framework. Numerical experiments are conducted for showing the validity of the new fractional Tikhonov method.

**Keywords:** Ill-posed problems; fractional Tikhonov regularization; error estimate; pseudo-differential operator

**AMS Subject Classification:** 65R35, 53C35, 22E46

## 1 Introduction

Many inverse problems arising from mathematical physics are defined in infinite domain or semi-infinite domain. These problems can be formulated as the operator

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equations in the frequency domain. The involved operators can be viewed as pseudo-differential operators. Several ill-posed problems fall into this category. For examples, the inverse heat conduction problem [2, 3, 10], the problem of analytic continuation [5, 6], the problem of numerical differentiation [23], the source identification problem for parabolic equation or elliptic equation [25, 29, 30].

When the inverse problems are ill-posed, the corresponding pseudo-differential operators are usually unbounded and a small perturbation in the input data might cause arbitrarily large changes in the solution. And regularization methods are required for stable numerical approximation. However, in some inverse problems we are more interested in the sharp or fine details in the exact solution, most of the classical regularization methods suffer from the over-smoothing of approximate solutions. Therefore the sharp or fine features of the solution by the classical methods are lost. Recently some fractional regularization methods are proposed for overcoming this disadvantage. Let's refer to the references [8, 13, 15, 16, 27, 28]. To the authors' knowledge, these studies on fractional regularization methods are restricted to compact operators. In this paper, we try to give a uniform regularization method for the ill-posed problems connected with pseudo-differential operators and provide a new fractional Tikhonov regularization method which is different from the existing fractional Tikhonov methods [8, 16]. Numerical results show that the new fractional Tikhonov method works better than the classical one.

It is well known that there is no universal method for solving ill-posed inverse problems. In each specific problem, the main difficulty is instability and has to be tackled in its own way. For illustrating this point, we give a survey for the problem of numerical differentiation as a specific example.

### 1.1 Numerical differentiation: an illustrative example

The Caputo fractional derivative of order  $k = n + \beta$ ,  $0 < \beta < 1$ ,  $n = 0, 1, 2, \dots$  is defined by

$$D^{(k)}h(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{h^{(n+1)}(s)}{(t-s)^\beta} ds, \quad 0 \leq t \leq T, 0 < \beta < 1, n = 0, 1, 2, \dots$$

$$D^{(k)}h(t) = \frac{d^n h(t)}{dt^n}, \quad 0 \leq t \leq T, k = n,$$

where  $\Gamma(\cdot)$  is the Euler Gamma function. We are interested in computing  $f(t) = D^{(k)}h(t)$ . However, the input data  $h(t)$  usually be replaced by the noisy data  $h_\delta(t)$  in practice. This is a classical ill-posed problem.

Let

$$\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\phi(s)ds}{(t-s)^\beta} = J^{(1-\beta)}\phi$$

be the Riemann-Liouville fractional integral of order  $1 - \beta$  [22]. Now using the properties of Riemann-Liouville fractional integral and the definition  $J^{(k)}f(t) = (a \star f)(t)$  where ' $\star$ ' denotes the convolution operator and  $a(t) = \frac{t^{k-1}}{\Gamma(k)}$ , one can easily get

$$J^{(k)}f(t) = J^{(k)}D^{(k)}h(t) = h(t) - \sum_{j=0}^n h^{(j)}(0) \frac{t^j}{j!} := g(t). \quad (1.1)$$

We assume that the derivative values  $h^{(j)}(0)$  are known for physical reasons [22] because  $h^{(j)}(0)$  denote the boundary datum. Thus the data  $h(t) - \sum_{j=0}^n h^{(j)}(0) \frac{t^j}{j!}$  is known and we defined it by  $g(t)$  in (1.1).

Therefore we reformulate (1.1) as an operator equation

$$Af = g, \quad (1.2)$$

where  $Af = \int_0^t \frac{(t-s)^{k-1}f(s)}{\Gamma(k)} ds$ .

In order to taking the Laplace transform, we extend the function  $f(t)$  from  $[0, T]$  to  $[0, +\infty)$ . This extension is plausible since the mapping from  $f$  to  $g$  is causal [22].

Considering the relationship of Fourier transform and Laplace transform, we extend the definitions of the functions (if needed) to the whole real  $t$ -axis by defining them to be zero for  $t < 0$ . Let

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi t} dt, \quad i = \sqrt{-1}.$$

Then the solution  $f$  of  $Af = g$  in (1.2) can be written in the frequency domain:

$$\widehat{f}(\xi) = (i\xi)^k \widehat{g}(\xi)$$

or equivalently computation of the pseudo-differential operator

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (i\xi)^k \widehat{g}(\xi) e^{i\xi t} d\xi.$$

It is obvious that ill-posedness of problem (1.2) is due to the high frequency components of  $(i\xi)^k$ . In this paper, we seek the solution  $f$  in  $L^2$  space i.e.  $\|f\|^2 = \int_{\mathbb{R}} |(i\xi)^k \widehat{g}(\xi)|^2 d\xi < +\infty$  which implies that the data function  $\widehat{g}(\xi)$  rapidly decays in the high frequency components. However, usually we only have the noisy data  $\widehat{g}_\delta(\xi) \in L^2(\mathbb{R})$  which has no such a property. This leads to a blow-up solution in the direct computation.

All kinds of regularization methods are given for solving this ill-posed problem. Murio *et al* [17, 18, 19, 20, 21] used a mollification regularization method with the

Gaussian kernel to deal with this problem. In the frequency domain, the regularization solution can be expressed as

$$\widehat{f}_\alpha(\xi) = \frac{(i\xi)^k}{e^{\alpha\xi^2}} \widehat{g}(\xi).$$

Hào *et al* [11] utilized another mollification regularization method with the de la Vallée Poussin kernel, and the regularization solution can be expressed in the frequency domain:

$$\widehat{f}_\alpha(\xi) = (i\xi)^k \psi_{1/\alpha}(\xi) \widehat{g}(\xi), \quad \psi_{1/\alpha}(\xi) = \begin{cases} 1, & |\xi| < \frac{1}{\alpha}, \\ 2 - \alpha\xi, & \frac{1}{\alpha} \leq |\xi| \leq \frac{2}{\alpha}, \\ 0, & |\xi| > \frac{2}{\alpha}. \end{cases}$$

Gorenflo and Yamamoto [9] introduced a Tikhonov-type method. In the frequency domain, the approximate solution is given by

$$\widehat{f}_\alpha(\xi) = \frac{(i\xi)^k}{1 + \alpha|\xi|^{2k}} \widehat{g}(\xi).$$

In these methods,  $\alpha$  denotes the regularization parameter.

We note that finding the regularization solutions can be uniformly written in a general form

$$\widehat{f}_\alpha(\xi) = F_\alpha(\xi) (i\xi)^k \widehat{g}(\xi).$$

where  $F_\alpha(\xi)$  is called the filter and satisfies  $F_\alpha(\xi) \rightarrow 1$  as  $\alpha \rightarrow 0$ .

## 1.2 Generalization and motivation

Above example makes us to propose a general regularization method for a general ill-posed problems connected with the pseudo-differential operators.

Let the  $n$ -dimensional Fourier transform of the function  $h(x) \in L^2(\mathbb{R}^n)$  be defined by

$$\hat{h}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} h(x) e^{-i\xi \cdot x} dx,$$

where  $x = (x_1, x_2, \dots, x_n)^T$ ,  $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$ . The Sobolev function space  $H^p(\mathbb{R}^n)$  is defined by

$$H^p(\mathbb{R}^n) = \{h(x) | h \in L^2(\mathbb{R}^n), \|h\|_p := \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^p |\hat{h}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty\},$$

where  $\hat{h}(\xi)$  is the Fourier transform of function  $h(x)$ ,  $|\xi| = (\xi_1^2 + \dots + \xi_n^2)^{1/2}$ , and  $\|\cdot\| := \|\cdot\|_0$  denotes the norm in  $L^2(\mathbb{R}^n)$ .

Consider the pseudo-differential operator  $A(D)$  with an unbounded symbol  $a(\xi)$  given by

$$f(x) := A(D)g(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} a(\xi) \hat{g}(\xi) e^{i\xi \cdot x} d\xi, \quad (1.3)$$

where  $D$  stands for  $\partial/\partial x$ ,  $\hat{g}(\xi)$  is the Fourier transform of the exact data  $g(x) \in L^2(\mathbb{R}^n)$ , the input data  $g(x)$  is only given approximately by  $g^\delta(x) \in L^2(\mathbb{R}^n)$  satisfying

$$\|g(\cdot) - g^\delta(\cdot)\| \leq \delta,$$

where  $\delta$  is the noise level and assumed to be known. Now we want to compute the solution  $f(x)$  from the noisy data  $g^\delta(x)$ .

From (1.3), we have

$$\hat{f}(\xi) = a(\xi) \hat{g}(\xi). \quad (1.4)$$

Throughout this paper, we assume the symbol  $a(\xi)$  satisfies the following property:  $|a(\xi)| = O(|\xi|^b)$  as  $|\xi| \rightarrow \infty$  with  $b > 0$ , i.e., the problem (1.3) is *mildly ill-posed problem*. Specifically, the symbol  $a(\xi)$  is assume to satisfy the following growth condition:

$$\tilde{C}_1 |\xi|^k \leq |a(\xi)| \leq \tilde{C}_2 |\xi|^k, \quad (1.5)$$

where  $\tilde{C}_1, \tilde{C}_2, k$  are known positive constants.

The earlier work on this kind of ill-posed problems dates back to Hào's work [12] where the mollification regularization method has been studied well in the case of one-dimensional setting. However, the main purpose of this paper is

1. to develop a new and very simple approach to the general problem (1.3).
2. to make many seemingly disparate approaches to be cast into the new framework proposed by us.

We stress that if the pseudo-differential operators are self-adjoint, the regularization methods [4, 14] based on the spectral theory are available but the source conditions on unknown solution are difficult to verify and the method proposed by us is simpler for use. If the pseudo-differential operators are not self-adjoint, the regularization methods [4, 14] based on the spectral theory are not available and the method proposed by us is an attractive alternative method. Moreover, the a-posteriori Fourier method in [6] is a special case of the method proposed in this paper.

The rest of this paper is organized as follows. In section 2, we provide a general regularization method and derive the error estimates, we also give some applications for the general regularization method; in section 3, we conduct some numerical tests to show the validity of the new fractional Tikhonov regularization method, finally we give some concluding remarks.

Throughout this paper, the symbols  $C, C_i, \tilde{C}_i$  are the generic constants independent on  $\alpha, \delta, E$  except for special instructions.

## 2 Regularization strategy

The a-priori information about unknown solution (or the so-called source condition) has been proved to be essential to obtain the convergence rates of regularization approximate solution for ill-posed problems in mathematical physics. Otherwise, without the a-priori information the convergence rate of the constructed regularization method is arbitrarily slow [4, 14]. However, in the general regularization theory the source condition usually is given in the form of the forward operator and is difficult to verify.

### 2.1 A general regularization method

In this paper, as usually we assume there holds the a-priori information for the unknown solution  $f$  with the form:

$$\|f(\cdot)\|_p \leq \tilde{E}, p > 0, \quad (2.1)$$

where  $\|\cdot\|_p$  denotes the Sobolev space norm,  $\tilde{E}$  is a finite positive real number. (2.1) means that the unknown solution has  $p$ -order weak derivative in  $L^2$ -space. This condition implies

$$\int_{\mathbb{R}} |\xi|^{2p} |\hat{f}(\xi)|^2 d\xi \leq \tilde{E}^2, p > 0,$$

which is equivalent to

$$\int_{\mathbb{R}} |a(\xi)|^{2p/k} |\hat{f}(\xi)|^2 d\xi \leq E^2,$$

where we used the assumption (1.5) and  $E$  is another finite positive number.

For proving Theorem 2.1 and Theorem 2.2 more clearly, we reformulate the source condition (2.1) as follows [26]:

there exists an element  $\omega(\xi) \in L^2(\mathbb{R}^n)$  such that  $\hat{f}(\xi) = a(\xi)^{-s} \omega(\xi)$  with  $s = p/k$ , and  $\omega(\xi)$  satisfies

$$\|\omega(\cdot)\|_{L^2} \leq E.$$

Motivated by the idea of filter-based method for the compact operators in [14, 15, 16], in the frequency domain we provide a general regularization method:

$$R_\alpha \hat{g}(\xi) = F_\alpha(\xi) a(\xi) \hat{g}(\xi), \quad (2.2)$$

where the continuous function  $F_\alpha(\xi)$  is called the filter with a regularization parameter  $\alpha$ .

If the filter  $F_\alpha(\xi)$  satisfies the following conditions:

$$\sup_{\xi} |F_\alpha(\xi) a(\xi)| = C(\alpha) < \infty, \forall \xi \in \mathbb{R}^n,$$

$$\begin{aligned} |F_\alpha(\xi)| &\leq \eta, \forall \xi \in \mathbb{R}^n \\ \lim_{\alpha \rightarrow 0} F_\alpha(\xi) &= 1, \forall \xi \in \mathbb{R}^n, \end{aligned}$$

where  $C(\alpha) \rightarrow 0$  as  $\alpha \rightarrow 0$ ,  $\eta$  is a positive constant (which will be used in Theorem 2.3), then (2.2) is called a regularization strategy (or method).

Furthermore, if  $F_\alpha(\xi)$  satisfies the following conditions with  $k > 0$ ,  $s = p/k > 0$ ,  $0 < p \leq p_0$  ( $p_0 > 0$ ):

$$\sup_{\xi} |F_\alpha(\xi)a(\xi)| \leq C_0\alpha^{-k}; \quad (2.3)$$

$$\sup_{\xi} |(1 - F_\alpha(\xi))a^{-s}(\xi)| \leq C_1\alpha^p, \quad (2.4)$$

where  $C_0, C_1$  are positive constants, in this case there holds error estimate for the method (2.2):

$$\|\hat{f}(\cdot) - R_\alpha \hat{g}^\delta\| \leq C_2 \delta^{1 - \frac{k}{p+k}} E^{\frac{k}{p+k}}, C_2 \geq 1.$$

And the method (2.2) is called an order-optimal regularization method.

## 2.2 Error estimate: the a-priori parameter choice

In this subsection, we will prove the method (2.2) where  $F_\alpha(\xi)$  satisfies (2.3) and (2.4) is order-optimal under the a-priori parameter choice rule for  $\alpha$ .

**Theorem 2.1.** *Let the exact solution  $f(x)$  fulfill the source condition (2.1), then the regularization method (2.2) is order-optimal for  $s = p/k > 0$  if the regularization parameter  $\alpha$  is given by*

$$\alpha = C_3(\delta/E)^{\frac{1}{k(s+1)}}. \quad (2.5)$$

**Proof.** By the Parseval's equality in Fourier analysis and (2.3), we have

$$\begin{aligned} \|R_\alpha \hat{g} - R_\alpha \hat{g}^\delta\| &= \|F_\alpha(\xi)a(\xi)\hat{g}(\xi) - F_\alpha(\xi)a(\xi)\hat{g}^\delta(\xi)\|_{L^2} \\ &\leq \sup_{\xi} |F_\alpha(\xi)a(\xi)| \cdot \|\hat{g}^\delta(\xi) - \hat{g}(\xi)\| \leq C_0\alpha^{-k}\delta. \end{aligned} \quad (2.6)$$

From (2.4), we find that

$$\begin{aligned} \|R_\alpha \hat{g} - \hat{f}(\xi)\| &= \|F_\alpha(\xi)a(\xi)\hat{g}(\xi) - \hat{f}(\xi)\|_{L^2} = \|F_\alpha(\xi, y)\hat{f}(\xi) - \hat{f}(\xi)\|_{L^2} \\ &\leq \sup_{\xi} |(1 - F_\alpha(\xi))a^{-s}(\xi)| \cdot \|\omega\| \leq C_1\alpha^{ks}E. \end{aligned}$$

If we take  $\alpha = \tilde{C}(\delta/E)^{\frac{1}{k(s+1)}}$  and noting  $s = p/k$ , then

$$\|\hat{f}(\xi) - R_\alpha \hat{g}^\delta\| \leq C\delta^{\frac{s}{s+1}} E^{\frac{1}{s+1}} = C\delta^{1 - \frac{k}{p+k}} E^{\frac{k}{p+k}}.$$



By Parseval's equality, this completes the proof.  $\square$

However, the exact bound  $E$  in (2.5) is seldom known. An incorrect guess on this value will lead to a bad approximate solution. Therefore, we turn to consider the a-posteriori parameter choice rule.

### 2.3 Error estimate : the a-posteriori parameter choice

According to Morozov's discrepancy principle, the regularization parameter  $\alpha(\delta, \hat{g}^\delta)$  is determined by the following equation with a fixed  $\tau > 1 + \eta$ :

$$\|a^{-1}(\xi)R_\alpha\hat{g}^\delta - \hat{g}^\delta\| = \|F_\alpha(\xi)\hat{g}^\delta(\xi) - \hat{g}^\delta(\xi)\| = \tau\delta. \quad (2.7)$$

Generally the above equation has a solution for  $\alpha$ . Let us refer to [4].

**Theorem 2.2.** *Let the exact solution  $f(x)$  fulfill the source condition (2.1), then the regularization method (2.2) is order-optimal if the regularization parameter  $\alpha$  is determined by (2.7).*

**Proof.** By the triangle inequality, we have

$$\begin{aligned} \|(F_\alpha(\xi) - 1)\hat{g}(\xi)\| &\leq \|(F_\alpha(\xi) - 1)(\hat{g}(\xi) - \hat{g}^\delta(\xi))\| + \|(F_\alpha(\xi) - 1)\hat{g}^\delta(\xi)\| \\ &\leq (C + 1)\delta + \tau\delta = \tilde{C}\delta. \end{aligned}$$

By the relationship  $R_\alpha\hat{g}(\xi) = F_\alpha(\xi)\hat{f}$ , and the source condition  $\hat{f} = a^{-s}(\xi)\omega$ , it yields

$$\begin{aligned} \|R_\alpha\hat{g}(\xi) - \hat{f}\| &= \|(F_\alpha(\xi) - 1)\hat{f}(\xi)\| = \|(F_\alpha(\xi) - 1)a^{-s}(\xi)\omega\| \\ &\leq \|(F_\alpha(\xi) - 1)\omega\|^{\frac{1}{s+1}} \|(F_\alpha(\xi) - 1)a^{-(s+1)}(\xi)\omega\|^{\frac{s}{s+1}}, \end{aligned}$$

where we used the Hölder inequality. On the other hand,

$$\begin{aligned} \|(F_\alpha(\xi) - 1)\hat{g}(\xi)\| &= \|(F_\alpha(\xi) - 1)a^{-1}(\xi)\hat{f}(\xi)\| \\ &= \|(F_\alpha(\xi) - 1)a^{-(s+1)}(\xi)\omega\|. \end{aligned}$$

Therefore

$$\|R_\alpha\hat{g}(\xi) - \hat{f}\| \leq C_3\|\omega\|^{\frac{1}{s+1}}(\tilde{C}\delta)^{\frac{s}{s+1}} \leq \tilde{C}_1 E^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}}. \quad (2.8)$$

Due to (2.6), we have

$$\|R_\alpha\hat{g}^\delta(\xi) - R_\alpha\hat{g}(\xi)\| \leq C_0\alpha^{-k}\delta. \quad (2.9)$$

Now we need to give a lower bound for  $\alpha$  in (2.9).

From the definition of  $\alpha$  in (2.7), it yields

$$\|a^{-1}(\xi)R_{2\alpha}\hat{g}^\delta - \hat{g}^\delta\| = \|F_{2\alpha}(\xi)\hat{g}^\delta(\xi) - \hat{g}^\delta(\xi)\| > \tau\delta.$$

By triangle inequality we have

$$\|a^{-1}(\xi)R_{2\alpha}\hat{g} - \hat{g}\| \geq \|a^{-1}(\xi)R_{2\alpha}\hat{g}^\delta - \hat{g}^\delta\| - \|a^{-1}(\xi)R_{2\alpha}\hat{g} - \hat{g} - (a^{-1}(\xi)R_{2\alpha}\hat{g}^\delta - \hat{g}^\delta)\|. \quad (2.10)$$

The second term on the right-hand side of (2.10) satisfies

$$\|a^{-1}(\xi)R_{2\alpha}\hat{g} - \hat{g} - (a^{-1}(\xi)R_{2\alpha}\hat{g}^\delta - \hat{g}^\delta)\| = \|(F_{2\alpha}(\xi) - 1)(\hat{g}^\delta - \hat{g})\| \leq (\eta + 1)\delta.$$

Therefore from (2.10), we have

$$\|a^{-1}(\xi)R_{2\alpha}\hat{g} - \hat{g}\| \geq (\tau - \eta - 1)\delta,$$

which implies

$$\begin{aligned} \delta &\leq C_3\|a^{-1}(\xi)R_{2\alpha}\hat{g} - \hat{g}\| = C_3\|(F_{2\alpha}(\xi) - 1)\hat{g}\| \\ &= C_3\|(F_{2\alpha}(\xi) - 1)a^{-1}(\xi)a^{-s}(\xi)\omega\| \\ &\leq C_3 \sup_{\xi} |(F_{2\alpha}(\xi) - 1)a^{-(s+1)}(\xi)| \cdot \|\omega\| \leq C_3(2\alpha)^{k(s+1)}E. \end{aligned} \quad (2.11)$$

From (2.11), the lower bound for  $\alpha$  is given by

$$\alpha \geq \tilde{C}_2 \left(\frac{\delta}{E}\right)^{\frac{1}{k(s+1)}}.$$

Therefore, (2.9) becomes

$$\|R_{\alpha}\hat{g}^\delta(\xi) - R_{\alpha}\hat{g}(\xi)\| \leq C_4 E^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}}. \quad (2.12)$$

Combining with (2.8), we have

$$\|R_{\alpha}\hat{g}^\delta(\xi) - \hat{f}(\xi)\| \leq C_5 E^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}} = C_5 \delta^{1-\frac{k}{k+p}} E^{\frac{k}{k+p}}. \quad (2.13)$$

Finally by Parseval's equality  $\|\hat{h}\| = \|h\|$ , the proof completes.  $\square$

## 2.4 Applications

There are many specific regularization methods that can be cast into the framework of the general regularization method. In this subsection, we list some order-optimal regularization methods. Especially, we give a new fractional Tikhonov regularization method which is different from the newly-developed fractional regularization methods [1, 8, 13, 16]. To verify the conditions (2.3) and (2.4) for the regularization methods, we need a simple inequality [24].

**Lemma 2.1.** *If  $0 < \mu < \nu$ ,  $C > 0$ ,  $z \geq 0$ , there holds*

$$\sup_{z \geq 0} \frac{z^\mu}{1 + Cz^\nu} \leq \frac{\nu - \mu}{\nu} \left( \frac{\mu}{C(\nu - \mu)} \right)^{\mu/\nu}.$$

**Example 2.1 (Fourier method).** According to [6], let  $\chi_\alpha(\xi)$  be the characteristic function on  $|\xi| \leq 1/\alpha$  with  $1/\alpha = \nu$ , and define the Fourier regularization solution with noisy data in the frequency domain as follows:

$$\hat{f}_{\delta, \alpha}^F = a(\xi) \hat{g}_\delta(\xi) \chi_\alpha(\xi).$$

Comparing with (2.2), we can see that this method is characterized by the choice  $F_\alpha(\xi) = \chi_\alpha(\xi)$ . By (1.5), it is easy to verify the  $F_\alpha(\xi)$  satisfies the conditions (2.3) and (2.4), i.e.,

$$\begin{aligned} \sup_{|\xi| \leq 1/\alpha} |a(\xi)| &\leq \sup_{|\xi| \leq 1/\alpha} \tilde{C}_2 |\xi|^k \leq \tilde{C}_2 \alpha^{-k}; \\ \sup_{|\xi| > 1/\alpha} |a^{-p/k}(\xi)| &\leq \sup_{|\xi| > 1/\alpha} \tilde{C}_1^{-p/k} |\xi|^{-p} \leq C \alpha^p. \end{aligned}$$

**Example 2.2. (Tikhonov method)** This method is characterized by the choice  $F_\alpha(\xi) = \frac{1}{1 + \alpha^{2k} |a(\xi)|^2}$ , then we get the classical Tikhonov regularization solution with noisy data in the frequency domain given by:

$$\hat{f}_{\delta, \alpha}^T = \frac{a(\xi)}{1 + \alpha^{2k} |a(\xi)|^2} \hat{g}_\delta(\xi).$$

Now we can verify that the  $F_\alpha(\xi)$  satisfies the conditions (2.3) and (2.4) by using (1.5) and Lemma 2.1, i.e.,

$$\begin{aligned} \sup_{\xi} \left| \frac{a(\xi)}{1 + \alpha^{2k} |a(\xi)|^2} \right| &\leq \frac{1}{2} \alpha^{-k}; \\ \sup_{\xi} \left| \frac{\alpha^{2k} |a(\xi)|^2}{1 + \alpha^{2k} |a(\xi)|^2} a^{-p/k}(\xi) \right| &\leq \alpha^p \sup_{\xi} \left| \frac{\alpha^{2k-p} |\xi|^{2k-p}}{1 + \tilde{C}_1^2 \alpha^{2k} |\xi|^{2k}} \right| \leq C \alpha^p, \quad 0 < p \leq 2k. \end{aligned}$$

**Example 2.3. (New fractional Tikhonov method)** Motivated by [27], we take  $F_\alpha(\xi) = \frac{1}{1 + \alpha^{\gamma k} |a(\xi)|^\gamma}$  with  $1 \leq \gamma \leq 2$ , then we get the new fractional Tikhonov regularization solution with noisy data in the frequency domain :

$$\hat{f}_{\delta, \alpha}^{NFT} = \frac{a(\xi)}{1 + \alpha^{\gamma k} |a(\xi)|^\gamma} \hat{g}_\delta(\xi). \quad (2.14)$$

Similarly we can verify the  $F_\alpha(\xi)$  satisfies the conditions (2.3) and (2.4) by using (1.5) and Lemma 2.1, i.e.,

$$\sup_{\xi} \left| \frac{a(\xi)}{1 + \alpha^{\gamma k} |a(\xi)|^\gamma} \right| \leq \tilde{C}_2^k \alpha^{-k} \sup_{\xi} \left| \frac{\alpha^k |\xi|^k}{1 + \tilde{C}_1^\gamma \alpha^{\gamma k} |\xi|^{\gamma k}} \right| \leq \tilde{C} \alpha^{-k};$$

$$\sup_{\xi} \left| \frac{\alpha^{\gamma k} |a(\xi)|^{\gamma}}{1 + \alpha^{\gamma k} |a(\xi)|^{\gamma}} a^{-p/k}(\xi) \right| \leq \alpha^p \sup_{\xi} \left| \frac{\alpha^{\gamma k - p} |\xi|^{\gamma k - p}}{1 + \tilde{C}_1^{\gamma} \alpha^{\gamma k} |\xi|^{\gamma k}} \right| \leq C \alpha^p, \quad 0 < p \leq \gamma k.$$

**Remark 2.1.** According to the idea in [8, 16], the existing fractional Tikhonov method is given by

$$\hat{f}_{\delta, \alpha}^{FT} = \frac{a(\xi)}{(1 + \alpha |a(\xi)|)^{\gamma}} \hat{g}_{\delta}(\xi),$$

where  $\gamma \geq 1$  and  $\alpha$  is the regularization parameter. Therefore the new fractional Tikhonov method in Example 2.3 is different from the method given in [8, 16].

### 3 Numerical examples

The fractional Tikhonov method is of particular interest for the problems where the sharp (or fine) features or the discontinuities of the solutions are of high priority to reconstruct. However, the classical Tikhonov method generally over-smooths the solution. In this section, we give some examples to show the validity of the new fractional Tikhonov method, especially we give the comparison with the classical Tikhonov method. The new fractional Tikhonov method ( $1 \leq \gamma < 2$ ) and the classical Tikhonov method ( $\gamma = 2$ ) are implemented in the Fourier frequency domain which have a uniform form (2.14). Therefore the computational cost for two methods is the same.

We give two ill-posed problems which satisfy (1.5) and apply the classical Tikhonov method and new fractional Tikhonov method for solving them. One is the numerical differentiation problem in one-dimensional setting, the second is the image de-blurring problem in two-dimensional case.

**Example 3.1.** Let's consider the  $\beta$ -th order numerical differentiation problem. We want to compute the  $\beta$ -th order numerical derivative from the noisy data. Suppose the function  $f(x) \in H^{\beta}(\mathbb{R})$ ,  $\beta > 0$ . The problem can be formulate as the

$$f^{(\beta)}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} (i\xi)^{\beta} \hat{f}(\xi) d\xi := A(D)f(x).$$

The symbol of  $A(D)$  is  $a(\xi) = (i\xi)^{\beta}$  which satisfies (1.5) with  $k = \beta$ . The fractional Tikhonov regularization solution is written as

$$\hat{f}^{NFT} = \frac{a(\xi)}{1 + \alpha^{\gamma k} |a(\xi)|^{\gamma}} \hat{f}(\xi), \quad 1 \leq \gamma < 2. \quad (3.1)$$

When  $\gamma = 2$ , the solution (3.1) is the classical Tikhonov solution. Inverse Fourier transform on (3.1) gives the regularization solution in the physical domain.

In this example, the data function  $f$  is sampled over a finite interval, i.e., the noise-free data vectors are  $F$  with the size  $m = 100$ . The noisy data vectors  $F^\delta$  are generated according to

$$F^\delta = F + \tilde{\delta} \cdot \max\{F\} \cdot \text{randn}(\text{size}(F)),$$

where  $\text{randn}(\text{size}(F))$  creates the random number which obeys the Gaussian distribution with mean 0, variance  $\sigma^2 = 1$  in MATLAB,  $\tilde{\delta}$  denotes the noise level added to the data. The error level  $\delta$  is computed by the discrete  $L^2$ -norm  $\|F^\delta - F\|$ .

Since the classical Tikhonov method recovers smooth solutions very well, but fails if the solution has discontinuities. We let  $f(x)$  be

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1/2, \\ 1 - x, & \text{if } 1/2 < x \leq 1. \end{cases}$$

And we can easily compute the first-order derivative:

$$f'(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1/2, \\ -1, & \text{if } 1/2 < x \leq 1. \end{cases}$$

The numerical results are shown in Fig. 1 with  $\tilde{\delta} = 0.01$  and the same regularization parameter  $\alpha = 0.02$  for two methods. From Fig.1, we can see that the approximation of the discontinuity by the new fractional Tikhonov method is better than the approximation by the classical Tikhonov method. From Fig.1a and Fig. 1b, the smaller  $\gamma$  is, the better is the approximation of the discontinuity. However, the new fractional Tikhonov method with smaller  $\gamma$  causes heavy oscillations of the approximate solution. This is because the less filter is used and the regularized solution cannot control the data error better.

**Example 3.2.** We consider the image de-blurring problem  $Bf = g$  with a known shift-invariant point spread function (psf)  $b(x, y)$  in the space  $L^2(\mathbb{R}^2)$ ,

$$Bf := b(x, y) \star f(x, y) = g(x, y), \quad (3.2)$$

where  $\star$  denotes convolution,  $g(x, y)$  is the noisy blurred image which is the sum of the exact blurred image  $g_e(x, y)$  and the noise  $n(x, y)$ . Assume the exact sharp image is given by  $f_e(x, y)$ , we have

$$Bf := b(x, y) \star f_e(x, y) = g_e(x, y).$$

In practice, we only have the equation (3.2) and have to seek an approximate solution  $f(x, y)$  satisfies  $Bf \approx g$ . Here we consider the case of uniform de-focus blur, where the psf is proportional to the characteristic function of a disc of radius  $D$ . If  $D > 0$  is the radius of the “circle of confusion”, the psf for de-focus blur is given by

$$b(x, y) = \begin{cases} (\pi D^2)^{-1}, & \text{if } x^2 + y^2 \leq D^2, \\ 0, & \text{if } x^2 + y^2 > D^2. \end{cases}$$

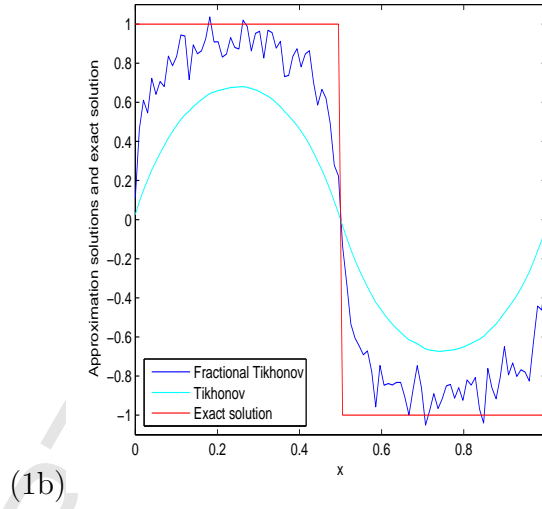
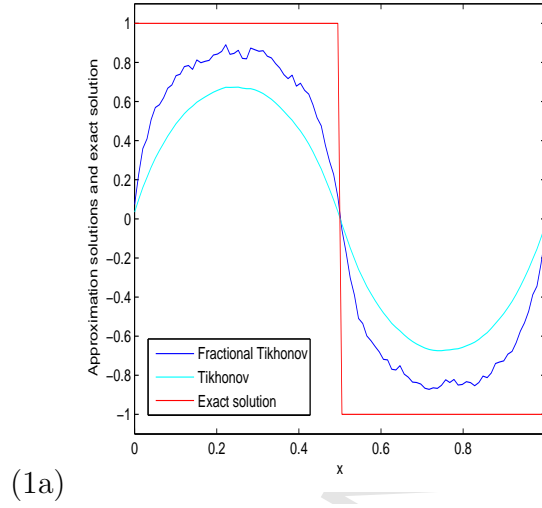


Figure 1: Example 3.1. Top(1a):  $\alpha = 0.02$ ,  $\gamma = 1.5$  (Fractional Tikhonov),  $\gamma = 2$  (Tikhonov). Bottom(1b):  $\alpha = 0.02$ ,  $\gamma = 1.2$  (Fractional Tikhonov),  $\gamma = 2$  (Tikhonov).

The  $b(x, y)$  has the Fourier transform which is given by the “sombbrero function” [7]

$$\hat{b}(\xi) = 2J_1(D|\xi|)/(D|\xi|), \quad |\xi| = \sqrt{\xi_1^2 + \xi_2^2},$$

where  $J_1(\cdot)$  is the Bessel function of the first kind with order 1. From (3.2), in the frequency domain we get

$$\hat{b}(\xi)\hat{f}(\xi) = \hat{g}(\xi),$$

which yields

$$\hat{f}(\xi) = \hat{b}^{-1}(\xi)\hat{g}(\xi) := a(\xi)\hat{g}(\xi).$$

According to the properties of Bessel function, we have  $\hat{b}^{-1}(\xi) = O((D|\xi|)^{3/2})$  as  $|\xi| \rightarrow \infty$  and  $\hat{b}^{-1}(\xi) = O(1)$  as  $|\xi| \rightarrow 0$ . Thus this model can be cast into the framework (1.4) and (1.5) with  $k = 3/2$ . Now the new fractional Tikhonov regularization solution is written:

$$\hat{f}^{NFT}(\xi) = \frac{a(\xi)}{1 + \alpha^{\gamma k} |a(\xi)|^{\gamma}} \hat{g}(\xi), 1 \leq \gamma < 2.$$

When  $\gamma = 2$ , the solution is the classical Tikhonov solution. Inverse Fourier transform gives the regularization solution in the physical domain.

Now *Text* image is the exact solution for our test. The original *Text* image  $f_e(x, y)$  consists of a  $256 \times 256$  array which is displayed in Fig 2a. Blurring was accomplished in the Fourier domain by using two-dimensional fast Fourier transform algorithm. Let  $n(x, y) = \tilde{\delta} \cdot \max\{g(x, y)\} \sigma(x, y)$  where  $\sigma(x, y)$  obeys the Gaussian distribution with mean 0, variance 1 in MATLAB,  $\tilde{\delta} = 1\%$ . The result  $g(x, y) = g_e(x, y) + n(x, y)$  is displayed in Fig 2b.

The numerical results are shown in Figure 3 with  $\tilde{\delta} = 1\%$  and the same regularization parameter  $\alpha = 0.1$  for two methods. From Fig.3, we can see that the new fractional Tikhonov method is better than the classical Tikhonov method. The classical Tikhonov method fails because its approximation is too smooth.

## 4 Concluding remarks

This article proposes and analyzes a new general regularization framework for solving ill-posed problems associated with pseudo-differential operators. The theoretical error estimates are proved under the a-priori and the a-posteriori regularization parameter strategies. The error bounds are order-optimal under the usual smoothness source conditions. Some specific classical regularization methods such as Fourier method and Tikhonov method are cast into the framework of regularization methods. Especially, we derive a new fractional Tikhonov method. From the numerical experiments, it is shown that the proposed fractional method works better than the classical one.

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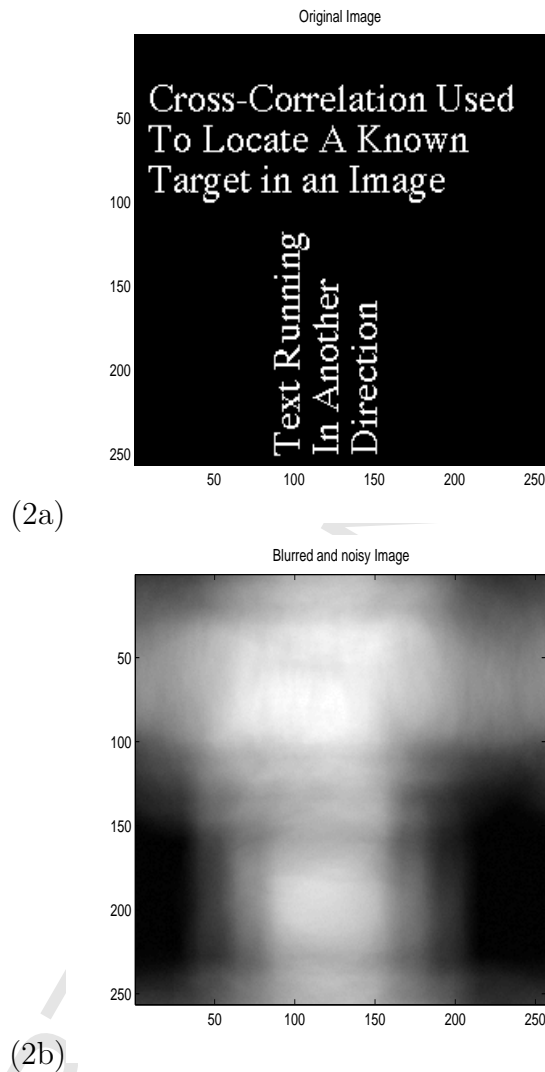


Figure 2: Example 3.2. Top(2a): the original image. Bottom(2b): the blurred and noisy image with  $D = 0.25$ ,  $\tilde{\delta} = 1\%$ .

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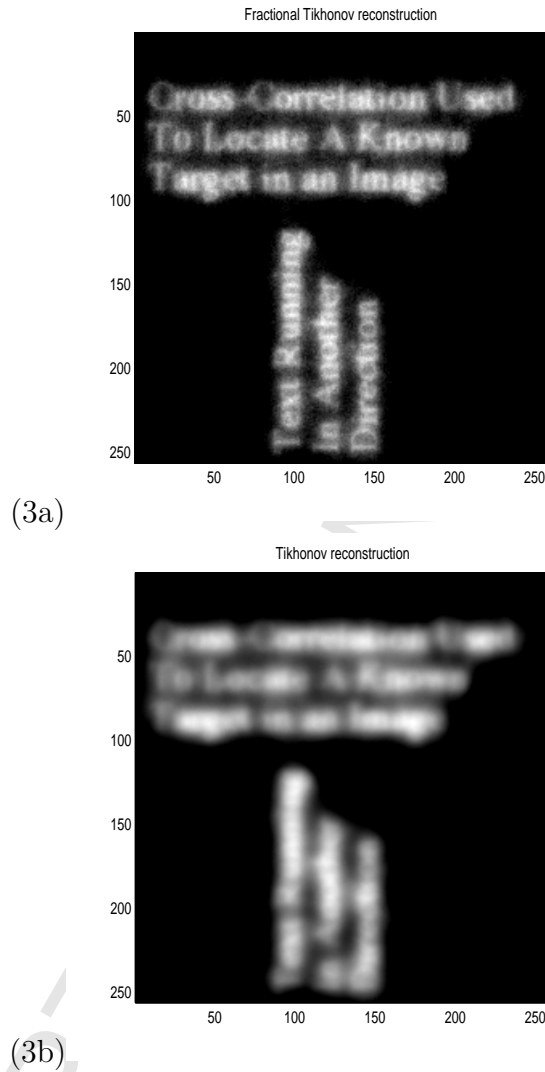


Figure 3: Example 3.2. Top(3a):  $\alpha = 0.1$ ,  $\gamma = 1.2$  (Fractional Tikhonov). Bottom(3b):  $\alpha = 0.1$ ,  $\gamma = 2$  (Tikhonov).

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