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Convergence analysis for derivative dependent Fredholm-Hammerstein integral equations with Green's kernel

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ABSTRACT

In this article, we consider a class of derivative dependent Fredholm-Hammerstein integral equations i.e., the integral equation, where the nonlinear function inside the integral sign is dependent on derivative and the kernel function is of Green's type. We propose the piecewise polynomial based Galerkin and iterated Galerkin methods to solve these type of derivative dependent Fredholm-Hammerstein integral equations. We discuss the convergence and error analysis of the proposed methods and also obtain the superconvergence results for iterated Galerkin approximations. Some numerical results are given to illustrate this improvement.

1. Introduction

Consider the following derivative dependent Hammerstein integral equation

$$u(x) = f(x) + \int_0^1 G(x,\xi)\psi(\xi, u(\xi), u'(\xi))\, d\xi, \tag{1.1}$$

where the right hand side function $f(x)$ and the Green's function $G(x,\xi)$ are known, u is the unknown function to be determined.

The integral equations of type (1.3) in general arise as a reformulation of the nonlinear boundary value problems. For example consider the following two point boundary value problem

$$(u'(x))' = \psi(x, u(x), u'(x)), \tag{1.2}$$

subject to the boundary conditions

$$u(0) = \alpha_1, \;\; \beta_1 u(1) + \gamma_1 u'(1) = \eta_1. \tag{1.3}$$

Then the transformed integral equation can be written as

$$u(x) = \alpha_1 + \frac{(\eta_1 - \alpha_1\beta_1)x}{\beta_1 + \gamma_1} + \int_0^1 G(x,\xi)\psi(\xi, u(\xi), u'(\xi))\, d\xi, \quad 0 \le x \le 1, \tag{1.4}$$

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where Green's function $G(x,\xi)$ is given by

$$G(x,\xi)=\begin{cases} x(1-\frac{\beta_1\xi}{\beta_1+\gamma_1}), & 0\le x\le\xi,\\ \xi(1-\frac{\beta_1 x}{\beta_1+\gamma_1}), & \xi\le x\le 1,\end{cases}$$

and $\alpha_1,\beta_1>0$, γ_1 and η_1 are any finite real constants.

Two point nonlinear boundary value problem for ordinary differential equations of type (1.2)–(1.3) arise very frequently in Poisson–Boltzmann equation [1], Stellar Structure [2], the equilibrium of a charged gas in a container [3], nonlinear heat conduction model of the human head [4], mathematical models of tumor growth [5], analytical bounding functions for diffusion problems [6] and singular diffusion problems in physiology [7]. In most of the cases, we can not solve boundary value problems analytically. So, these problems must be solved by different kinds of numerical techniques. The numerical approaches to solve the boundary value problems such as decomposition method, Adomian decomposition method, modified decomposition method are well documented in [8–14], where the authors considered the boundary value problem (1.2) with nonlinear function ψ independent of derivatives. In [15], a survey of some of the theoretical developments on projection methods for the numerical solution of boundary value problems is given along with their convergence. Although, these numerical methods have many advantages, an immense amount of computational work is involved as it requires the computation of undetermined coefficients in a sequence of nonlinear algebraic or more difficult transcendental equations, which increases the computational work (see. [16–18]). Moreover, the undetermined coefficients may not be uniquely determined in some cases. This may be the major disadvantage of these methods for solving nonlinear boundary value problems. Hence one can solve an equivalent integral equation instead of solving boundary value problems, which leads to derivative dependent nonlinear Hammerstein integral equations, i.e., the nonlinear function depends on the derivative. There is huge literature available on the nonlinear Fredholm-Hammerstein integral equations, where the nonlinear function is independent of derivative (See. [19–25]). In [26–28], projection and iterated projection methods are applied to nonlinear Fredholm integral equation with special classes of kernels. In ([19–24,29–33]), the authors discussed the projection methods such as Galerkin, collocation, multi-Galerkin and multi-collocation methods to solve the nonlinear integral equations of Hammerstein type, where the nonlinear function is independent of derivatives for the smooth as well as weakly singular kernels and discussed the superconvergence results. In [34–36], M. Turkyilmazoglu discussed some important numerical methods to obtain the approximation of the Volterra Fredholm-Hammerstein integro-differential equations and initial and boundary value problems. The main motivation to consider this article for Fredholm-Hammerstein integral equations with derivative dependent nonlinear function is to obtain the superconvergence results.

In this article, our aim is to solve the derivative dependent Hammerstein integral equation of type (1.3) and investigate the error analysis and order of convergence for the approximations in Galerkin method and iterated Galerkin method. We illustrate our theoretical results with numerical examples.

We organize this article as follows. In Section 2, we discuss the Galerkin and iterated Galerkin methods to solve the Eq. (1.3). In Section 3, we obtain convergence results. In Section 4, numerical results are given to illustrate the theoretical results. Throughout this paper, we assume that c is a generic constant.

2. Galerkin method: Derivative dependent Hammerstein integral equations with Green's kernel

Let $\mathbb{X}=L^\infty[0,1]$. We consider the derivative dependent Hammerstein integral equation as follows

$$u(x)=f(x)+\int_0^1 G(x,\xi)\psi(\xi,u(\xi),u'(\xi))\,d\xi, \tag{2.1}$$

where G, f and ψ are known functions and u is the unknown function to be determined.

For a fixed $x\in[0,1]$, we denote $G_x(\xi)=G(x,\xi)$, $\ell_x(\xi)=\ell(x,\xi)=\frac{\partial G}{\partial x}(x,\xi)$, and

$$G_{1x}(\xi)=G_x(\xi),\quad 0\le x\le\xi, \tag{2.2}$$
$$G_{2x}(\xi)=G_x(\xi),\quad \xi\le x\le 1. \tag{2.3}$$

We assume for $0\le x\le 1$, $G_{1x}\in\mathcal{C}^{k_1}[0,x]$, $G_{2x}\in\mathcal{C}^{k_1}[x,1]$ and $G(x,\xi)\in\mathcal{C}^{k_1}(0,x)\cap\mathcal{C}^{k_1}(x,1)\cap\mathcal{C}^{\gamma}(0,1)$ and $\ell(x,\xi)=\frac{\partial G}{\partial x}(x,\xi)\in\mathcal{C}^{k_1-1}(0,x)\cap\mathcal{C}^{k_1-1}(x,1)\cap\mathcal{C}^{\gamma-1}(0,1)$, with $k_1\ge 1$ and $k_1\ge\gamma\ge -1$. We assume $f\in\mathcal{C}^{k_1}[0,1]$. According to the Theorem 4.1 and corollary 4.2 of [27], $u\in\mathcal{C}^{k_1}([0,1])$. Denote

$$\|v\|_{k_1,\infty}=\max\{\|v^{(i)}\|_\infty : 0\le i\le k_1\},$$

where $v^{(i)}$ be the ith derivative of v.

Throughout the paper, we make the following assumptions on f, $G(x,\xi)$ and $\psi(.,u(.),u'(.))$:

(i) $f \in \mathcal{C}^{k_1}[0,1]$.

(ii) $M_1 = \sup_{x,\xi\in[0,1]} |G(x,\xi)| < \infty$, $M_2 = \sup_{x,\xi\in[0,1]} |\ell(x,\xi)| < \infty$.

(iii) The nonlinear function $\psi(\xi,u,u')$ is Lipschitz continuous in u and u', i.e., for any $u_1,u_2,u_1',u_2' \in \mathbb{X}$, $\exists\ c_1 > 0$ such that

$$|\psi(\xi,u_1,u_1') - \psi(\xi,u_2,u_2')| \le c_1\{|u_1(\xi)-u_2(\xi)| + |u_1'(\xi)-u_2'(\xi)|\}, \forall\ \xi \in [0,1].$$

(iv) The partial derivatives $\psi^{(0,1,0)}(\xi,u,u')$, $\psi^{(0,0,1)}(\xi,u,u')$ of ψ w.r.t the second and third variables exists and are Lipschitz continuous in u and u', i.e., for any $u_1,u_2,u_1',u_2' \in \mathbb{X}$, $\exists\ c_2,c_3 > 0$ such that

$$|\psi^{(0,1,0)}(\xi,u_1,u_1') - \psi^{(0,1,0)}(\xi,u_2,u_2')| \le c_2\{|u_1(\xi)-u_2(\xi)| + |u_1'(\xi)-u_2'(\xi)|\}, \forall\ \xi \in [0,1],$$
$$|\psi^{(0,0,1)}(\xi,u_1,u_1') - \psi^{(0,0,1)}(\xi,u_2,u_2')| \le c_3\{|u_1(\xi)-u_2(\xi)| + |u_1'(\xi)-u_2'(\xi)|\}, \forall\ \xi \in [0,1],$$

and $\psi^{(0,1,0)}, \psi^{(0,0,1)} \in \mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R})$.

(v) We assume that $M = M_1 + M_2$ and c_1 satisfy the condition that $Mc_1 < 1$.

For any $v \in \mathbb{X}$ and $x \in [0,1]$, let

$$\mathcal{K}v(x) = \int_0^1 G(x,\xi)v(\xi)\,d\xi,$$

and

$$\mathcal{L}v(x) = \frac{d}{dx}(\mathcal{K}v)(x) = \int_0^1 \ell(x,\xi)v(\xi)\,d\xi.$$

Note that $\mathcal{K}:\mathbb{X}\to\mathbb{X}$ and $\mathcal{L}:\mathbb{X}\to\mathbb{X}$ are compact operators and

$$\|\mathcal{K}\|_\infty \le M_1 \quad \text{and} \quad \|\mathcal{L}\|_\infty \le M_2. \tag{2.4}$$

Now Kumar and Sloan [24] technique will be used for approximating the Eq. (2.1). We apply the Galerkin method to an equivalent equation for the function z described by

$$z(\xi) = \psi(\xi,u(\xi),u'(\xi)),\ \ \xi \in [0,1]. \tag{2.5}$$

Note that if $\psi(.,.,.) \in \mathcal{C}^{k_1}([0,1]\times[0,1]\times[0,1])$, then using the chain rule for higher derivative, we can see that $z \in \mathcal{C}^{k_1}[0,1]$.

The solution u of (2.1) satisfies the following

$$u(x) = f(x) + \int_0^1 G(x,\xi)z(\xi)\,d\xi, \quad 0 \le x \le 1. \tag{2.6}$$

Then the Eq. (2.6) will take the form

$$u = f + \mathcal{K}z. \tag{2.7}$$

For our convenience, we consider a nonlinear operator $\Psi:\mathbb{X}\to\mathbb{X}$ defined by

$$\Psi(u)(\xi) = \psi(\xi,u(\xi),u'(\xi)). \tag{2.8}$$

Then Eq. (2.5) becomes

$$z = \Psi(f + \mathcal{K}z). \tag{2.9}$$

Letting $\mathcal{T}(v) = \Psi(f+\mathcal{K}v)$, $v \in \mathbb{X}$, then the Eq. (2.9) can be written as

$$z = \mathcal{T}z. \tag{2.10}$$

The following theorem gives the condition for the existence of the unique solution of Eq. (2.10) in $\mathbb{X}$.

Theorem 1. *Let $\psi(\xi,u(\xi),u'(\xi)) \in \mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R})$ satisfies Lipschitz condition in the second and third variables, i.e., for any $u_1,u_2,u_1',u_2' \in \mathbb{X}$, $\exists\ c_1 > 0$ such that*

$$|\psi(\xi,u_1,u_1') - \psi(\xi,u_2,u_2')| \le c_1\{|u_1(\xi)-u_2(\xi)| + |u_1'(\xi)-u_2'(\xi)|\}, \forall\ \xi \in [0,1],$$

with $(M_1+M_2)c_1 < 1$. Then the operator equation $z = \mathcal{T}z$ possess an isolated solution $z_0 \in \mathbb{X}$, i.e., $z_0 = \mathcal{T}z_0$.

Proof. Let $z_1, z_2 \in \mathcal{C}[0,1]$ and using Lipschitz's continuity of $\psi(\xi, u(\xi), u'(\xi))$, we obtain

$$\begin{aligned} \|\mathcal{T}z_1 - \mathcal{T}z_2\|_\infty &= \|\Psi(f+\mathcal{K}z_1) - \Psi(f+\mathcal{K}z_2)\|_\infty \\ &= \|\psi(\xi, f+\mathcal{K}z_1, f'+\mathcal{L}z_1) - \psi(\xi, f+\mathcal{K}z_2, f'+\mathcal{L}z_2)\|_\infty \\ &\le c_1\{\|\mathcal{K}(z_1-z_2)\|_\infty + \|\mathcal{L}(z_1-z_2)\|_\infty\} \\ &\le \{M_1\|z_1-z_2\|_\infty + M_2\|z_1-z_2\|_\infty\} \\ &\le Mc_1\|z_1-z_2\|_\infty. \end{aligned} \tag{2.11}$$

By assumption $Mc_1 < 1$, hence $\mathcal{T}$ is a contraction mapping on $\mathbb{X}$ and by Banach contraction theorem, $\mathcal{T}$ has a unique fixed point z_0 in $\mathbb{X}$.

This completes the proof. □

Next, we will apply Galerkin method to solve Eq. (2.7). For this, we consider $\Pi_n : 0 = t_0 < t_1 < \cdots < t_n = 1$, a partition of $[0,1]$ and $h_i = \{t_i - t_{i-1} : 1 \le i \le n\}$. Let $h = \max h_i$ denotes the norm of the partition. We assume that $h \to 0$, as $n \to \infty$. We denote, $\triangle_i = [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$ and define $\mathcal{C}_\triangle := \prod_{i=1}^n \mathcal{C}(\triangle_i)$. Then $g \in \mathcal{C}_\triangle$ consists of n components $g_i \in \mathcal{C}(\triangle_i)$ and is a piecewise-continuous function having (possibly) different left and right values at the partition points t_i. Note that $\mathcal{C}_\triangle$ is a Banach space with the norm $\|.\|_\infty$ defined by $\|g\|_\triangle = \max_i \|g_i\|_\infty$, and since $\|g\|_\triangle = \|g\|_\infty$, for $g \in \mathcal{C}_\triangle$, we have $\mathcal{C}_\triangle \subset \mathcal{C}[0,1]$. More generally, we denote for a positive integer k, $\mathcal{C}^k_\triangle := \prod_{i=1}^n \mathcal{C}^k(\triangle_i)$, $g_i \in \mathcal{C}^k_\triangle$ iff its kth derivative $g_i^{(k)}$ is continuous on $\triangle_i$.

Here we let the approximating subspaces

$$\mathbb{X}_n = \mathbb{P}_{r,\triangle} = \{u : u|_{(t_{i-1},t_i)} \in \mathbb{P}_r, 1 \le i \le n\},$$

where $\mathbb{P}_r$ denotes, for given $r \ge 1$, the space of (real) polynomials of order r (i.e., degree less than $r+1$). For $g \in \mathbb{P}_{r,\triangle}$, if the value at t_i is defined by continuity, then $\mathbb{P}_{r,\triangle} \subset \mathcal{C}_\triangle$ and the projection $\mathcal{P}_n$ is defined from $\mathcal{C}_\triangle$ onto $\mathbb{P}_{r,\triangle}$ with $g = (g_1, g_2, \ldots, g_n) \to \mathcal{P}_n g = (\mathcal{P}g_1, \mathcal{P}g_2, \ldots, \mathcal{P}g_n)$, where $\mathcal{P}g_i$ is the orthogonal projection of $g_i \in \mathcal{C}(\triangle_i)$ on the polynomial of degree less than $r+1$ on $\triangle_i$.

We first quote the following Lemma from Chatelin [37].

Lemma 1. *Let $\mathcal{P}_n : \mathcal{C}_\triangle \to \mathbb{X}_n$ be the orthogonal projection operator. Then there hold*

(i) $\mathcal{P}_n$ is uniformly bounded in infinity norm, i.e, $\exists$ a constant p independent of n such that

$$\|\mathcal{P}_n\|_\infty \le p < \infty. \tag{2.12}$$

(ii) $\|\mathcal{P}_n u - u\|_\infty \to 0$, as $n \to \infty$, $u \in \mathcal{C}_\triangle$.

(iii) In particular if $u \in \mathcal{C}^{r+1}_\triangle$, then

$$\|(\mathcal{I} - \mathcal{P}_n)u\|_\infty \le ch^{r+1}\|u^{(r+1)}\|_\infty. \tag{2.13}$$

The Galerkin method for Eq. (2.9) is seeking an approximate solution $z_n \in X_n$ such that

$$z_n = \mathcal{P}_n\Psi(f + \mathcal{K}z_n). \tag{2.14}$$

Let $\mathcal{T}_n$ be the operator defined by

$$\mathcal{T}_n(u) := \mathcal{P}_n\Psi(f + \mathcal{K}u). \tag{2.15}$$

Then Eq. (2.14) becomes

$$z_n = \mathcal{T}_n(z_n). \tag{2.16}$$

Corresponding approximate solution u_n of u is defined by

$$u_n = f + \mathcal{K}(z_n). \tag{2.17}$$

In order to obtain more accurate approximation solution for (2.9), the iterated solution be defined as

$$\tilde{z}_n = \Psi(f + \mathcal{K}z_n). \tag{2.18}$$

Applying $\mathcal{P}_n$ on both sides of the Eq. (2.18), we obtain

$$\mathcal{P}_n\tilde{z}_n = \mathcal{P}_n\Psi(f + \mathcal{K}z_n). \tag{2.19}$$

From Eqs. (2.14) and (2.19), it follows that $\mathcal{P}_n\tilde{z}_n = z_n$. Using this, we see that the iterated solution $\tilde{z}_n$ satisfies the following equation

$$\tilde{z}_n = \Psi(f + \mathcal{K}\mathcal{P}_n\tilde{z}_n). \tag{2.20}$$

Letting

$$\widetilde{\mathcal{T}}_n(u) := \Psi(f + \mathcal{K}\mathcal{P}_n u). \tag{2.21}$$

The Eq. (2.20) becomes

$$\tilde{z}_n = \widetilde{\mathcal{T}}_n \tilde{z}_n. \tag{2.22}$$

The Corresponding approximation $\tilde{u}_n$ of u is defined by

$$\tilde{u}_n = \mathcal{K}\tilde{z}_n + f. \tag{2.23}$$

3. Convergence rates

In this section, we discuss the existence and convergence of the approximate and iterated approximate solutions in the Galerkin method.

To do this, first we quote the following theorem from [38], which gives us the condition under which the solvability of one equation leads to the solvability of other equation.

Theorem 2 *([38]). Let $\widehat{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ be continuous operators over an open set Ω in a Banach space $\mathbb{X}$. Let the equation $u = \widetilde{\mathcal{F}}u$ has an isolated solution $\tilde{u}_0 \in \Omega$ and let the following conditions be satisfied.*

(a) The operator $\widehat{\mathcal{F}}$ is Frechet differentiable in some neighborhood of the point $\tilde{u}_0$, while the linear operator $\mathcal{I} - \widehat{\mathcal{F}}'(\tilde{u}_o)$ is continuously invertible.

(b) Suppose that for some $\delta > 0$ and $0 < q < 1$, the following inequalities are valid (the number δ is assumed to be so small that the sphere $\|u - \tilde{u}_0\| \le \delta$ is contained within Ω)

$$\sup_{\|u-\tilde{u}_0\|\le\delta} \|(\mathcal{I} - \widehat{\mathcal{F}}'(\tilde{u}_o))^{-1}(\widehat{\mathcal{F}}'(u) - \widehat{\mathcal{F}}'(\tilde{u}_o))\| \le q, \tag{3.1}$$

$$\alpha = \|(\mathcal{I} - \widehat{\mathcal{F}}'(\tilde{u}_o))^{-1}(\widehat{\mathcal{F}}(\tilde{u}_o) - \widetilde{\mathcal{F}}(\tilde{u}_o))\| \le \delta(1-q). \tag{3.2}$$

Then the equation $u = \widehat{\mathcal{F}}u$ has a unique solution $\hat{u}_0$ in the sphere $\|u - \tilde{u}_0\| \le \delta$. Moreover, the inequality

$$\frac{\alpha}{1+q} \le \|\hat{u}_0 - \tilde{u}_0\| \le \frac{\alpha}{1-q} \tag{3.3}$$

is valid.

Now we discuss the existence and uniqueness of the approximate and iterated approximate solutions.

Lemma 2. *Let $z_0 \in \mathcal{C}^{k_1}[0,1]$ be an isolated solution of the Eq. (2.7). Then there hold*

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty = \mathcal{O}(h^{\beta+\beta^*}),$$

and

$$\|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty = \mathcal{O}(h^{\beta+\beta^{**}}),$$

where $\beta = \min\{k_1, r+1\}$, $\beta_1 = \min\{\beta, \gamma+1\}$, $\beta_2 = \min\{k_1 - 1, r+1\}$, $\beta^* = \min\{\beta, \beta_1 + 1\}$ and $\beta^{**} = \min\{\beta_1, \beta_2\}$. Hence we obtain

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty \to 0, \quad \text{as} \quad n \to \infty,$$

and

$$\|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty \to 0, \quad \text{as} \quad n \to \infty.$$

Proof. Using orthogonality of $\mathcal{P}_n$, we have

$$\begin{aligned}
\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty &= \sup_{x\in[0,1]} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0(x)| \\
&= \sup_{x\in[0,1]} \left| \int_0^1 G_x(\xi)(\mathcal{I} - \mathcal{P}_n)z_0(\xi)\, d\xi \right| \\
&= \sup_{x\in[0,1]} |\langle G_x(.), (\mathcal{I} - \mathcal{P}_n)z_0(.)\rangle| \\
&= \sup_{x\in[0,1]} \sum_{i=1}^{n} \left|\left\langle (\mathcal{I} - \mathcal{P}^G)(G_x)_i, (\mathcal{I} - \mathcal{P}^G)(z_0)_i \right\rangle\right| \\
&\le \sum_{i=1}^{n} [\|(\mathcal{I} - \mathcal{P}^G)(G_x)_i\|_{2,\triangle_i} \|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{2,\triangle_i}].
\end{aligned} \tag{3.4}$$

Since $z_0 \in \mathcal{C}^{k_1}[0,1]$, using the estimate (2.13), we have for $i = 1, 2, \ldots, n$

$$\|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{\infty,\triangle_i} \le c h_i^{\beta} \|(z_0)_i^{(\beta)}\|_\infty. \tag{3.5}$$

From this, we have

$$\|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{2,\triangle_i} \le c h_i^{\frac{1}{2}} \|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{\infty,\triangle_i} \le c h_i^{\beta+\frac{1}{2}} \|(z_0)_i^{(\beta)}\|_\infty = \mathcal{O}(h^{\beta+\frac{1}{2}}), \tag{3.6}$$

where $\beta = \min\{k_1, r+1\}$.

Next we consider $x \notin \triangle$, i.e., $x \in (t_{i-1}, t_i)$, for some $i \in \{1, 2, \ldots, n\}$ and $(G_{1x})_j, (G_{2x})_j \in \mathcal{C}^{k_1}(\triangle_j)$, for $j \neq i$, then from Lemma 7.8 of [37] (pp 330–331), we have for $j \neq i$ and $j = 1, 2, \ldots, n$

$$\begin{aligned}\|(\mathcal{I} - \mathcal{P}^G)(G_x)_j\|_{2,\triangle_j} &\le c h_j^{\beta} \max(\|(G_{1x})_j^{(\beta)}\|_{2,\triangle_j}, \|(G_{2x})_j^{(\beta)}\|_{2,\triangle_j}) \\ &\le c h_j^{\beta+\frac{1}{2}} \max(\|G_{1x}^{(\beta)}\|_\infty, \|G_{2x}^{(\beta)}\|_\infty) = \mathcal{O}(h^{\beta+\frac{1}{2}}),\end{aligned} \tag{3.7}$$

and on $\triangle_i$,

$$\begin{aligned}\|(\mathcal{I} - \mathcal{P}^G)(G_x)_i\|_{2,\triangle_j} &\le c h_i^{\beta_1} \left[(\|G_{1x}^{(\beta_1)}\|_{2,[t_{i-1},x]})^2 + (\|G_{2x}^{(\beta_1)}\|_{2,[x,t_i]})^2\right]^{\frac{1}{2}} \\ &\le c h_i^{\beta_1+\frac{1}{2}} \left[(\|G_{1x}^{(\beta_1)}\|_\infty)^2, (\|G_{2x}^{(\beta_1)}\|_\infty)^2\right]^{\frac{1}{2}} = \mathcal{O}(h^{\beta_1+\frac{1}{2}}),\end{aligned} \tag{3.8}$$

where $\beta_1 = \min\{\beta, \gamma+1\}$ and $\beta = \min\{k_1, r+1\}$.

Hence from estimates (3.4) and (3.6)–(3.8) it follows that

$$\begin{aligned}\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty &\le \sum_{i=1}^{n} [\|(\mathcal{I} - \mathcal{P}^G)(G_x)_i\|_{2,\triangle_i} \|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{2,\triangle_i}] \\ &\le \sum_{\substack{i=1 \\ j\neq i}}^{n} [\|(\mathcal{I} - \mathcal{P}^G)(G_x)_j\|_{2,\triangle_j} \|(\mathcal{I} - \mathcal{P}^G)(z_0)_j\|_{2,\triangle_j}] \\ &+ \|(\mathcal{I} - \mathcal{P}^G)(G_x)_i\|_{2,\triangle_i} \|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{2,\triangle_i} \\ &= \sum_{\substack{i=1 \\ j\neq i}}^{n} [\mathcal{O}(h^{\beta+\frac{1}{2}})\mathcal{O}(h^{\beta+\frac{1}{2}})] + \mathcal{O}(h^{\beta+\frac{1}{2}})\mathcal{O}(h^{\beta_1+\frac{1}{2}}) \\ &= \mathcal{O}(h^{\min(2\beta, \beta+\beta_1+1)}) = \mathcal{O}(h^{\beta+\beta^*}),\end{aligned} \tag{3.9}$$

where $\beta^* = \min\{\beta, \beta_1 + 1\} = \min\{\beta, \gamma + 2\}$.

Now consider

$$\begin{aligned}\|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty &= \sup_{x\in[0,1]} |\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0(x)| \\ &= \sup_{x\in[0,1]} \sum_{i=1}^{n} \left|\left\langle(\mathcal{I} - \mathcal{P}^G)(\ell_x)_i, (\mathcal{I} - \mathcal{P}^G)(z_0)_i\right\rangle\right| \\ &\le \sum_{i=1}^{n} [\|(\mathcal{I} - \mathcal{P}^G)(\ell_x)_i\|_{2,\triangle_i} \|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{2,\triangle_i}].\end{aligned} \tag{3.10}$$

Since $z_0 \in \mathcal{C}^{k_1}[0,1]$, using the estimate (2.13), we have for $i = 1, 2, \ldots, n$

$$\|(\mathcal{I} - \mathcal{P}^G)(z_0(.))_i\|_{2,\triangle_i} \le c h_i^{\frac{1}{2}} \|(\mathcal{I} - \mathcal{P}^G)(z_0)_i\|_{\infty,\triangle_i} \le c h_i^{\beta+\frac{1}{2}} \|(z_0)_i^{(\beta)}\|_\infty = \mathcal{O}(h^{\beta+\frac{1}{2}}), \tag{3.11}$$

where $\beta = \min\{k_1, r+1\}$.

Next we consider $x \notin \triangle$, i.e., $x \in (t_{i-1}, t_i)$, for $i = 1, 2, \ldots, n$ and $(\ell_{1x})_j, (\ell_{2x})_j \in \mathcal{C}^{k_1-1}(\triangle_j)$, for $j \neq i$, then from Lemma 7.8 of [37] (pp 330–331), we have for $j \neq i$ and $j = 1, 2, \ldots, n$

$$\begin{aligned}\|(\mathcal{I} - \mathcal{P}^G)(\ell_x)_j\|_{2,\triangle_j} &\le c h_j^{\beta_2} \max(\|(\ell_{1x})_j^{\beta_2}\|_{2,\triangle_j} \|(\ell_{2x})_j^{\beta_2}\|_{2,\triangle_j}) \\ &\le c h_j^{\beta_2+\frac{1}{2}} \max(\|\ell_{1x}^{\beta_2}\|_\infty, \|\ell_{2x}^{\beta_2}\|_\infty) = \mathcal{O}(h^{\beta_2+\frac{1}{2}}),\end{aligned} \tag{3.12}$$

and on $\triangle_i$,

$$\begin{aligned}\|(\mathcal{I}-\mathcal{P}^G)(\ell_x)_i\|_{2,\triangle_j} &\leq ch_i^{\beta_1-1}\left[(\|\ell_{1x}^{(\beta_1-1)}\|_{2,[t_{i-1},x]})^2+(\|\ell_{2x}^{(\beta_1-1)}\|_{2,[x,t_i]})^2\right]^{\frac{1}{2}}\\ &\leq ch_i^{\beta_1-\frac{1}{2}}\left[(\|\ell_{1x}^{(\beta_1-1)}\|_\infty)^2,(\|\ell_{2x}^{(\beta_1-1)}\|_\infty)^2\right]^{\frac{1}{2}}=\mathcal{O}(h^{\beta_1-\frac{1}{2}}),\end{aligned} \tag{3.13}$$

where $\beta_1=\min\{k_1,\gamma+1\}$ and $\beta_2=\min\{k_1-1,r+1\}$.

Hence from estimates (3.10)–(3.13), it follows that

$$\begin{aligned}\|\mathcal{L}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty &\leq \sum_{i=1}^n[\|(\mathcal{I}-\mathcal{P}^G)(\ell_x)_i\|_{2,\triangle_i}(\mathcal{I}-\mathcal{P}^G)(z_0)_i\ \|_{2,\triangle_i}]\\ &\leq \sum_{\substack{i=1\\ j\neq i}}^n[\|(\mathcal{I}-\mathcal{P}^G)(\ell_x)_j\|_{2,\triangle_j}(\mathcal{I}-\mathcal{P}^G)(z_0)_j\ \|_{2,\triangle_j}]\\ &+\|(\mathcal{I}-\mathcal{P}^G)(\ell_x)_i\|_{2,\triangle_i}(\mathcal{I}-\mathcal{P}^G)(z_0)_i\ \|_{2,\triangle_i}\\ &=\sum_{\substack{i=1\\ j\neq i}}^n[\mathcal{O}(h^{\beta+\frac{1}{2}})\mathcal{O}(h^{\beta_2+\frac{1}{2}})]+\mathcal{O}(h^{\beta+\frac{1}{2}})\mathcal{O}(h^{\beta_1-\frac{1}{2}})\\ &=\mathcal{O}(h^{\min(\beta+\beta_2,\beta+\beta_1)})=\mathcal{O}(h^{\beta+\beta^{**}}),\end{aligned} \tag{3.14}$$

where $\beta^{**}=\min\{\beta_1,\beta_2\}$.

This completes the proof.

Lemma 3. *Let the Fréchet derivatives of $\mathcal{T}(z)$ and $\widetilde{\mathcal{T}}_n(z)$ at z_0 be $\mathcal{T}'(z_0)$ and $\widetilde{\mathcal{T}}_n'(z_0)$, respectively. Then*

$$\begin{aligned}&\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\to 0,\quad n\to\infty,\\ &\|(\mathcal{I}-\mathcal{P}_n)\mathcal{T}'(z_0)\|_\infty\to 0,\quad n\to\infty.\end{aligned}$$

Proof.

We have

$$\widetilde{\mathcal{T}}_n'(z_0)=\Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz_0)\mathcal{K}\mathcal{P}_n+\Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz_0)\mathcal{L}\mathcal{P}_n. \tag{3.15}$$

Now using the Lipschitz's continuity of $\psi^{(0,1,0)}(.,u(.),u'(.))$, $\psi^{(0,0,1)}(.,u(.),u'(.))$, Lemma 2 and boundedness of $\|\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\|_\infty$ and $\|\Psi^{(0,0,1)}(f+\mathcal{K}z_0)\|_\infty$, we have

$$\begin{aligned}\|\Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz_0)\|_\infty &\leq \|\Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz_0)-\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\|_\infty+\|\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\|_\infty\\ &\leq c_2\{\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\}+\|\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\|_\infty\\ &\leq B_1<\infty.\end{aligned} \tag{3.16}$$

and similarly we can show that

$$\|\Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz_0)\|_\infty\leq B_2<\infty, \tag{3.17}$$

where B_1, B_2 are constants independent of n.

Next, let $\bar{B}:=\{x\in\mathbb{X}:\|x\|_\infty\leq 1\}$ be the closed unit ball in $\mathbb{X}$. Since $\{\mathcal{K}\mathcal{P}_n\}$ and $\{\mathcal{L}\mathcal{P}_n\}$ are sequence of compact operators. Using (3.15), one can show that the relatively compactness of the set $S=\{\widetilde{\mathcal{T}}_n'(z_0)u : u\in\bar{B}, n\in N\}$. From Lemma 1, it is concluded that

$$\begin{aligned}\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_\infty &=\sup\{\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)u\|_\infty : u\in\bar{B}\}\\ &=\sup\{\|(\mathcal{I}-\mathcal{P}_n)v\|_\infty : v\in S\}\to 0,\ \ as\ \ n\ \to\infty.\end{aligned} \tag{3.18}$$

Similarly, since $\Psi^{(0,1,0)}(f+\mathcal{K}z_0)$ and $\Psi^{(0,1,0)}(f+\mathcal{K}z_0)$ are bounded and $\mathcal{K}$ and $\mathcal{L}$ are compact, $\mathcal{T}'(z_0)=\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\mathcal{K}+\Psi^{(0,0,1)}(f+\mathcal{K}z_0)\mathcal{L}$ is also compact and we have

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{T}'(z_0)\|_\infty\to 0,\quad as\ \ n\to\infty.$$

This complete the proof. □

Theorem 3. *Let the isolated solution $z_0\in\mathcal{C}^{k_1}[0,1]$ be defined by Eq. (2.9). Let 1 is not an eigenvalue of the operator $[\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\mathcal{K}+\Psi^{(0,0,1)}(f+\mathcal{K}z_0)\mathcal{L}]$, which is the Fréchet derivative of $\Psi(f+\mathcal{K}z_0)$ at z_0. Let $\mathcal{P}_n:\mathbb{X}\to\mathbb{X}_n$ be orthogonal* *projection operator. Then Eq. (2.14) has a unique solution* $z_n \in B(z_0, \delta) = \{z : \|z - z_0\|_\infty < \delta\}$ *for some* $\delta > 0$ *and for sufficiently large n. Moreover, there exists a constant* $0 < q < 1$, *independent of n such that*

$$\frac{\alpha_n}{1+q} \le \|z_n - z_0\|_\infty \le \frac{\alpha_n}{1-q},$$

where $\alpha_n = \|(\mathcal{I} - \mathcal{T}_n'(z_0))^{-1}(\mathcal{T}_n(z_0) - \mathcal{T}(z_0))\|_\infty$. *Further, we obtain*

$$\|z_n - z_0\|_\infty \le c\|(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty = \mathcal{O}(h^\beta),$$

where $\beta = \min\{k_1, r+1\}$.

Proof. Using Lemma 3, we have

$$\begin{aligned}
&\|\mathcal{T}_n'(z_0) - \mathcal{T}'(z_0)\|_\infty \\
&= \|\mathcal{P}_n\Psi^{(0,1,0)}(f + \mathcal{K}z_0)\mathcal{K} + \mathcal{P}_n\Psi^{(0,0,1)}(f + \mathcal{K}z_0)\mathcal{L} - \Psi^{(0,1,0)}(f + \mathcal{K}z_0)\mathcal{K} - \Psi^{(0,0,1)}(f + \mathcal{K}z_0)\mathcal{L}\|_\infty \\
&= \|(\mathcal{I} - \mathcal{P}_n)\Psi^{(0,1,0)}(f + \mathcal{K}z_0)\mathcal{K} + (\mathcal{I} - \mathcal{P}_n)\Psi^{(0,0,1)}(f + \mathcal{K}z_0)\mathcal{L}\|_\infty \\
&= \|(\mathcal{I} - \mathcal{P}_n)\mathcal{T}'(z_0)\|_\infty \to 0 \ \ as \ \ n \ \to \infty.
\end{aligned}$$

Using this and since 1 is not an eigen value of $\mathcal{T}'(z_0)$, we obtain $(\mathcal{I} - \mathcal{T}_n'(z_0))^{-1}$ exists and is uniformly bounded on $\mathbb{X}$, for some sufficiently large n, i.e., there exists some $A_1 > 0$ such that $\|(\mathcal{I} - \mathcal{T}_n'(z_0))^{-1}\|_\infty \le A_1 < \infty$.

Now from estimate (2.12), for any $z \in B(z_0, \delta)$ and $v \in \mathbb{X}$, we have

$$\begin{aligned}
\|[\mathcal{T}_n'(z_0) - \mathcal{T}_n'(z)]v\|_\infty &= \| \, [\mathcal{P}_n\Psi^{(0,1,0)}(f + \mathcal{K}z_0)\mathcal{K} + \mathcal{P}_n\Psi^{(0,0,1)}(f + \mathcal{K}z_0)\mathcal{L}] \\
&- [\mathcal{P}_n\Psi^{(0,1,0)}(f + \mathcal{K}z)\mathcal{K} + \mathcal{P}_n\Psi^{(0,0,1)}(f + \mathcal{K}z)\mathcal{L}]v \, \|_\infty \\
&\le \|\mathcal{P}_n\|_\infty\|[\Psi^{(0,1,0)}(f + \mathcal{K}z_0) - \Psi^{(0,1,0)}(f + \mathcal{K}z)]\mathcal{K}v\|_\infty \\
&+ \|\mathcal{P}_n\|_\infty\|[\Psi^{(0,0,1)}(f + \mathcal{K}z_0) - \Psi^{(0,0,1)}(f + \mathcal{K}z)]\mathcal{L}v\|_\infty \\
&\le p[M_1\|\Psi^{(0,1,0)}(f + \mathcal{K}z_0) - \Psi^{(0,1,0)}(f + \mathcal{K}z)\|_\infty \\
&+ M_2\|\Psi^{(0,0,1)}(f + \mathcal{K}z_0) - \Psi^{(0,0,1)}(f + \mathcal{K}z)\|_\infty]\|v\|_\infty.
\end{aligned} \tag{3.19}$$

Taking use of the Lipschitz continuity of $\psi^{(0,1,0)}(., u(.), u'(.))$, $\psi^{(0,0,1)}(., u(.), u'(.))$, we have

$$\begin{aligned}
\|\Psi^{(0,1,0)}(f + \mathcal{K}z_0) - \Psi^{(0,1,0)}(f + \mathcal{K}z)\|_\infty &\le c_2\{\|\mathcal{K}(z_0 - z)\|_\infty + \|\mathcal{L}(z_0 - z)\|_\infty\} \\
&\le c_2\{M_1 + M_2\}\|z_0 - z\|_\infty \le Mc_2\delta,
\end{aligned} \tag{3.20}$$

and similarly we can obtain

$$\|\Psi^{(0,0,1)}(f + \mathcal{K}z_0) - \Psi^{(0,0,1)}(f + \mathcal{K}z)\|_\infty \le Mc_3\delta. \tag{3.21}$$

Using the estimates (3.20), (3.21) in (3.19), we obtain

$$\|[\mathcal{T}_n'(z_0) - \mathcal{T}_n'(z)]v\|_\infty \le pM^2c\delta\|v\|_\infty.$$

Thus we have

$$\sup_{\|z - z_0\| \le \delta} \|(\mathcal{I} - \mathcal{T}_n'(z_0))^{-1}(\mathcal{T}_n'(z_0) - \mathcal{T}_n'(z))\|_\infty \le A_1pM^2c\delta \le q.$$

Here we choose δ in such a way that, $0 < q < 1$. This proves Eq. (3.1) of Theorem 2.

We have

$$\begin{aligned}
\alpha_n &= \|(\mathcal{I} - \mathcal{T}_n'(z_0))^{-1}(\mathcal{T}_n(z_0) - \mathcal{T}(z_0))\|_\infty \\
&\le A_1\|(\mathcal{T}_n(z_0) - \mathcal{T}(z_0))\|_\infty \\
&\le A_1\|\mathcal{P}_n\Psi(f + \mathcal{K}z_0) - \Psi(f + \mathcal{K}z_0)\|_\infty \\
&\le A_1\|(\mathcal{I} - \mathcal{P}_n)\Psi(f + \mathcal{K}z_0)\|_\infty \\
&\le A_1\|(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty.
\end{aligned} \tag{3.22}$$

Using the estimate (3.5), we get

$$\|(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty = \max_i \|(\mathcal{I} - \mathcal{P})(z_0)_i\|_{\infty, \triangle_i} = \max_i\{ch_i^{(\beta)}\|(z_0)_i^{(\beta)}\|_\infty\} = \mathcal{O}(h^\beta), \tag{3.23}$$

where $\beta = \min\{k_1, r+1\}$. Hence from the estimates (3.22) and (3.23), we get

$$\alpha_n \le A_1\|(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty \to 0, \quad as \ \ n \to \infty.$$

By choosing n large enough such that $\alpha_n \le \delta(1-q)$, Eq. (3.3) of Theorem 2 is satisfied. Hence by applying Theorem 2, we obtain

$$\frac{\alpha_n}{1+q} \le \|z_n - z_0\|_\infty \le \frac{\alpha_n}{1-q},$$

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and

$$\|z_n - z_0\|_\infty \le \frac{\alpha_n}{1-q} \le c\|(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty.$$

Hence from estimate (3.23), we have

$$\|z_n - z_0\|_\infty = \mathcal{O}(h^\beta),$$

where $\beta = \min\{k_1, r+1\}$. $\square$

Next we discuss the existence and convergence of the iterated approximate solutions $\tilde{z}_n$ to z_0.

Theorem 4. *$\widetilde{\mathcal{T}}_n'(z_0)$ is ν-convergent to $\mathcal{T}'(z_0)$ in infinity norm.*

Proof. Consider

$$\begin{aligned} |\widetilde{\mathcal{T}}_n'(z_0)z(t)| &= |\Psi^{(0,1,0)}(f + \mathcal{K}\mathcal{P}_n z_0)\mathcal{K}\mathcal{P}_n z(t) + \Psi^{(0,0,1)}(f + \mathcal{K}\mathcal{P}_n z_0)\mathcal{L}\mathcal{P}_n z(t)| \\ &\le |\Psi^{(0,1,0)}(f + \mathcal{K}\mathcal{P}_n z_0) - \Psi^{(0,1,0)}(f + \mathcal{K}z_0)||\mathcal{K}\mathcal{P}_n z(t)| + |\Psi^{(0,1,0)}(f + \mathcal{K}z_0)||\mathcal{K}\mathcal{P}_n z(t)| \\ &+ |\Psi^{(0,0,1)}(f + \mathcal{K}\mathcal{P}_n z_0) - \Psi^{(0,0,1)}(f + \mathcal{K}z_0)||\mathcal{L}\mathcal{P}_n z(t)| + |\Psi^{(0,0,1)}(f + \mathcal{K}z_0)||\mathcal{L}\mathcal{P}_n z(t)|. \end{aligned} \tag{3.24}$$

Next using Lemma 2 and Lipschitz continuity of $\psi^{(0,1,0)}(., u(.), u'(.))$, $\psi^{(0,0,1)}(., u(.), u'(.))$, we have

$$\begin{aligned} \|\Psi^{(0,1,0)}(f + \mathcal{K}\mathcal{P}_n z_0) - \Psi^{(0,1,0)}(f + \mathcal{K}z_0)\|_\infty &\le c_2\{\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty + \|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty\} \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \|\Psi^{(0,0,1)}(f + \mathcal{K}\mathcal{P}_n z_0) - \Psi^{(0,0,1)}(f + \mathcal{K}z_0)\|_\infty &\le c_3\{\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty + \|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty\} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.26}$$

From estimate (2.12), we have

$$\|\mathcal{K}\mathcal{P}_n z\|_\infty \le M_1 p\|z\|_\infty, \tag{3.27}$$

and

$$\|\mathcal{L}\mathcal{P}_n z\|_\infty \le M_2 p\|z\|_\infty. \tag{3.28}$$

Hence using the estimates (3.25)–(3.28) in (3.24), we obtain

$$\begin{aligned} \|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty \le\ & M_1 p(c_2\{\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty + \|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty\} + |\Psi^{(0,1,0)}(f + \mathcal{K}z_0)|_\infty) \\ &+ M_2 p(c_3\{\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty + \|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty\} + |\Psi^{(0,0,1)}(f + \mathcal{K}z_0)|_\infty) < \infty. \end{aligned} \tag{3.29}$$

This shows that $\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty$ is uniformly bounded.

Next consider

$$\begin{aligned} |(\widetilde{\mathcal{T}}_n'(z_0) - \mathcal{T}'(z_0))\widetilde{\mathcal{T}}_n'(z_0)z(t)| &= |\{\Psi^{(0,1,0)}(f + \mathcal{K}\mathcal{P}_n z_0)\mathcal{K}\mathcal{P}_n + \Psi^{(0,0,1)}(f + \mathcal{K}\mathcal{P}_n z_0)\mathcal{L}\mathcal{P}_n \\ &- \Psi^{(0,1,0)}(f + \mathcal{K}z_0)\mathcal{K} + \Psi^{(0,0,1)}(f + \mathcal{K}z_0)\mathcal{L}\}\widetilde{\mathcal{T}}_n'(z_0)z(t)| \\ &= |\Psi^{(0,1,0)}(f + \mathcal{K}\mathcal{P}_n z_0)(\mathcal{K}\mathcal{P}_n - \mathcal{K})\widetilde{\mathcal{T}}_n'(z_0)z(t)| \\ &+ |\{\Psi^{(0,1,0)}(f + \mathcal{K}\mathcal{P}_n z_0) - \Psi^{(0,1,0)}(f + \mathcal{K}z_0)\}\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z(t)| \\ &+ |\Psi^{(0,0,1)}(f + \mathcal{K}\mathcal{P}_n z_0)(\mathcal{L}\mathcal{P}_n - \mathcal{L})\widetilde{\mathcal{T}}_n'(z_0)z(t)| \\ &+ |\{\Psi^{(0,0,1)}(f + \mathcal{K}\mathcal{P}_n z_0) - \Psi^{(0,0,1)}(f + \mathcal{K}z_0)\}\mathcal{L}\widetilde{\mathcal{T}}_n'(z_0)z(t)|. \end{aligned} \tag{3.30}$$

Now for the first and third term in the above estimate (3.30), we have

$$\begin{aligned} \|(\mathcal{K}\mathcal{P}_n - \mathcal{K})\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty &= \sup_{x\in[0,1]} \left|\int_0^1 G(x,\xi)(\mathcal{I} - \mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)z(\xi)\,d\xi\right| \\ &\le M_1\|(\mathcal{I} - \mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\|z\|_\infty, \end{aligned} \tag{3.31}$$

and similarly we can obtain

$$\|(\mathcal{L}\mathcal{P}_n - \mathcal{L})\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \le M_2\|(\mathcal{I} - \mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\|z\|_\infty. \tag{3.32}$$

For the second term and fourth term of the estimate (3.30), using Lipschitz continuity, we have

$$\begin{aligned} &\|\{\Psi^{(0,1,0)}(f + \mathcal{K}\mathcal{P}_n z_0) - \Psi^{(0,1,0)}(f + \mathcal{K}z_0)\}\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \\ &\le c_2\{\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty + \|\mathcal{L}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty\}\|\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \end{aligned}$$

$$\begin{aligned} &\leq c_2\{M_1\|(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty + M_2\|(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\}\|\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \\ &\leq c_2M^2\|(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\|z\|_\infty, \end{aligned} \tag{3.33}$$

and similarly on same lines, we can prove that

$$\|\{\Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz_0) - \Psi^{(0,0,1)}(f+\mathcal{K}z_0)\}\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \leq c_3M^2\|(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\|z\|_\infty. \tag{3.34}$$

Hence using the estimates (3.16), (3.17) and (3.31)–(3.34) in (3.30), we see that

$$\begin{aligned} \|(\widetilde{\mathcal{T}}_n'(z_0) - \mathcal{T}'(z_0))\widetilde{\mathcal{T}}_n'(z_0)z\| \leq\ & \{B_1M_1\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_\infty + c_2M^2\|(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty \\ & + B_2M_2\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_\infty + c_3M^2\|(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\}\|z\|_\infty. \end{aligned}$$

Hence using Lemmas 1, 3 and the uniform boundedness of $\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty$, we obtain

$$\|(\widetilde{\mathcal{T}}_n'(z_0) - \mathcal{T}'(z_0))\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \to 0, \quad as \quad n \to \infty.$$

Following the similar steps we can prove that

$$\|(\widetilde{\mathcal{T}}_n'(z_0) - \mathcal{T}'(z_0))\mathcal{T}'(z_0)\|_\infty \to 0, \quad as \quad n \to \infty.$$

This shows that $\widetilde{\mathcal{T}}_n'(z_0)$ is ν-convergent to $\mathcal{T}'(z_0)$ in infinity norm. This completes the proof. □

Using [39], Theorem 4 and since 1 is not an eigenvalue of $[\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\mathcal{K} + \Psi^{(0,0,1)}(f+\mathcal{K}z_0)\mathcal{L}]$, we have for sufficiently large n, the operator $(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))$ is invertible on $\mathbb{X}$ and there exists a constant $\mathbb{L} > 0$ independent of n such that $\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}\|_\infty \leq \mathbb{L} < \infty$.

Theorem 5. *Let $z_0 \in \mathcal{C}^{k_1}[0,1]$ be solution of Eq. (2.9) and 1 is not an eigenvalue of $[\Psi^{(0,1,0)}(f+\mathcal{K}z_0)\mathcal{K} + \Psi^{(0,0,1)}(f+\mathcal{K}z_0)\mathcal{L}]$, which is the Fréchet derivative of $\Psi(f+\mathcal{K}z_0)$ at z_0. Let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ is the orthogonal projection operator. Then for sufficiently large n and for some $\delta > 0$, Eq. (2.18) has an isolated solution $\tilde{z}_n \in B(z_0,\delta) = \{z : \|z - z_0\|_\infty < \delta\}$. Moreover, ∃ a constant $0 < q < 1$, ind. of n such that*

$$\frac{\beta_n}{1+q} \leq \|\tilde{z}_n - z_0\|_\infty \leq \frac{\beta_n}{1-q},$$

where $\beta_n = \|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0))\|_\infty$. Further, we obtain

$$\|\tilde{z}_n - z_0\|_\infty = \mathcal{O}(h^{\beta+\beta^{**}}), \tag{3.35}$$

where $\beta^{**} = \min\{\beta_1, \beta_2\}$.

Proof. We know that $\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}\|_\infty \leq \mathbb{L} < \infty$

Consider for any $z \in B(z_0,\delta)$ and $v \in \mathbb{X}$,

$$\begin{aligned} \|[\mathcal{T}_n'(z_0) - \mathcal{T}_n'(z)]v\|_\infty =\ & \|[\Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz) - \Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz_0)]\mathcal{K}\mathcal{P}_nv\|_\infty \\ & + \|\Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz) - \Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz_0)\mathcal{L}\mathcal{P}_nv\|_\infty \\ & \leq \|\Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz) - \Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz_0)\|_\infty\|\mathcal{K}\mathcal{P}_nv\|_\infty \\ & + \|\Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz) - \Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz_0)\|_\infty\|\mathcal{L}\mathcal{P}_nv\|_\infty. \end{aligned} \tag{3.36}$$

Using Lipschitz continuity and estimates (3.27) and (3.28), we have

$$\begin{aligned} \|\Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz) - \Psi^{(0,1,0)}(f+\mathcal{K}\mathcal{P}_nz_0)\|_\infty &\leq c_2\{\|\mathcal{K}\mathcal{P}_n(z-z_0)\|_\infty + \|\mathcal{L}\mathcal{P}_n(z-z_0)\|_\infty\} \\ &\leq c_2p\{M_1+M_2\}\|z-z_0\|_\infty \leq Mpc_2\delta, \end{aligned} \tag{3.37}$$

similarly

$$\|\Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz) - \Psi^{(0,0,1)}(f+\mathcal{K}\mathcal{P}_nz_0)\|_\infty \leq Mpc_3\delta. \tag{3.38}$$

Hence using the estimates (3.27), (3.28), (3.37) and (3.38) in (3.36), we obtain

$$\|[\mathcal{T}_n'(z_0) - \mathcal{T}_n'(z)]v\|_\infty \leq p^2M^2c\delta\|v\|_\infty.$$

Thus we have

$$\sup_{\|z-z_0\|\leq\delta} \|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n'(z) - \widetilde{\mathcal{T}}_n'(z_0))\|_\infty \leq \mathbb{L}p^2M^2c\delta \leq q.$$

Here we choose δ in such a way that, $0 < q < 1$. This proves Eq. (3.1) of Theorem 2.

Hence applying the Lipschitz continuity and Lemma 2, we have

$$\begin{aligned}\beta_n = \|(\mathcal{I}-\widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0)-\mathcal{T}(z_0))\|_\infty &\le \|(\mathcal{I}-\widetilde{\mathcal{T}}_n'(z_0))^{-1}\|_\infty\|\widetilde{\mathcal{T}}_n(z_0)-\mathcal{T}(z_0)\|_\infty\\ &\le \mathbb{L}\|\Psi(f+\mathcal{K}\mathcal{P}_nz_0)-\Psi(f+\mathcal{K}z_0)\|_\infty\\ &\le \mathbb{L}c_1\{\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\}\\ &\to 0,\quad as\quad n\to\infty.\end{aligned}\tag{3.39}$$

By choosing n large enough such that $\beta_n \le \delta(1-q)$, Eq. (3.3) of Theorem 2 is satisfied. Hence by applying Theorem 2, we obtain

$$\frac{\beta_n}{1+q} \le \|\tilde{z}_n - z_0\|_\infty \le \frac{\beta_n}{1-q},$$

and

$$\|\tilde{z}_n - z_0\|_\infty \le \frac{\beta_n}{1-q} \le c\mathbb{L}\{\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty+\|\mathcal{L}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\}.$$

Hence from Lemma 2, we have

$$\|\tilde{z}_n - z_0\|_\infty = \mathcal{O}(h^{\beta+\beta^{**}}),$$

where $\beta^{**} = \min\{\beta_1, \beta_2\}$.

This completes the proof. □

Theorem 6. *Let $z_0 \in \mathcal{C}^{k_1}[0,1]$ be an isolated solution of Eq. (2.9). Assume that 1 is not an eigenvalue of the linear operator $\mathcal{T}'(z_0)$. Then for n large enough, the iterated Galerkin solution $\tilde{z}_n$ of (2.18) will satisfy*

$$\sup_{x\in\triangle}|\tilde{z}_n(x)-z_0(x)| = \mathcal{O}(h^{\beta+\beta^*}),$$

where $\beta = \min\{k_1, r+1\}$, $\beta^* = \min\{\beta, \gamma+2\}$.

Proof. The following identity is easily verified:

$$\begin{aligned}(\mathcal{I}-\mathcal{T}'(z_0))(\tilde{z}_n - z_0) &= [\mathcal{I}-\mathcal{T}'(z_0)(\mathcal{I}-\mathcal{P}_n)][\mathcal{T}z_n-\mathcal{T}z_0-\mathcal{T}'(z_0)(z_n-z_0)]\\ &\quad -\mathcal{T}'(z_0)(\mathcal{I}-\mathcal{P}_n)-\mathcal{T}'(z_0)(z_n-z_0)\mathcal{T}'(z_0)(\mathcal{I}-\mathcal{P}_n)z_0.\end{aligned}\tag{3.40}$$

Applying $(\mathcal{I}-\mathcal{T}'(z_0))^{-1}$ and defining $M = (\mathcal{I}-\mathcal{T}'(z_0))^{-1}\mathcal{T}'(z_0)$, we have

$$\begin{aligned}(\tilde{z}_n - z_0) &= [\mathcal{I}-M\mathcal{P}_n][\mathcal{T}z_n-\mathcal{T}z_0-\mathcal{T}'(z_0)(z_n-z_0)] - M(\mathcal{I}-\mathcal{P}_n)\mathcal{T}'(z_0)(z_n-z_0)\\ &\quad - M(\mathcal{I}-\mathcal{P}_n)z_0.\end{aligned}\tag{3.41}$$

Using Lipschitz continuity of $\mathcal{T}(z_0)$ and boundedness of $\mathcal{T}'(z_0)$ and Theorem 5, we can show that

$$[\mathcal{T}z_n-\mathcal{T}z_0-\mathcal{T}'(z_0)(z_n-z_0)] = \mathcal{O}(\|z_n-z_0\|_\infty^2) = \mathcal{O}(h^{2\beta}),$$

where $\beta^* = \min\{\beta, \beta_1+1\}$, $\beta_1 = \min\{\beta, \gamma+1\}$.

According to the Lemma 5.1 and Theorem 5.2 of [28], we can write Eq. (3.41) as

$$\tilde{z}_n(x) - z_0(x) = \phi_n(x) + \psi_n(x) + \mathcal{O}(h^{2\beta}),\tag{3.42}$$

with

$$\phi_n(x) = -\langle(\mathcal{I}-\mathcal{P}_n)\mathcal{T}'(z_0)(z_n-z_0), \bar{m}_x\rangle,$$

$$\psi_n(x) = -\langle(\mathcal{I}-\mathcal{P}_n)z_0, \bar{m}_x\rangle.$$

Following the similar steps as in the proof of Theorem 5.2 of [28], we can show that

$$\sup_{x\in\triangle}|\psi_n(x)| = \mathcal{O}(h^{2\beta}),\tag{3.43}$$

and

$$\sup_{x\in\triangle}|\phi_n(x)| = \mathcal{O}(h^{2\beta+\beta_1}),\tag{3.44}$$

where $\beta = \min\{k_1, r+1\}$ and $\beta_1 = \min\{\beta, \gamma+1\}$.

Hence combining the results (3.42)–(3.44), we get

$$\sup_{x\in\triangle}|\tilde{z}_n(x)-z_0(x)| = \mathcal{O}(h^{\beta+\beta^*}).$$

Hence the proof follows. □

Table 1
Galerkin and iterated Galerkin methods.

n	$\|x-x_n^G\|_\infty$	α	$\|x-\tilde{x}_n^G\|_\infty$	a
2	$1.064283944\times 10^{-2}$	–	$1.215226743\times 10^{-4}$	
4	$2.6015808426\times 10^{-3}$	2.03	$1.581178576\times 10^{-5}$	2.94
8	$6.452034988\times 10^{-4}$	2.01	$1.815377407\times 10^{-6}$	3.12
16	$1.595941766\times 10^{-4}$	2.01	$2.266913439\times 10^{-7}$	3.00
32	$3.910143702\times 10^{-5}$	2.02	$3.470940979\times 10^{-8}$	2.70
64	$9.387825960\times 10^{-6}$	2.05	$5.195472384\times 10^{-9}$	2.74
128	$2.161365511\times 10^{-6}$	2.11	$7.106715881\times 10^{-10}$	2.87

Theorem 7. *Let $u_0 \in \mathcal{C}^{k_1}[0,1]$ be an isolated solution of Eq. (2.1) and u_n and $\tilde{u}_n$ be the Galerkin and iterated Galerkin approximations of u_0. Then there hold*

$$\|u_n-u_0\|_\infty = \mathcal{O}(h^{\beta}),$$

and

$$\|\tilde{u}_n-u_0\|_\infty = \mathcal{O}(h^{\beta+\beta^{**}}),$$

where $\beta^{**}=\min\{\beta_1,\beta_2\}$.

Proof. From the Eqs. (2.7), (2.17), (2.23) and Theorems 3, 5 and 6, we obtain

$$\|u_n-u_0\|_\infty = \|\mathcal{K}(z_0-z_n)\|_\infty \le M_1\|z_0-z_n\|_\infty = \mathcal{O}(h^{\beta}),$$

and

$$\|\tilde{u}_n-u_0\|_\infty = \|\mathcal{K}(z_0-\tilde{z}_n)\|_\infty \le M_1\|z_0-\tilde{z}_n\|_\infty = \mathcal{O}(h^{\beta+\beta^{**}}),$$

where $\beta^{**}=\min\{\beta_1,\beta_2\}$. This completes the proof.

4. Numerical outcome

In this unit, we present the numerical outcomes. For that we take the piecewise polynomials as the basis functions for the subspace $\mathbb{X}_n$. We present the errors of the approximate and iterated approximate solutions of Galerkin and iterated Galerkin methods in uniform norm. We denote the Galerkin and iterated Galerkin solutions by x_n^G and $\tilde{x}_n^G$, respectively. Also, we denote $\|x-x_n^G\|_\infty = \mathcal{O}(h^{\alpha})$ and $\|x-\tilde{x}_n^G\|_\infty = \mathcal{O}(h^{a})$. The numerical tests were performed on a PC Intel(R) Core (TM) i5-3470 CPU @ 3.20 GHz Processor, 16.00 GB RAM and 62-bit operating system on Matlab (R2012b).

Consider the uniform partition of [0, 1]:

$$0=t_0<t_1<t_2<\cdots<t_n=1$$

where $t_i=\frac{i}{n},\ i=0,1,2,\ldots,n$.

We choose the approximating subspaces as the space of piecewise linear functions ($r=1$), which has dimension $(n+1)$. Then for $r=1$, the expected orders of convergence are $\alpha=2$ and $a=3$. We present the errors in Galerkin method and iterated Galerkin method in Tables 1 and 2.

Example 1. Consider the following problem (Table 1)

$$(u'(x))' = -(2xe^{u}u'+2e^{u})$$
$$u(0)=ln(\frac{1}{4}),\quad u(1)=ln(\frac{1}{5})$$

which is equivalent to the Hammerstein integral equation

$$u(x)=f(x)+\int_0^1 G(x,\xi)\psi(\xi,u(\xi),u'(\xi))\,d\xi,\quad 0\le x\le 1,$$

where $f(x)=ln(\frac{1}{4})+ln(\frac{4}{5})x$, $\psi(\xi,u(\xi),u'(\xi))=-(2\xi e^{u}u'+2e^{u})$ and the exact solution is given by $u(x)=ln(\frac{1}{4+x^2})$, and the kernel function

$$G(x,\xi)=\begin{cases} -x(1-\xi), & 0\le x\le \xi,\\ -\xi(1-x), & \xi\le x\le 1.\end{cases}$$

Table 2
Galerkin and iterated Galerkin methods.

n	$\|x - x_n^G\|_\infty$	α	$\|x - \tilde{x}_n^G\|_\infty$	a
2	$4.520675529 \times 10^{-3}$	–	$1.053294795 \times 10^{-4}$	–
4	$1.211726903 \times 10^{-3}$	1.89	$1.439106499 \times 10^{-5}$	2.87
8	$3.131417535 \times 10^{-4}$	1.95	$1.878608957 \times 10^{-6}$	2.93
16	$7.924930613 \times 10^{-5}$	1.98	$2.386264426 \times 10^{-7}$	2.97
32	$1.978111375 \times 10^{-5}$	2.00	$3.129019143 \times 10^{-8}$	2.93
64	$4.867363857 \times 10^{-6}$	2.02	$5.100316836 \times 10^{-9}$	2.61
128	$1.171047227 \times 10^{-6}$	2.05	$7.172693413 \times 10^{-10}$	2.83

Example 2. Consider the following Volterra integral equation of second kind (Table 2)

$$(u'(x))' = -u'e^u$$
$$u(0) = ln(\frac{1}{2}), \quad u(1) = ln(\frac{1}{3})$$

which is equivalent to the Hammerstein integral equation

$$u(x) = f(x) + \int_0^1 G(x,\xi)\psi(\xi, u(\xi), u'(\xi))\,d\xi, \quad 0 \le x \le 1,$$

where $f(x) = ln(\frac{1}{2}) + ln(\frac{2}{3})x$, $\psi(\xi, u(\xi), u'(\xi)) = -e^u u'$ and the exact solution is given by $u(x) = ln(\frac{1}{2+x})$, and the kernel function

$$G(x,\xi) = \begin{cases} -x(1-\xi), & 0 \le x \le \xi, \\ -\xi(1-x), & \xi \le x \le 1. \end{cases}$$

Example 3. Consider the following derivative-dependent boundary value problem (Tables 3 and 4)

$$(u'(x))' = x^{\lambda-2}\lambda u(xu' + u(\lambda-1))$$
$$u(0) = 0, \quad u(1) = e, \quad \lambda > 0$$

which is equivalent to the Hammerstein integral equation

$$u(x) = f(x) + \int_0^1 G(x,\xi)\psi(\xi, u(\xi), u'(\xi))\,d\xi, \quad 0 \le x \le 1,$$

where $f(x) = ex$, $\psi(\xi, u(\xi), u'(\xi)) = \xi^{\lambda-2}\lambda u(\xi u' + u(\lambda-1))$ and the kernel function

$$G(x,\xi) = \begin{cases} -x(1-\xi), & 0 \le x \le \xi, \\ -\xi(1-x), & \xi \le x \le 1. \end{cases}$$

Example 4. Consider the following Volterra integral equation of second kind (Tables 5 and 6)

$$(u'(x))' = x^{\lambda-2}\lambda u e^u(-xu' - \lambda + 1)$$
$$u(0) = 0, \quad u(1) = -ln(5), \quad \lambda > 0$$

which is equivalent to the Hammerstein integral equation

$$u(x) = f(x) + \int_0^1 G(x,\xi)\psi(\xi, u(\xi), u'(\xi))\,d\xi, \quad 0 \le x \le 1,$$

where $f(x) = -ln(5)x$, $\psi(\xi, u(\xi), u'(\xi)) = \xi^{\lambda-2}\lambda u e^u(-\xi u' - \lambda + 1)$ and the kernel function

$$G(x,\xi) = \begin{cases} -x(1-\xi), & 0 \le x \le \xi, \\ -\xi(1-x), & \xi \le x \le 1. \end{cases}$$

Example 5. We consider the following problem which arises in the study of finite deflections of an elastic string under a transverse load (Table 7) **(see** [40]**)**

$$(u'(x))' = -(1 + a^2(u')^2), x \in [0, 1]$$
$$u(0) = 0, \quad u(1) = 0,$$

Table 3
Galerkin and iterated Galerkin methods, when $\lambda = 1$.

n	$\|x - x_n^G\|_\infty$	α	$\|x - \tilde{x}_n^G\|_\infty$	a
2	$2.098654785 \times 10^{-1}$	–	$1.653201445 \times 10^{-2}$	
4	$5.246636962 \times 10^{-2}$	2.00	$1.875389973 \times 10^{-3}$	3.14
8	$1.284665683 \times 10^{-2}$	2.03	$2.264384133 \times 10^{-4}$	3.05
16	$3.059553362 \times 10^{-3}$	2.07	$2.830480166 \times 10^{-5}$	3.00
32	$7.439724288 \times 10^{-4}$	2.04	$3.441350671 \times 10^{-6}$	3.04
64	$1.735377052 \times 10^{-4}$	2.10	$4.155157042 \times 10^{-7}$	3.05
128	$4.190659347 \times 10^{-5}$	2.05	$5.051917568 \times 10^{-8}$	3.04

Table 4
Galerkin and iterated Galerkin methods, when $\lambda = 2$.

n	$\|x - x_n^G\|_\infty$	α	$\|x - \tilde{x}_n^G\|_\infty$	a
2	$1.125463325 \times 10^{-2}$	–	$3.032561021 \times 10^{-3}$	
4	$5.136946604 \times 10^{-3}$	2.04	$3.661576074 \times 10^{-4}$	3.05
8	$1.240490836 \times 10^{-3}$	2.05	$4.608805438 \times 10^{-5}$	2.99
16	$3.101227090 \times 10^{-4}$	2.00	$5.642446992 \times 10^{-6}$	3.03
32	$7.699513376 \times 10^{-5}$	2.01	$6.812805667 \times 10^{-7}$	3.05
64	$1.846466729 \times 10^{-5}$	2.06	$7.675055324 \times 10^{-8}$	3.15
128	$4.367159922 \times 10^{-6}$	2.08	$9.076305972 \times 10^{-9}$	3.08

Table 5
Galerkin and iterated Galerkin methods, when $\lambda = 1$.

n	$\|x - x_n^G\|_\infty$	α	$\|x - \tilde{x}_n^G\|_\infty$	a
2	$2.025642104 \times 10^{-2}$	–	$2.023654120 \times 10^{-3}$	–
4	$5.427573629 \times 10^{-3}$	1.90	$2.529567652 \times 10^{-4}$	3.00
8	$1.424353647 \times 10^{-3}$	1.93	$3.140118312 \times 10^{-5}$	3.01
16	$3.487602183 \times 10^{-4}$	2.03	$3.844369514 \times 10^{-6}$	3.03
32	$8.306055904 \times 10^{-5}$	2.00	$4.906434863 \times 10^{-7}$	2.97
64	$2.090957273 \times 10^{-5}$	1.99	$6.175702270 \times 10^{-8}$	2.99
128	$5.155426116 \times 10^{-6}$	2.02	$7.613349440 \times 10^{-9}$	3.02

Table 6
Galerkin and iterated Galerkin methods, when $\lambda = 2$.

n	$\|x - x_n^G\|_\infty$	α	$\|x - \tilde{x}_n^G\|_\infty$	a
2	$5.236545214 \times 10^{-3}$	–	$1.053294795 \times 10^{-4}$	–
4	$1.393405646 \times 10^{-3}$	1.91	$1.401369466 \times 10^{-5}$	2.91
8	$3.483514115 \times 10^{-4}$	2.00	$1.763895952 \times 10^{-6}$	2.99
16	$8.529561079 \times 10^{-5}$	2.03	$2.159494373 \times 10^{-7}$	3.03
32	$2.147222217 \times 10^{-5}$	1.99	$2.8139986809 \times 10^{-8}$	2.94
64	$5.221265781 \times 10^{-6}$	2.04	$3.743920361 \times 10^{-9}$	2.91
128	$1.278453421 \times 10^{-6}$	2.03	$4.551928328 \times 10^{-10}$	3.04

which is equivalent to the Hammerstein integral equation

$$u(x) = f(x) + \int_0^1 G(x,\xi)\psi(\xi, u(\xi), u'(\xi))\,d\xi, \quad 0 \le x \le 1,$$

where $f(x) = 0$, $\psi(\xi, u(\xi), u'(\xi)) = -(1 + a^2(u')^2)$, $u(x) = \frac{1}{a^2}\log\left[\frac{\cos a(x-1/2)}{\cos a/2}\right]$ and the kernel function

$$G(x,\xi) = \begin{cases} -x(1-\xi), & 0 \le x \le \xi, \\ -\xi(1-x), & \xi \le x \le 1. \end{cases}$$

From Tables 1–7 we see that the iterated approximate solution gives better convergence rates than the approximate solution in Galerkin method. Also, to verify computational improvement, we have calculated CPU time for the above numerical results. We observe the CPU times for evaluating the numerical results for the Galerkin and iterated methods given in Table 8.

Table 7
Galerkin and iterated Galerkin methods, when $a = \frac{1}{7}$.

n	$\|x - x_n^G\|_\infty$	α	$\|x - \tilde{x}_n^G\|_\infty$	a
2	$5.154286542 \times 10^{-1}$	–	$1.125412361 \times 10^{-2}$	–
4	$4.231056513 \times 10^{-3}$	2.03	$5.021048963 \times 10^{-4}$	3.12
8	$1.456225614 \times 10^{-4}$	2.06	$3.024101471 \times 10^{-6}$	3.05
16	$9.03201581 \times 10^{-6}$	1.97	$1.047851144 \times 10^{-7}$	3.01
32	$1.484125743 \times 10^{-7}$	2.01	$2.014542411 \times 10^{-8}$	3.04
64	$5.102102441 \times 10^{-8}$	2.03	$4.214655441 \times 10^{-9}$	3.16
128	$3.021025414 \times 10^{-10}$	2.06	$6.021540125 \times 10^{-11}$	3.08

Table 8
CPU time for the numerical results.

Numerical example	λ	Galerkin method time	Iterated Galerkin method time
Example-1	–	211.67 s	356.34 s
Example-2	–	184.67 s	378.60 s
Example-3	$\lambda = 1$	124:92 s	225.56 s
Example-3	$\lambda = 2$	123:41 s	227:31 s
Example-4	$\lambda = 1$	190:41 s	346:74 s
Example-4	$\lambda = 2$	195:67 s	365:34 s
Example-5	–	99.12 s	150.70 s

References

[1] P.L. Chambre, On the solution of the Poisson–Boltzmann equation with application to the theory of thermal explosions, J. Chem. Phys. 20 (11) (1952) 1795–1797.
[2] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, Vol. 74, Ciel et Terre, 1958, p. 379.
[3] J.B. Keller, B. Joseph, Electrohydrodynamics I. The equilibrium of a charged gas in a container, J. Ration. Mech. Anal. (1956) 715–724.
[4] R.C. Duggan, A.M. Goodman, Pointwise bounds for a nonlinear heat conduction model of the human head, Bull. Math. Biol. 48 (2) (1986) 229–236.
[5] J.A. Adam, S.A. Maggelakis, Mathematical models of tumor growth. IV. Effects of a necrotic core, Math. Biosci. 97 (1) (1989) 121–136.
[6] N. Anderson, A.M. Arthurs, Analytical bounding functions for diffusion problems with Michaelis–Menten kinetics, Bull. Math. Biol. 47 (1) (1985) 145–153.
[7] N.S. Asaithambi, J.B. Garner, Pointwise solution bounds for a class of singular diffusion problems in physiology, Appl. Math. Comput. 30 (3) (1989) 215–222.
[8] G. Adomian, Solution of the Thomas–Fermi equation, Appl. Math. Lett. 11 (3) (1998) 131–133.
[9] G. Ben-Yu, Spectral Methods and their Applications, World Scientific, 1998.
[10] Z. Cen, Numerical study for a class of singular two-point boundary value problems using Green's functions, Appl. Math. Comput. 183 (1) (2006) 10–16.
[11] B. Gray, The distribution of heat sources in the human head theoretical considerations, J. Theoret. Biol. 82 (3) (1980) 473–476.
[12] M. Inç, D.J. Evans, The decomposition method for solving of a class of singular two-point boundary value problems, Int. J. Comput. Math. 80 (7) (2003) 869–882.
[13] R. Singh, J. Kumar, G. Nelakanti, Numerical solution of singular boundary value problems using Green's function and improved decomposition method, J. Appl. Math. Comput. 43 (1–2) (2013) 409–425.
[14] R. Singh, J. Kumar, G. Nelakanti, Approximate series solution of singular boundary value problems with derivative dependence using Green's function technique, Comput. Appl. Math. 33 (2) (2014) 451–467.
[15] G. Reddien, Projection methods for two-point boundary value problems, SIAM Rev. 22 (2) (1980) 156–171.
[16] M. Danish, S. Kumar, S. Kumar, A note on the solution of singular boundary value problems arising in engineering and applied sciences: use of oham, Comput. Chem. Eng. 36 (2012) 57–67.
[17] S.A. Khuri, A. Sayfy, A novel approach for the solution of a class of singular boundary value problems arising in physiology, Math. Comput. Modelling 52 (3–4) (2010) 626–636.
[18] M. Kumar, N. Singh, Modified adomian decomposition method and computer implementation for solving singular boundary value problems arising in various physical problems, Comput. Chem. Eng. 34 (11) (2010) 1750–1760.
[19] P. Das, G. Nelakanti, G. Long, Discrete Legendre spectral projection methods for Fredholm–Hammerstein integral equations, J. Comput. Appl. Math. 278 (2015) 293–305.
[20] P. Das, M.M. Sahani, G. Nelakanti, G. Long, Legendre spectral projection methods for Fredholm–Hammerstein integral equations, J. Sci. Comput. 68 (1) (2016) 213–230.
[21] H. Kaneko, R.D. Noren, Y. Xu, Numerical solutions for weakly singular Hammerstein equations and their superconvergence, J. Integral Equ. Appl. 4 (1992) 391–407.
[22] H. Kaneko, Y. Xu, Superconvergence of the iterated Galerkin methods for Hammerstein equations, SIAM J. Numer. Anal. 33 (1996) 1048–1064.
[23] S. Kumar, Superconvergence of a collocation-type method for Hammerstein equations, IMA J. Numer. Anal. 7 (1987) 313–325.
[24] S. Kumar, I.H. Sloan, A new collocation-type method for Hammerstein integral equations, Math. Comp. 48 (178) (1987) 585–593.
[25] M. Mandal, G. Nelakanti, Superconvergence of Legendre spectral projection methods for Fredholm–Hammerstein integral equations, J. Comput. Appl. Math. 319 (2017) 423–439.
[26] K. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Vol. 4, Cambridge University Press, 1997.
[27] K. Atkinson, F. Potra, et al., The discrete Galerkin method for nonlinear integral equations, J. Integral Equ. Appl. 1 (1) (1988) 17–54.
[28] K.E. Atkinson, F.A. Potra, Projection and iterated projection methods for nonlinear integral equations, SIAM J Numer. Anal. 24 (6) (1987) 1352–1373.
[29] Z. Chen, J. Li, Y. Zhang, A fast multiscale solver for modified Hammerstein equations, Appl. Math. Comput. 218 (2011) 3057–3067.
[30] H. Kaneko, P. Padilla, Y. Xu, Superconvergence of the iterated degenerate kernel method, Appl. Anal. 80 (2001) 331–351.

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[31] M. Mandal, B.L. Panigrahi, G. Nelakanti, Legendre multiGalerkin methods for fredholm integral equations with weakly singular kernel and the corresponding eigenvalue problem, J. Comput. Appl. Math. 346 (2018) 224–236.
[32] M. Mandal, G. Nelakanti, Superconvergence results for weakly singular Fredholm–Hammerstein integral equations, Numer. Funct. Anal. Optim. 40 (5) (2019) 548–570.
[33] M. Mandal, G. Nelakanti, Superconvergence results of Legendre spectral projection methods for weakly singular Fredholm–Hammerstein integral equations, J. Comput. Appl. Math. 349 (2019) 114–131.
[34] M. Turkyilmazoglu, High-order nonlinear Volterra–Fredholm–Hammerstein integro-differential equations and their effective computation, Appl. Math. Comput. 247 (2014) 410–416.
[35] M. Turkyilmazoglu, An effective approach for numerical solutions of high-order Fredholm integro-differential equations, Appl. Math. Comput. 227 (2014) 384–398.
[36] M. Turkyilmazoglu, Solution of initial and boundary value problems by an effective accurate method, Int. J. Comput. Methods 14 (06) (2017) 1750–1769.
[37] F. Chatelin, Spectral Approximation of Linear Operators Ci, SIAM, 1983.
[38] G.M. Vainikko, Galerkin's perturbation method and the general theory of approximate methods for non-linear equations, USSR Comput. Math. Math. Phys. 7 (4) (1967) 1–41.
[39] M. Ahues, A. Largillier, B. Limaye, Spectral Computations for Bounded Operators, CRC Press, 2001.
[40] R.E. Bellman, R.E. Kalaba, Quasilinearization and Nonlinear Boundary-Value Problems, Rand Corporation, 1965.