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A new class of two-step P-stable TFPL methods for the numerical solution of second order IVPs with oscillating solutions

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Abstract

A new class of two-step linear symmetric methods are introduced in this paper for the numerical solution of second order IVPs that having highly oscillatory solutions. In this class, for the first time in the literature, we calculate the coefficients of the method by composition of TF (trigonometrically-fitting) and VSDPL (vanished some of derivatives of phase-lag) technique, and we construct the new class of methods which have both properties of TF methods and VSDPL methods, we say TFPL methods. This method is of algebraic order 8 and has an important P-stability property. The main structure of the method is multiderivative, and the combined phases have been applied for expanding stability interval and for achieving P-stability. The advantage of the method in comparison with similar method, in terms of efficiency, accuracy and stability, has been showed by the implementation of it in some important problems, undamped Duffing equation, etc.

Key words: Phase-fitting, Phase-lag, Ordinary differential equations, P-stable, Multiderivative methods.

1. Introduction and Preliminaries

In this paper, we are concerned with the numerical integration of second order ODEs modeled by initial value problems of the form

$$y'' = f(x,y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1}$$

whose solutions exhibit a pronounced oscillatory character. Such problems are usually encountered in many scientific researches, engineering applications and so on. If the exact solutions of these equations are not available, the numerical solutions become very important and interesting. However, it is difficult to choose the most suitable method for a given oscillatory problem with frequencies in the solution. Obviously, the analysis of phase properties for numerical methods is of prime importance when dealing with an oscillatory problem. The main aim of this paper is the construction of fast and reliable methods such as multiderivative methods for the solution of the second-order initial value problems (IVPs) with oscillating solutions and related problems (1). These methods can be divided into two main categories: methods with constant coefficients and methods with coefficients depending on the frequency of the problem. Moreover, the second class of above methods also can be divided into two classes of problems: problems which the frequency ω is given (even approximately) and problems which the frequency ω is not known. Our method in this paper was designed for the numerical solution of the problems which its frequency ω is given (even approximately). Note that to solve problems with unknown frequency ω, the determination of ω is a critical issue, as was shown in the article by Vigo-Aguiar and Ramos [11]. The knowledge of an estimation to the unknown frequency ω is needed in order to apply the numerical method efficiently, since its coefficients depend on the value of this parameter. Usually, the value for the frequency ω that appears in the trigonometrically fitted and vanished phase-lag and some

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of its derivatives methods is chosen near the angular frequency ω, but this is not the best choice as has been shown in the numerical examples in [11]. In order to provide an estimation of the parameter, Vigo-Aguiar and Ramos in [11] consider two formulations of a trigonometrically fitted method of Falkner-type for solving nonlinear oscillators, and they present a strategy for the practical estimation of the parameter. Their estimation is based on the minimization of the total energy of the system over a selected interval corresponding to a few times the period.

The last decades much research has been done on the construction of efficient, fast and reliable algorithms for the approximate solution of (1) and related problems because these problems are usually encountered in celestial mechanics, quantum mechanical scattering theory, theoretical physics and chemistry, and electronics. Generally, the solution of (1) is periodic, so it is expected that the result produced by some numerical methods preserves the analogical periodicity of the analytic solution. In the following, we mention some bibliography:

- Multistep methods with vanishing of phase-lag and some of its derivatives have been obtained in [2, 12, 13, 14, 15].
- In [1,10,14-25] minimal phase-lag, exponentially and trigonometrically fitted methods are constructed.
- Review papers have been presented in [3-9] and [26-31].

Consider the multiderivative method of the form (see for details [14]):

$$\sum_{i=0}^{k} \alpha_i y_{n-j+1} = \sum_{i=1}^{l} h^{2i} \sum_{j=0}^{k} \beta_{ij} y_{n-j+1}^{(2i)}, \tag{2}$$

for the numerical integration of the problem (1). The method (2) is symmetric when $\alpha_j = \alpha_{k-j},\ \beta_j = \beta_{k-j},\ j = 0, 1, 2, \cdots, k$, and it is of order q if the truncation error associated with the linear difference operator is given as

$$TE = C_{q+2} h^{q+2} y^{(q+2)}, \qquad x_{n-k+1} < \eta < x_{n+1},$$

where C_{q+2} is a constant dependent on h. To investigate the stability properties of the methods for solving the initial value problem (1), Lambert and Watson [5] introduced the scalar test equation

$$y'' = -\omega^2 y, \quad \omega \in \mathbb{R}.$$

When the method (2) is applied to the test equation, we get the characteristic equation as

$$\rho(\xi) - \sum_{i=1}^{l} (-1)^i v^{2i} \sigma_i(\xi) = 0, \tag{3}$$

where $v = \lambda h$ and

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^{k-j}, \quad \sigma_i(\xi) = \sum_{j=0}^{k} \beta_{ij} \xi^{k-j}, \quad i = 1, 2, \cdots, l. \tag{4}$$

Definition 1.1. The method (2) is said to have interval of periodicity $(0, v_0^2)$ if for all $v^2 \in (0, v_0^2)$ the roots of Eq. (3) are complex and at least two of them lie on the unit circle and the others lie inside the unit circle and the method (2) is said to be P-stable if its interval of periodicity is $(0, \infty)$.

Definition 1.2. For any symmetric multistep methods, the phase-lag (frequency distortion) of order q is given by

$$t(v) = v - \theta(v) = C v^{q+1} + O(v^{q+2}), \tag{5}$$

where C is the phase lag constant and q is the phase-lag order.

The characteristic equation of the method (2) is given by

$$\Omega(s:v^2) = A(v)s^2 - 2B(v)s + A(v) = 0, \tag{6}$$

where

$$A(v) = 1 + \sum_{i=1}^{m}(-1)^i \beta_{i0} v^{2i}, \quad B(v) = 1 + \sum_{i=1}^{m}(-1)^i \beta_{i1} v^{2i}. \tag{7}$$

During the recent decades, in the basis of these classes, varietal methods presented by different people, that we ensign some important types of them. The papers had written by Achar [2], Wang et al [31], Quinlan et al [10] and Lambert and Watson [5] in which these are relate to grade class 1 and in all of them the coefficient obtained from Taylor series of their difference operators, and each one of these methods that mentioned, consist of properties, that we can refer to related papers for more details. At 1961, implementation of the exponential basis of the form

$$\left\{1, x, \cdots, x^K, e^{\pm\mu x}, \cdots, x^P e^{\pm\mu x}\right\}, \tag{8}$$

(which it coefficients be as a result of coefficients output depends on the frequency), that at first discuss by Goutshi [4] which in that time there was no support, but after 1990, mathematicians focus their attention on this subject and different people presented notable things in this required, which they are relate to the class 2 (if $\mu = i\omega$, then the biases used in this regard will be trigonometric which $i = \sqrt{-1}$). An important property of trigonometrically fitted algorithms is that they tend to the corresponding classical ones when the involved frequencies tend to zero, a fact which allows us to say that trigonometrical fitting represents a natural extension of the classical polynomial fitting. The examination of the convergence of trigonometrical and exponential fitted multistep methods is included in Lyches theory [1]. There is a large number of significant methods presented with high practical importance that have been presented in the bibliography. Therefore as an example we can mention the papers presented by Simos [16], Vigo-Aguiar et al [11] and Shokri et al [12, 13], and Wang et al [31]. Implementation of phase-lag error and some of its derivatives to linear multistep methods, at first presented by Brusa et al [3], that in fact the angle between the analytical solution and the numerical solution. But when we speak about the problems which have periodic or oscillatory solutions, the accuracy of the numeral methods (in terms of quality, not algebraic order position) is very important. In other words, using trigonometric basis are suitable when we have good approximation of the frequency and because of it, small perturbation at the frequency, cause to the critical disturbance in solution so, it is better in these problems, furthermore we consider the phase-lag of the numeral method too. Vlachos et al suggested, avoiding form this matter, we can obtain the coefficients of methods by solving of obtained nonlinear system from vanishing of phase-lag and some of its derivatives [30]. Some of other peoples have been published some papers in this field, for instance we can mention the papers that presented by Shokri [14]. Our Main idea in this paper is producing symmetric Obrechkoff multistep method for the numerical solving of general differential equation systems that have oscillatory solutions. Now for the first time in the literature, we produce the coefficients of our method by solving of system through combination of trigonometrically fitting and vanishing of phase-lag and some of its derivatives in which besides enhancing level of algebraic order and enlarge the stability region of method, improve produced approximations in terms of quality. More accurately, this method that we discuss about it, is a implicit two-step method of eight algebraic order and consists of very important property of P-stability that we can reach it through solving of system of four nonlinear equation arising from one trigonometric basis (one equation) and vanishing of phase-lag and its first, second and third derivations (three equations).

This paper is organized as follows. In Section 2, we derive a new two-step multiderivative method for the numerical integration of (1). The coefficients of the new method are calculated, for the first time in the literature, by combination of TF (trigonometrically fitted) methods and vanishing of phase-lag and its derivatives technique, we say TFPL method. In Section 3, the numerical experiments are reported. Finally, we are devoted to some conclusive remarks.

2. Development and analysis of TFPL method

2.1. Development

From the form (2) and without loss of generality we assume $\alpha_j = \alpha_{m-j}$, $\beta_{i,j} = \beta_{i,m-j}$, $j = 0(1)\lfloor \frac{m}{2} \rfloor$ and we can write

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^{m} h^{2i} \left[\beta_{i0} y_{n+1}^{(2i)} + \beta_{i1} y_n^{(2i)} + \beta_{i0} y_{n-1}^{(2i)}\right], \tag{9}$$

when $m = 2$ we get

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} =& h^2 \left[\beta_{10}(y_{n+1}^{(2)} + y_{n-1}^{(2)}) + \beta_{11} y_n^{(2)}\right] \\ &+ h^4 \left[\beta_{20}(y_{n+1}^{(4)} + y_{n-1}^{(4)}) + \beta_{21} y_n^{(4)}\right]. \end{aligned} \tag{10}$$

$M-3$ for method (10) is 7 so that if $P = -1$, $K = 9$ we obtain classic method and the coefficients of this method are

$$\beta_{1,0} = \frac{11}{252}, \quad \beta_{1,1} = \frac{115}{126}, \quad \beta_{2,0} = -\frac{13}{15120}, \quad \beta_{2,1} = \frac{313}{7560}, \tag{11}$$

and its local truncation error is given by

$$LTE_{clas} = \frac{59}{76204800} y^{(10)} h^{10} + O\left(h^{12}\right).$$

If $P = 4$, $K = -1$ we obtain the method with zero phase-lag (PL), and the coefficients of this case are given in [13]. Moreover h is the step-length of the integration, n is the number of steps, y_n is the numerical solution on the point x_n, $x_n = x_0 + nh$ and x_0 is the starting point of integration. The difference equation (10), includes four free parameters that must be calculated. Obviously, all the characteristics expected from the method will be achieved by calculation of β_{ij}, ($i = 1, 2$ and $j = 0, 1$). To this, in this paper, β_{10} is calculated through trigonometrically-fitting (i.e. with trigonometric basis $\cos(v)$, where $v = \omega h$ with frequency ω and step-length h). Then we have

$$\beta_{10} = \frac{2v^4 \cos(v) \beta_{20} + \beta_{21} v^4 - v^2 - 2\cos(v) + 2}{2v^2(\cos(v) - 1)}, \tag{12}$$

and the rest of the free parameters by the manufacturer system through vanishing of phase-lag and its first and second derivatives. Then we have

$$\begin{aligned} PL &= \frac{\left(2 - 2\beta_{2,0} v^4 + (-\beta_{1,1} + 1) v^2\right) \cos(v) - \beta_{2,1} v^4 + \beta_{1,1} v^2 - 2}{(2\beta_{2,0} + \beta_{2,1}) v^6 - v^4} = 0, \\ PL' &= \frac{T_1}{v^5 (2v^2 \beta_{20} + v^2 \beta_{21} - 1)^2} = 0, \\ PL'' &= \frac{T_2}{v^6 \left(2v^2 \beta_{20} + v^2 \beta_{21} - 1\right)^3} = 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} T_1 =& \left(8 + \left(8{\beta_{20}}^2 + 4\beta_{20}\beta_{21}\right) v^6 + 8(\beta_{11} - 1)(1/2\beta_{21} + \beta_{20}) v^4 + (-24\beta_{20} - 2\beta_{11} - 12\beta_{21} + 2) v^2\right) \cos(v) \\ &+ 4\left(-1 + \beta_{20} v^4 + (1/2\beta_{11} - 1/2) v^2\right)\left(-1/2 + (1/2\beta_{21} + \beta_{20}) v^2\right) v \sin(v) \\ &- 8 + \left(4\beta_{20}\beta_{21} + 2{\beta_{21}}^2\right) v^6 - 8\beta_{11}(1/2\beta_{21} + \beta_{20}) v^4 + (24\beta_{20} + 2\beta_{11} + 12\beta_{21}) v^2, \end{aligned}$$

$$T_2 = \Big(40 + 8\left(\frac{1}{2}\beta_{21} + \beta_{20}\right)^2 \beta_{20} v^{10} - 48\left(\beta_{20}^2 + \left(-\frac{1}{12}\beta_{11} + \frac{1}{2}\beta_{21} + \frac{1}{4}\right)\beta_{20} - \frac{1}{24}\beta_{21}(\beta_{11} - 1)\right)$$
$$+ \left(\frac{1}{2}\beta_{21} + \beta_{20}\right) v^8 + \Big(\left(-80\beta_{11} + 64\right)\beta_{20}^2 + \left(\left(-80\beta_{11} + 68\right)\beta_{21} - 4\beta_{11} + 6\right)\beta_{20}$$
$$+ \left(-20\beta_{11} + 18\right)\beta_{21}^2 + \left(-2\beta_{11} + 2\right)\beta_{21}\Big) v^6$$
$$+ \left(336\beta_{20}^2 + \left(36\beta_{11} + 336\beta_{21} - 28\right)\beta_{20} + 84\beta_{21}^2 + \left(18\beta_{11} - 14\right)\beta_{21} + \beta_{11} - 1\right) v^4$$
$$+ \left(-216\beta_{20} - 6\beta_{11} - 108\beta_{21} + 4\right) v^2\Big) \cos(v) - 32\left(-\frac{1}{2} + \left(\frac{1}{2}\beta_{21} + \beta_{20}\right) v^2\right) v$$
$$\cdot \left(1 + \left(\frac{1}{2}\beta_{21} + \beta_{20}\right)\beta_{20} v^6 + (\beta_{11} - 1)\left(\frac{1}{2}\beta_{21} + \beta_{20}\right) v^4 + \left(-3\beta_{20} - \frac{1}{4}\beta_{11} - \frac{3}{2}\beta_{21} + \frac{1}{4}\right) v^2\right) \sin(v)$$
$$- 40 - 24\left(\frac{1}{2}\beta_{21} + \beta_{20}\right)^2 \beta_{21} v^8 + 80\left(\beta_{20}\beta_{11} + \frac{1}{2}\left(\beta_{11} - \frac{1}{10}\right)\beta_{21}\right)\left(\frac{1}{2}\beta_{21} + \beta_{20}\right) v^6$$
$$- 336\left(\beta_{20} + \frac{3\beta_{11}}{28} + \frac{1}{2}\beta_{21}\right)\left(\frac{1}{2}\beta_{21} + \beta_{20}\right) v^4 + \left(216\beta_{20} + 6\beta_{11} + 108\beta_{21}\right) v^2.$$

By solving the above system of equations, we obtain the coefficients of the new two-step symmetric multiderivative method that is a combination of trigonometrically-fitted methods and methods based on vanishing of phase-lag and some of its derivatives, (so we called TFPL method) as follow

$$\beta_{10} = \frac{1}{6}\frac{T_3}{T_4}, \quad \beta_{11} = \frac{1}{4}\frac{T_5}{T_6},$$
$$\beta_{20} = \frac{1}{2}\frac{T_7}{T_8}, \quad \beta_{21} = \frac{1}{3}\frac{T_9}{T_{10}},$$

where

$$T_3 = 96(\sin(v))^6 + \left(\left(-20v^3 + 40v\right)\cos(v) + 36v^3 - 184v\right)(\sin(v))^5$$
$$+ \Big(\left(4v^4 - 124v^2 + 96\right)(\cos(v))^2 + \left(-v^6 - 10v^4 - 16v^2\right)\cos(v) - 45v^4 + 188v^2 - 96\Big)(\sin(v))^4$$
$$- 2v\Big(\left(-10v^2 + 40\right)(\cos(v))^3 + \left(v^4 - 53v^2 + 32\right)(\cos(v))^2 + \left(-5v^4 + 6v^2 - 40\right)\cos(v)$$
$$- 10v^4 + 57v^2 - 32\Big)(\sin(v))^3 + \Big(\left(-17v^4 + 48v^2\right)(\cos(v))^3 + \left(3v^6 - 40v^4\right)(\cos(v))^2$$
$$+ \left(17v^4 - 48v^2\right)\cos(v) - 4v^6 + 40v^4\Big)(\sin(v))^2 + 8(\cos(v) - 1)(\cos(v) + 7/4)v^5(\cos(v) + 1)\sin(v)$$
$$- 3(\cos(v) - 1)(\cos(v) + 4/3)v^6(\cos(v) + 1),$$

$$T_4 = (\cos(v) - 1)\left(\left(\left(-\frac{1}{3}v^2 + 2\right)\sin^2(v) - \frac{1}{3}(5\cos(v) + 7)v\sin(v) + v^2(\cos(v) + 1)\right)\sin^2(v)v^3\Big(-3\sin(v)\right.$$
$$\left. + v(\cos(v) + 2)\Big)\right),$$

$$T_5 = \left(-v^5 + 17v^3 + 124v\right)(\sin(v))^4 + \left(\left(-8v^4 - 22v^2 + 120\right)\cos(v) - 16v^4 - 88v^2 - 120\right)(\sin(v))^3$$
$$+ \left(\left(3v^5 - 5v^3 - 64v\right)\cos(v) + 2v^5 + 11v^3 + 64v\right)(\sin(v))^2$$
$$+ \left(\left(4v^4 + 8v^2\right)\cos(v) + 2v^4 - 8v^2\right)\sin(v) + 3v^5(\cos(v) + 1),$$

$$T_6 = \Bigg(\left(-1/4\,v^3 + \frac{21\,v}{4}\right)(\sin(v))^3 + \left(-2\cos(v)\,v^2 - 9/2\,v^2 + 9/2\cos(v) - 9/2\right)(\sin(v))^2 + \left(\left(v^2-3\right)\cos(v) + 5/4\,v^2 + 3\right)v\sin(v) + 1/2\,v^2(\cos(v)-1) \Bigg) v^2\sin(v),$$

$$T_7 = 14\,v(\sin(v))^3 + \left(v^4 - 2\cos(v)\,v^2 - 4\,v^2 + 16\cos(v) - 16\right)(\sin(v))^2 + v^3(\cos(v)+1)\sin(v) - v^4(\cos(v)+1),$$

$$T_8 = (\sin(v))^2\,v^5\,(-3\sin(v) + v(\cos(v)+2)),$$

$$\begin{aligned} T_9 = &-32\,(\sin(v))^5 - v\left(v^4 + 4\,v^2 - 16\cos(v) - 28\right)(\sin(v))^4 \\ &+ \left(\left(-v^4 - 18\,v^2 - 32\right)\cos(v) - 22\,v^2 + 32\right)(\sin(v))^3 \\ &+ \left(\left(2\,v^5 + 8\,v^3\right)\cos(v) + v^5 + 8\,v^3\right)(\sin(v))^2 \\ &- 4\,v^4(\cos(v)+1)\sin(v) + 2\,v^5(\cos(v)+1), \end{aligned}$$

$$T_{10} = \left(\left(-1/3\,v^2 + 2\right)(\sin(v))^2 - 1/3\,(5\cos(v)+7)\,v\sin(v) + v^2(\cos(v)+1)\right)(\sin(v))^2\,v^5.$$

The following Taylor series expansions should be used in the cases that the coefficients are subject to heavy cancelations for some values of $|v|$:

$$\begin{aligned} \beta_{10} = &\frac{11}{252} + \frac{59}{105840}v^2 - \frac{61}{9779616}v^4 - \frac{130457}{177989011200}v^6 - \frac{149491}{4485323082240}v^8 \\ &- \frac{2136118429}{1761386374395648000}v^{10} - \frac{3738192043}{93705755117848473600}v^{12} - \cdots, \end{aligned}$$

$$\begin{aligned} \beta_{11} = &\frac{115}{126} - \frac{59}{52920}v^2 + \frac{61}{4889808}v^4 + \frac{130457}{88994505600}v^6 + \frac{149491}{2242661541120}v^8 \\ &+ \frac{2136118429}{880693187197824000}v^{10} + \frac{3738192043}{46852877558924236800}v^{12} + \cdots, \end{aligned}$$

$$\begin{aligned} \beta_{20} = &-\frac{13}{15120} - \frac{59}{1270080}v^2 - \frac{10579}{5867769600}v^4 - \frac{6103}{100118818800}v^6 - \frac{2579623}{1345596924672000}v^8 \\ &- \frac{813902203}{14091090995165184000}v^{10} - \frac{38062279841}{22489381228283633664000}v^{12} - \cdots, \end{aligned}$$

$$\begin{aligned} \beta_{21} = &\frac{313}{7560} - \frac{59}{127008}v^2 + \frac{28879}{2933884800}v^4 + \frac{12917}{160190110080}v^6 + \frac{2740123\,v^8}{672798462336000} \\ &+ \frac{6456929}{56364363980660736}v^{10} + \frac{3460952431}{1022244601285619712000}v^{12} + \cdots, \end{aligned}$$

where $v = \omega h$. The behavior of the coefficients are given in figures 1 and 2. In the related figures to β_{ij}, ($i = 1,2$ and $j = 0,1$), we can see the behavior of coefficients and intuitively figure out that in what areas of v these coefficients are smooth or in what areas have high volatility, and they may even have some asymptotic in some states (when the denominator of the ratio is targeted zero). Obviously, when the coefficient for every region of the v is an asymptote, or has a high fluctuation, it would be better to use Taylor series. The new TFPL method, has two steps, eight algebraic order and phase-lag and its first and second derivatives equal to zero, moreover this method has a local truncation error which is given by:

$$LTE_{New} = \frac{59}{76204800}\left(\omega^6 y^{(4)}(x) + 3\omega^4 y^{(6)}(x) + 3\omega^2 y^{(8)}(x) + y^{(10)}(x)\right)h^{10} + O\left(h^{12}\right).$$

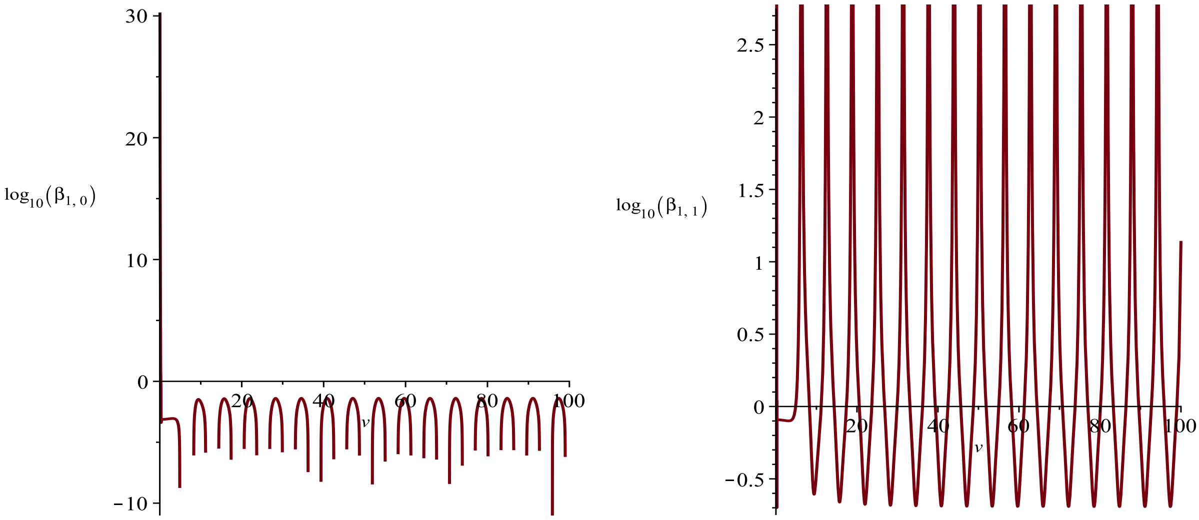

Figure 1: Behavior of the coefficients β_{10} and β_{11} of new method.

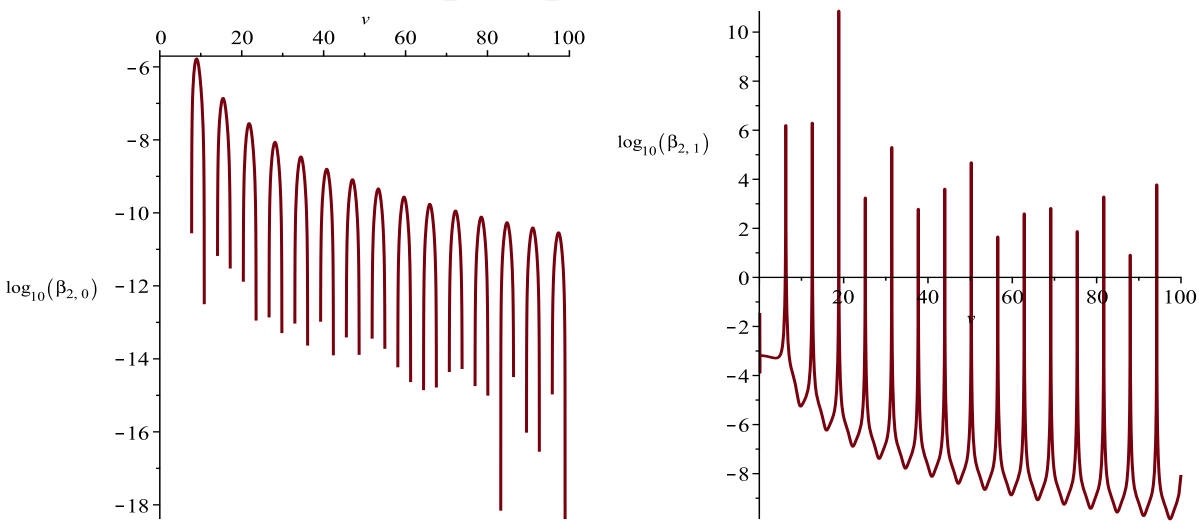

Figure 2: Behavior of the coefficients β_{20} and β_{21} of new method.

2.2. Periodicity analysis

In this section, we will analyze the stability and periodicity properties of the new method (10). In order to achieve the study of the stability of a symmetric multistep method, applying our new method constructed in Section 2.1 to the test equation:

$$y''(x) = -\phi^2 y(x), \tag{14}$$

yields:

$$A_1(s,\nu)\,(y_{n+1}+y_{n-1}) + A_0(s,\nu)y_n = 0, \tag{15}$$

where

$$\begin{aligned} A_0(s,\nu) &= \frac{T_{00}}{T_{01}}, \\ A_1(s,\nu) &= \frac{1}{2}\frac{T_{10}}{T_{11}}, \end{aligned} \tag{16}$$

where the frequency used in the scalar test equation for the stability analysis (ϕ) is not equal to the frequency of the scalar test equation used for the phase-lag analysis (ω), i.e. $\phi \neq \omega$ and $\nu = \omega h$, $s = \phi h$ and

$$\begin{aligned} T_{00} =& 16\cos(v)^3 s^4 + ((s^2-2)v^6 + (-s^4+4s^2)v^4 - 2s^4v^2 - 32s^4)\cos(v)^2 \\ &+ (((-3s^2+6)v^5 + (-s^4-20s^2)v^3 + 14s^4v)\sin(v) - 2s^4v^2 + 4s^2v^4 - 2v^6 + 16s^4)\cos(v) \\ &+ (2(s-v))((-7s^2+3v^2)\sin(v) + (s^2-2)v^3 + 2s^2v)v(v+s), \\ T_{01} =& (\cos(v)-1)(v\cos(v) - 3\sin(v) + 2v)v^5. \end{aligned}$$

Moreover

$$\begin{aligned} T_{10} =& (2s^4v^2 - 4s^2v^4 + 2v^6 - 16s^4)\cos(v)^2 + ((-14s^4v + 20s^2v^3 - 6v^5)\sin(v) \\ &+ (s^2+2)v^6 + (-s^4-4s^2)v^4 + 2s^4v^2 + 32s^4)\cos(v) \\ &+ ((3s^2+6)v^5 + (s^4-20s^2)v^3 + 14s^4v)\sin(v) \\ &- 4s^4v^2 + 8s^2v^4 - 4v^6 - 16s^4, \\ T_{11} =& (\cos(v)-1)(v\cos(v) - 3\sin(v) + 2v)v^5. \end{aligned}$$

Remark 2.1. We note that the terms P-stable and singularly almost P-stable method are hold in the case $\omega = \phi$, i.e. only when the frequency of the scalar test equation for the stability analysis is equal with the frequency of the scalar test equation for the phase-lag analysis, i.e. the surroundings of the first diagonal of the $s - v$ plane.

In Fig. 3, we plot the stability region of the new multiderivative method derived in Section 2.1. It is clear that the diagonal line $s = v$, (i.e. the fitted frequency ϕ equals the test frequency ω) is a stability boundary.

Theorem 2.2. The new two-step TFPL method (10) is P-stable.

Proof. In the case $s = v$, the characteristic equation (ChE) for the new method (10) is given by:

$$ChE = -\frac{2T_0}{T_1}\left(\lambda^2 - 2\cos(v)\lambda + 1\right),$$

where

$$T_0 = (v^3 + 4\sin(v))\cos(v) + v^3 - 4\sin(v),$$

and

$$T_1 = (v\sin(v)\cos(v) + 3\cos(v)^2 + 2v\sin(v) - 3)v.$$

Hence obviously the interval of periodicity of the new method is $(0,\infty)$, and then the new eight algebraic order two-step TFPL method (10) is P-stable. □

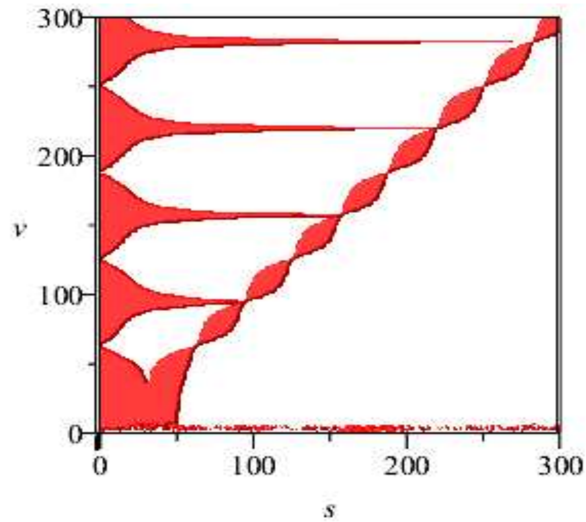

Figure 3: The periodicity region of new P-stable method where s is frequency of test problem and v is frequency of method.

3. Numerical results

In this section, we will compare the numerical performance of the new multiderivative method with some existing multistep methods proposed in the scientific literature as

- The 8th order Obrechkoff method of Saldanha and Achar [2] which indicated as Achar.
- The 12th order Obrechkoff method of Van Daele [17] which indicated as Daele.
- The 12th order Obrechkoff method of Simos [16] which indicated as Simos.
- The 12th order Obrechkoff method of Wang [31] which indicated as Wang.
- The 8th order Obrechkoff method of Shokri [14] which indicated as Shokri.
- The new 8th order TFPL method proposed in this paper which indicated as TFPL.

Example 3.1. We consider the periodically forced nonlinear problem (undamped Duffing's equation)

$$y'' = -y - y^3 + B\cos(\omega x), \quad y(0) = 0.200426728067, \quad y'(0) = 0, \tag{17}$$

where $B = 0.002$, $\omega = 1.01$ and $x \in \left[0, \frac{40.5\pi}{1.01}\right]$. We use the following exact solution for (17) from [9],

$$g(x) = \sum_{i=0}^{3} K_{2i+1} \cos((2i+1)\omega x),$$

where

$$\{K_1, K_3, K_5, K_7\} = \{0.200179477536, 0.246946143 \times 10^{-3}, 0.304016 \times 10^{-6}, 0.374 \times 10^{-9}\}.$$

In order to integrate this equation by a Obrechkoff method, one needs the values of y', which occur in calculating $y^{(4)}$. These higher order derivatives can all be expressed in terms of $y(x)$ and $y'(x)$ through (17), i.e.

$$\begin{aligned} y^{(3)}(x) &= -(1+3y^2(x))y'(x) - B\omega\sin(\omega x), \\ y^{(4)}(x) &= -(1+3y^2(x))y''(x) - 6y(x)y'(x)^2 - B\omega^2\cos(\omega x), \end{aligned}$$

h	TFPL	Shokri	Simos	Daele	Achar	Wang
$\frac{M}{500}$	4.12e-14	3.28e-12	3.15e-4	4.06e-5	4.09e-5	4.08e-5
$\frac{M}{1000}$	6.35e-16	7.62e-14	1.81e-5	1.87e-6	1.27e-6	1.27e-6
$\frac{M}{2000}$	4.93e-16	3.31e-14	1.08e-6	3.83e-8	3.94e-8	3.93e-8
$\frac{M}{3000}$	6.26e-16	5.16e-14	2.09e-7	5.13e-9	5.18e-9	5.17e-9
$\frac{M}{4000}$	2.74e-16	5.86e-14	6.55e-8	3.19e-9	1.23e-9	1.23e-9
$\frac{M}{5000}$	9.53e-17	6.37e-14	2.67e-8	9.89e-10	4.09e-10	4.07e-10

Table 1: Comparison of the end-point absolute error in the approximations obtained by using Methods: Shokri, Simos, Daele, Achar, Wang and the new TFPL method for Example 3.1.

h	TFPL	Shokri	Simos	Daele	Achar	Wang
$\frac{M}{500}$	0.9	1.1	1.4	1.5	1.2	1.4
$\frac{M}{1000}$	1.8	2.1	2.9	2.9	2.3	2.9
$\frac{M}{2000}$	3.2	3.7	6.2	6.3	4.8	6.2
$\frac{M}{3000}$	4.8	5.6	9.8	9.7	7.5	9.5
$\frac{M}{4000}$	6.1	7.2	13.5	13.3	10	13
$\frac{M}{5000}$	7.3	8.1	17	17	12.9	16.5

Table 2: CPU time (in seconds) for the example 3.1, are calculated for comparison among eight methods: Shokri, Simos, Daele, Achar, Wang and the new TFPL methods.

The absolute errors at $x = \frac{40.5\pi}{1.01}$, for the new method, in comparison with methods of Shokri method, classical method, zero phase-lag method, Simos, Daele, Achar, Wang and the new methods are given in Table 1 and the CPU times are listed in Table 2.

Example 3.2. We consider the inhomogeneous equation

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \quad y'(0) = 11,$$

whose exact solution is given by $y(t) = \sin(t) + \sin(10t) + \cos(10t)$. In our test we choose $\omega = 1$ and it has been solved numerically for $0 \le x \le 10\pi$ using exact starting values. In the numerical experiment, we take the step lengths $h = \pi/50$, $\pi/100$, $\pi/200$, $\pi/300$, $\pi/400$ and $\pi/500$. In Table 3, we present the absolute errors at the end-point and the CPU times are listed in Table 4.

Example 3.3. Consider the initial value problem

$$y'' = \frac{8y^2}{1+2x}, \quad y(0) = 1, \quad y'(0) = -2, \quad x \in [0, 4.5],$$

h	TFPL	Shokri	Simos	Daele	Achar
$\frac{\pi}{50}$	7.26e-19	8.21e-17	3.05e-11	1.20e-11	5.79e-13
$\frac{\pi}{100}$	3.52e-22	2.32e-20	2.28e-13	7.34e-13	5.79e-13
$\frac{\pi}{200}$	9.58e-28	8.16e-25	4.40e-13	8.62e-13	1.32e-12
$\frac{\pi}{300}$	1.17e-28	7.39e-26	2.11e-12	2.63e-12	1.96e-12
$\frac{\pi}{400}$	5.63e-30	6.14e-28	1.38e-12	2.93e-12	4.78e-12
$\frac{\pi}{500}$	2.87e-31	1.18e-29	6.46e-12	2.89e-12	7.50e-12

Table 3: Comparison of the end-point absolute error in the approximations obtained by using Methods: Shokri, Simos, Daele, Achar and the new TFPL methods for Example 3.2.

h	TFPL	Shokri	Simos	Daele	Achar
$\frac{\pi}{50}$	0.11	0.13	0.17	0.25	0.19
$\frac{\pi}{100}$	0.27	0.36	0.51	0.53	0.45
$\frac{\pi}{200}$	0.42	0.53	0.86	0.83	0.75
$\frac{\pi}{300}$	0.76	0.92	1.14	1.15	0.95
$\frac{\pi}{400}$	1.01	1.21	1.39	1.40	1.23
$\frac{\pi}{500}$	1.32	1.76	1.70	1.78	1.47

Table 4: CPU time (in seconds) for the example 3.2, are calculated for comparison among seven methods: Shokri, Simos, Daele, Achar and the new TFPL methods.

h	TFPL	Shokri	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	9.32e-19	7.32e-16	1.24e-7	1.26e-7	1.26e-7	1.24e-7
$\frac{4.5}{1000}$	6.21e-21	5.38e-18	3.82e-9	3.90e-9	3.85e-9	3.82e-9
$\frac{4.5}{2000}$	8.19e-23	3.68e-20	1.19e-10	1.23e-10	1.20e-10	1.19e-10
$\frac{4.5}{3000}$	7.46e-24	2.97e-21	1.92e-11	2.02e-11	1.40e-11	1.92e-11
$\frac{4.5}{4000}$	8.32e-25	3.42e-22	7.85e-12	7.85e-12	2.68e-12	7.85e-12
$\frac{4.5}{5000}$	1.01e-25	1.61e-23	1.63e-12	1.63e-12	7.47e-14	1.63e-12

Table 5: Comparison of the end-point absolute error in the approximations obtained by using Methods: Shokri, Simos, Daele, Achar, Wang and the new TFPL methods for Example 3.3.

with the exact solution

$$y(x) = \frac{1}{1+2x}.$$

The absolute errors at $x = 4.5$ for the new methods, in comparison with methods of Wang, Shokri, Simos, Daele, Achar, classical and zero phase-lag are given in the Table 5. The relative CPU times of computation of the new methods in comparison with the other seven referred methods are given in Table 6.

Conclusions

In this paper, a new technique for calculating of the coefficients in the multistep methods is introduced. The new constructed method in this paper has two steps, multiderivative type and eight algebraic order. This technique is based on the combination of the TF (trigonometrically fitted) and vanishing of phase-lag and its derivatives methods, say TFPL methods, that have improved the local truncation error, phase-lag error, interval of periodicity, periodicity region and CPU time for the classes of two-step multistep methods. The results of the numerical experiments confirm

h	TFPL	Shokri	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	0.12	0.16	0.369	0.34	0.19	0.31
$\frac{4.5}{1000}$	0.22	0.31	0.62	0.61	0.76	1.23
$\frac{4.5}{2000}$	0.48	0.61	0.62	0.61	0.76	1.23
$\frac{4.5}{3000}$	0.63	0.87	1.23	1.92	1.20	1.87
$\frac{4.5}{4000}$	1.06	1.35	1.89	2.59	1.62	2.56
$\frac{4.5}{5000}$	1.36	1.54	2.59	3.29	2.06	3.24

Table 6: CPU time (in seconds) for the example 3.3, are calculated for comparison among eight methods: Shokri, Simos, Daele, Achar, Wang and the new TFPL methods.

that our new method work better than those well-known high quality codes in the sense of efficiency, accuracy and stability (see Refs. [2, 14, 16, 17, 31]).

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