



ELSEVIER

Journal of Computational and Applied Mathematics 133 (2001) 535–544

JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Modification of a method using Szegő polynomials in frequency analysis: the V-process

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Received 3 November 1999; received in revised form 14 June 2000

Dedicated to Professor Theodore Chihara on the occasion of his 70th birthday

Abstract

Recently, a method has been established for determining the n_0 unknown frequencies ω_j in a trigonometric signal by using Szegő polynomials; $\rho_n(\psi_N; z)$. Essential in the study is the asymptotic behavior of the zeros. If $n \geq n_0$ then n_0 of the zeros in the limit polynomial will tend to the frequency points $e^{\pm i\omega_j}$. The remaining $(n - n_0)$ are bounded away from the unit circle. Several modifications of this method are developed. The modifications are of two main types: Modifying the observed signal values or modifying the moments. In the present paper we will replace the moment sequence $\{\mu_m^{(N)}/N\}$ by a new sequence $\{(\mu_m^{(N)}/N)R^{m^2}\}$, where $R \in (0, 1)$. In this situation we prove the surprising result that a multiple of the n_0 zeros tend to the frequency points. We also prove the rate at which certain Toeplitz determinants tend to zero. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Frequency analysis; Szegő polynomial; Zeros; Toeplitz determinant

1. Background

Determination of the unknown frequencies ω_j in a trigonometric signal when the signal values $x(m)$ from N observations are known, is sometimes referred to as the frequency analysis problem. One method of determining those frequencies is by using certain orthogonal polynomials; the Szegő polynomials. This method has its root back to Wiener [11] and Levinson [3], and it is developed independently by Jones et al. [1] and Pan and Saff [6]. The method starts with N observations of a

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continuous trigonometric signal at equally spaced time. The observations lead to a discrete form of the signal given by

$$x(m) = \sum_{j=-I}^I \alpha_j e^{im\omega_j}, \quad m = 0, 1, \dots, N-1. \quad (1.1)$$

The observed signal values $x(m)$ are known. The normalized frequencies ω_j , the amplitudes α_j and the number $n_0 = 2I + 1$ (see Theorem 1) of frequencies are unknown. A certain absolutely continuous measure, $\psi_N(\theta)$, on the unit circle (see (1.2) in [1]) is constructed from the signal values (1.1). For any fixed N , this measure gives rise to a sequence of moments $\{\mu_m^{(N)}\}$ and in turn to a sequence of monic orthogonal polynomials, the Szegő polynomials $\{\rho_n(\psi_N; z)\}$. These polynomials are orthogonal on the unit circle with respect to the measure $\psi_N(\theta)$, and have all zeros located inside the unit circle. The weak star convergence of this measure to a measure ψ_0 with support in the n_0 frequency points $e^{\pm i\omega_j}$ is one of the essential properties in the method. By using asymptotic behavior of the Szegő polynomials, this property ensures us that n_0 of the zeros in the limit polynomial of degree $n \geq n_0$ will tend to the frequency points $e^{\pm i\omega_j}$.

The key to the solution of the frequency analysis problem is the fact that if we know the number n_0 of frequencies, then the limit polynomial in question will be

$$\lim_{N \rightarrow \infty} \rho_{n_0}(\psi_N; z) = (z-1)^L \prod_{j=1}^I (z - e^{i\omega_j})(z - e^{-i\omega_j}). \quad (1.2)$$

If we do not know the number of frequencies, which is the usual situation, we have to pick an n such that $n \geq n_0$. In this case we have to go to certain subsequences to obtain convergence. The crucial point is that in each convergence case (1.2) will be a factor. Hence n_0 of the zeros will tend to the frequency points. In addition, we have $(n - n_0)$ zeros, sometimes referred to as the “uninteresting” zeros. From [6], it can be extracted that there exists a number $K_n < 1$ such that

$$|z_p^{(N_k)}| < K_n < 1, \quad p = n_0 + 1, \dots, n. \quad (1.3)$$

Hence we can separate the “uninteresting” zeros from the frequency points. The method is called the N -process.

2. Modifications

Several extensions or modifications of the N -process are dealt with in a number of papers throughout the last ten years. The modifications can be classified as two main types:

2.1. Modification of the signal

In this situation the crucial step is to multiply the observed signal values (1.1) by certain *weight functions*. The first modification of this type was to multiply $\{x(m)\}$ by T^m where $T \in (0, 1)$ [5]. In this situation we can still pick out the frequency points, and the “uninteresting” zeros are bounded away from the unit circle. Another method using this type of modification is described in [9,10].

2.2. Modification of the moments

In this situation the moments

$$\left\{ \frac{\mu_m^{(N)}}{N} \right\} = \{ \tilde{\mu}_m^{(N)} \} \quad (2.1)$$

from the N -process or

$$\left\{ \lim_{N \rightarrow \infty} \frac{\mu_m^{(N)}}{N} \right\} = \sum_{j=-l}^l |\alpha_j|^2 e^{im\omega_j} =: \mu_m \quad (2.2)$$

are multiplied by *certain factors*. The first modification of this type was to replace the moment sequence (2.1) by a new moment sequence $\{(\mu_m^{(N)}/N)R^{|m|}\}$, where $R \in (0, 1)$ [2]. Then we have a new corresponding measure $\psi_N^{(R)}$ [4]. The properties mentioned in 2.1 hold. Another method involving this moment-modification can be found in [7].

2.3. Tools

To establish the proof of a later theorem, we will need some tools. We use the determinant formula for the Szegő polynomials: where $\Delta_{n-1}^{(N)}$ is the Toeplitz determinant of order n .

The expressions of the Szegő polynomials and Toeplitz determinant of order $(n+1)$ are

$$\rho_n(\psi_N; z) = \frac{1}{\Delta_{n-1}^{(N)}} \begin{vmatrix} \tilde{\mu}_0^{(N)} & \tilde{\mu}_{-1}^{(N)} & \cdots & \tilde{\mu}_{-n}^{(N)} \\ \tilde{\mu}_1^{(N)} & \tilde{\mu}_0^{(N)} & \cdots & \tilde{\mu}_{-n+1}^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mu}_{n-1}^{(N)} & \tilde{\mu}_{n-2}^{(N)} & \cdots & \tilde{\mu}_{-1}^{(N)} \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad (2.3a)$$

$$\Delta_n^{(N)} = \begin{vmatrix} \tilde{\mu}_0^{(N)} & \tilde{\mu}_{-1}^{(N)} & \tilde{\mu}_{-2}^{(N)} & \cdots & \tilde{\mu}_{-n}^{(N)} \\ \tilde{\mu}_1^{(N)} & \tilde{\mu}_0^{(N)} & \tilde{\mu}_{-1}^{(N)} & \cdots & \tilde{\mu}_{-n+1}^{(N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\mu}_n^{(N)} & \tilde{\mu}_{n-1}^{(N)} & \tilde{\mu}_{n-2}^{(N)} & \cdots & \tilde{\mu}_0^{(N)} \end{vmatrix}. \quad (2.3b)$$

The moment sequence is positive definite meaning that

$$\tilde{\mu}_n^{(N)} = \tilde{\mu}_{-n}^{(N)} \quad \text{and} \quad \Delta_{n-1}^{(N)} > 0. \quad (2.3c)$$

The identity

$$\cos m\omega_j = 2 \cos \omega_j \cos(m-1)\omega_j - \cos(m-2)\omega_j \quad (2.3d)$$

will play an important role. For the sake of simplicity we will use the notation $\cos m\omega_j = T_m(x_j)$, so that (2.3d) can be written as

$$T_m(x_j) = 2x_j T_{m-1}(x_j) - T_{m-2}(x_j). \quad (2.3e)$$

3. The V-process

In the present paper we introduce a method called the V-process. We modify *the moments* in the following way:

Replace $\{\lim_{N \rightarrow \infty} \mu_m^{(N)}/N\}$ by $\{(\lim_{N \rightarrow \infty} \mu_m^{(N)}/N)R^{m^2}\}$. Then the “moments” (see Final remarks) can be written

$$\begin{aligned}\mu_m^{(V)} &:= \left(\lim_{N \rightarrow \infty} \frac{\mu_m^{(N)}}{N} \right) R^{m^2} = \sum_{j=-I}^I |\alpha_j|^2 e^{im\omega_j} R^{m^2} \\ &= \left(\alpha_0^2 + 2 \sum_{j=0}^I |\alpha_j|^2 \cos m\omega_j \right) R^{m^2}.\end{aligned}$$

With $R = 1 - d$ we can write

$$\mu_m^{(V)} = \left(\alpha_0^2 + 2 \sum_{j=0}^I |\alpha_j|^2 \cos m\omega_j \right) (1 - d)^{m^2}. \quad (3.1)$$

Theorem 1. Let $n = \beta n_0 + \gamma$ where $\beta = 1, 2, \dots$; $\gamma = 0, 1, \dots, n_0 - 1$; $n_0 = 2I + L$, $L = 1$ if $\alpha_0 \neq 0$, $L = 0$ otherwise.

Furthermore let $\mu_m^{(V)} := \mu_m R^{m^2}$ where $R \in (0, 1)$, $d = 1 - R$.

Then the limit

$$\lim_{R \uparrow 1} \left(\lim_{n \rightarrow \infty} \rho_N(\psi_N^{(V)}; z) \right) =: \hat{\rho}_n(z)$$

exists and for $n \geq n_0$, the polynomial $\hat{\rho}_n(z)$ is of the form

$$\hat{\rho}_n(z) = (z - 1)^{\beta L} \prod_{j=1}^I ((z - e^{i\omega_j})(z - e^{-i\omega_j}))^\beta \prod_{p=1}^\gamma (z - z_p^{(\beta, n)}).$$

Proof. In the proof we will first consider a simple signal ($n_0 = 2$)

$$x(m) = \frac{1}{\sqrt{2}} e^{im\omega} + \frac{1}{\sqrt{2}} e^{-im\omega}.$$

The moments in the V-process are computed from (3.1):

$$\mu_m^{(V)} = \cos m\omega (1 - d)^{m^2} = T_m(x)(1 - d)^{m^2}.$$

We consider the Szegő polynomial of degree n . Then we have the determinant formula (2.3a):

$$\begin{vmatrix} \mu_0^{(V)} & \mu_{-1}^{(V)} & \cdots & \mu_{-n}^{(V)} \\ \mu_1^{(V)} & \mu_0^{(V)} & \cdots & \mu_{-n+1}^{(V)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}^{(V)} & \mu_{n-2}^{(V)} & \cdots & \mu_{-1}^{(V)} \\ 1 & z & \cdots & z^n \end{vmatrix} \frac{1}{\Delta_{n-1}^{(V)}} = \rho_n(\psi_N^{(V)}; z). \quad (3.2)$$

From left to right we use the following numbers of the columns: $k = \{1, 2, \dots, n+1\}$. Define, $T_{-m}(x) := T_m(x)$ where $m = \{-n+1, -n+2, \dots, -1, 0, 1, \dots, n-1, n\}$. Crucial in the proof is the fact (2.3e):

$$T_m(x) - 2xT_{m-1}(x) + T_{m-2}(x) = 0. \quad (3.3)$$

For the sake of simplicity we abbreviate $T_m(x) =: T_m$, in the rest of the proof. Notice that $(k+m)$ is *constant* in each row, and equal to the *row* number. Let

$$t := 2k + 2m = 2(k+m) \quad (3.4)$$

and observe that $t \geq 1$ for all t . We simplify the determinant in (3.2) by using column operations. Replace column k by

$$\text{col}(k) - 2x(1-d)^{2k-1} \text{col}(k-1) + (1-d)^{4k-4} \text{col}(k-2). \quad (3.5)$$

Since the indices of the moments are increasing from right to left, we get just one case to consider if we ignore the last row. The column operations lead to the elements.

$$\begin{aligned} & T_m(1-d)^{m^2} - 2xT_{m+1}(1-d)^{2k-1}(1-d)^{(m+1)^2} + T_{m+2}(1-d)^{4k-4}(1-d)^{(m+2)^2} \\ &= T_m(1-d)^{m^2} - 2xT_{m+1}(1-d)^{m^2+2m+2k} + T_{m+2}(1-d)^{m^2+4m+4k} \\ &= T_m(1-d)^{m^2} - 2xT_{m+1}(1-d)^{m^2+t} + T_{m+2}(1-d)^{m^2+2t} \\ &= (1-d)^{m^2} \{T_m - 2xT_{m+1}(1-d)^t + T_{m+2}(1-d)^{2t}\}. \end{aligned}$$

We observe that the terms without d will vanish because of the identity (3.3). Hence we may write:

$$\begin{aligned} & (1-d)^{m^2} \left\{ -2x(-td + \frac{t(t-1)}{2}d^2 + O(d^3))T_{m+1} + \left(-2td + \frac{2t(2t-1)}{2}d^2 + O(d^3)\right)T_{m+2} \right\} \\ &= (1-d)^{m^2}td \left\{ -2x\left(-1 + \frac{(t-1)}{2}d + O(d^2)\right)T_{m+1} + \left(-2 + \frac{(2t-1)}{2}d + O(d^2)\right)T_{m+2} \right\} \\ &= (1-d)^{m^2}2td \left\{ (xT_{m+1} - T_{m+2}) - x\left(\frac{t-1}{2}d + O(d^2)\right)T_{m+1} + \left(\frac{2t-1}{2}d + O(d^2)\right)T_{m+2} \right\}. \end{aligned}$$

Observe that we can pull out d in each column except for the *two first* columns. They are left unchanged. We simplify the determinant once more using (3.5). This time we get

$$\begin{aligned} & (1-d)^{m^2}2td \left\{ \left((xT_{m+1} - T_{m+2}) - x\left(\frac{t-1}{2}d + O(d^2)\right)T_{m+1} + \left(\frac{2t-1}{2}d + O(d^2)\right)T_{m+2}\right) \right. \\ & \quad - 2x(1-d)^t \left((xT_{m+2} - T_{m+3}) - x\left(\frac{t-1}{2}d + O(d^2)\right)T_{m+2} \right. \\ & \quad \left. \left. + \left(\frac{2t-1}{2}d + O(d^2)\right)T_{m+3}\right) \right. \\ & \quad \left. + (1-d)^{2t} \left((xT_{m+3} - T_{m+4}) - x\left(\frac{t-1}{2}d + O(d^2)\right)T_{m+3} \right. \right. \\ & \quad \left. \left. + \left(\frac{2t-1}{2}d + O(d^2)\right)T_{m+4}\right) \right\}. \end{aligned}$$

We observe that one more d can be pulled out. Simple computation leads to

$$(1-d)^{m^2}(2td)^2\{(x^2T_{m+2}-xT_{m+3})-(xT_{m+3}-T_{m+4})+O(d)\}.$$

Using the fact that $T_{m+4}=2xT_{m+3}-T_{m+2}$, we may rewrite the expression

$$(1-d)^{m^2}(2td)^2\{(x^2-1)T_{m+2}+O(d)\}.$$

We may continue the process. The next time we can pull out one more d , and for each time we continue the process, one extra d can be extracted. This is the crucial property which will lead to the stated theorem. We will now make the same replacement (3.5) in the last row of the determinant. First we obtain

$$\begin{aligned} z^n-2x(1-d)^{2k-1}z^{n-1}+(1-d)^{4k-4}z^{n-2} &= z^{n-2}(z^2-2x(1-d)^{2k-1}z+(1-d)^{4k-4}) \\ &= z^{n-2}(z^2-2xz+1)+O(d). \end{aligned}$$

If we continue the process neglecting the O-terms, we get

$$\begin{aligned} z^{n-2}(z^2-2xz+1)-z^{n-3}(z^2-2xz+1)(1-d)^{2k-1}+z^{n-4}(z^2-2xz+1)(1-d)^{4k-4} \\ = z^{n-4}(z^2-2xz+1)^2 \end{aligned}$$

and so on. In the rest of the proof we will use the notation

$$z^2-2xz+1=(z-e^{i\omega})(z-e^{-i\omega})=:w. \quad (3.6)$$

We have to consider two cases:

Case (i): If n is of the form $n=\beta n_0+\gamma$ where $\gamma=0$ (here $n=2\beta$ since $n_0=2$), we get a determinant of the following form (neglecting the O-terms in the last row):

$$\begin{vmatrix} \mu_0^{(V)} & \mu_{-1}^{(V)} & d(*) & d(*) & \cdots & d^{\beta-1}(*) & d^{\beta-1}(*) & d^{\beta}(*) \\ \mu_1^{(V)} & \mu_0^{(V)} & d(*) & d(*) & \cdots & d^{\beta-1}(*) & d^{\beta-1}(*) & d^{\beta}(*) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mu_n^{(V)} & \mu_{n-1}^{(V)} & d(*) & d(*) & \cdots & d^{\beta-1}(*) & d^{\beta-1}(*) & d^{\beta}(*) \\ 1 & z & w & wz & \cdots & w^{\beta-1} & w^{\beta-1}z & w^{\beta} \end{vmatrix}. \quad (3.7)$$

The terms $(*)$ consist of polynomials in $\cos m\omega$. They are of the form $(\cos m\omega + O(\text{higher order terms in } d))$. We expand the determinant by the *last row*. We notice that only *the last term* will give contribution in *the lowest power of d* . We use the notation $\tilde{\Delta}_{n-1}^{(V)}$ for the determinant in question (the one in the upper-left corner). Then we obtain

$$\hat{\rho}_n(z) = (w^{\beta} \tilde{\Delta}_{n-1}^{(V)}) \frac{1}{\Delta_{n-1}^{(V)}}.$$

Notice that $\tilde{\Delta}_{n-1}^{(V)} = \Delta_{n-1}^{(V)}$ since the column operations could be done starting with $\Delta_{n-1}^{(V)}$.

Hence (3.7) can be written (provided that $\Delta_{n-1}^{(V)} \neq 0$) as

$$\hat{\rho}_n(z) = w^{\beta} = (z^2 - 2xz + 1)^{\beta} = ((z - e^{i\omega})(z - e^{-i\omega}))^{\beta}.$$

Case (ii): If n is of the form $n = \beta n_0 + \gamma$ where $\gamma \neq 0$ (here $n = 2\beta + 1$), we get the following situation:

$$\begin{vmatrix} \mu_0^{(V)} & \mu_{-1}^{(V)} & d(*) & d(*) & \cdots & d^{\beta-1}(*) & d^{\beta-1}(*) & d^\beta(*) & d^\beta(*) \\ \mu_1^{(V)} & \mu_0^{(V)} & d(*) & d(*) & \cdots & d^{\beta-1}(*) & d^{\beta-1}(*) & d^{\beta-1}(*) & d^\beta(*) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mu_n^{(V)} & \mu_{n-1}^{(V)} & d(*) & d(*) & \cdots & d^{\beta-1}(*) & d^{\beta-1}(*) & d^\beta(*) & d^\beta(*) \\ 1 & z & w & wz & \cdots & w^{\beta-1} & w^{\beta-1}z & w^\beta & w^\beta z \end{vmatrix}. \quad (3.8)$$

We expand the determinant in the same way as in (3.7). We observe that this time the contribution in the lowest power of d will come from the two last terms. We use the notation $\tilde{D}_{n-1}^{(V)}$ for the new determinant in question.

Then we get the expression:

$$\begin{aligned} \hat{\rho}_n(z) &= (w^\beta z \tilde{A}_{n-1}^{(V)} - w^\beta \tilde{D}_{n-1}^{(V)}) \frac{1}{\tilde{A}_{n-1}^{(V)}} = w^\beta (z \tilde{A}_{n-1}^{(V)} - \tilde{D}_{n-1}^{(V)}) \frac{1}{\tilde{A}_{n-1}^{(V)}} \\ &= w^\beta \left(z - \frac{\tilde{D}_{n-1}^{(V)}}{\tilde{A}_{n-1}^{(V)}} \right) = ((z - e^{i\omega})(z - e^{-i\omega}))^\beta \left(z - \frac{\tilde{D}_{n-1}^{(V)}}{\tilde{A}_{n-1}^{(V)}} \right). \end{aligned}$$

In the general case we use exactly the same type of column operations as in the special case. For the sake of simplicity, let $\alpha_0 = 0$. Then the moments (3.1) are

$$\mu_m^{(V)} = \left(2 \sum_{j=1}^I |\alpha_j|^2 T_m(x_j) \right) (1-d)^{m^2}. \quad (3.9)$$

The column operations will be as follows:

Replace column k by

$$\text{col}(k) - 2x_j(1-d)^{2k-1} \text{col}(k-1) + (1-d)^{4k-4} \text{col}(k-2). \quad (3.10)$$

In this situation we let the index j run through $j = \{1, 2, \dots, I\}$. The first time we get

$$(1-d)^{m^2} \left\{ 2 \sum_{j=1}^I T_m(x_j) - 2x_1 T_{m+1}(x_j)(1-d)^t + T_{m+2}(x_j)(1-d)^{2t} \right\}.$$

If we look at the above expression, we will see that the first time we can pull out *no* d . We get rid of the term

$$T_m(x_1) - 2x_1 T_{m-1}(x_1) - T_{m-2}(x_1)$$

since this is equal to zero (3.3). The second time we get rid of x_2 and so on, until we reach x_I after $I = n_0/2$ operations. Hence we will need to do this type of operation I times before we can extract the first d . In the determinant we will hence have n_0 columns with d extracted. The next column operation will have the same pattern. Then we will have n_0 columns with d extracted and so on. Each time we do this operation, n_0 columns will be left unchanged with respect to the power of d . In the last γ columns, d^β can be pulled out. We continue this type of operations in the way same as in the special case. We have to do the column operations $\beta n_0/2 = \beta I$ times if $\gamma = 0$. If $\gamma \neq 0$ we have to continue the process.

In the last row the following pattern will occur:

$$z^n - 2x_1(1-d)^{2k-1}z^{n-1} + (1-d)^{4k-4}z^{n-2} = z^{n-2}(z^2 - 2x_1(1-d)^{2k-1}z + (1-d)^{4k-4}) \\ = z^{n-2}(z^2 - 2x_1z + 1) + O(d).$$

We will need $I = n_0/2$ operations to obtain the factor

$$\prod_{j=1}^I (z - e^{i\omega_j})(z - e^{-i\omega_j}) = : \prod_{j=1}^I w_j.$$

We will, as in the previous example, have two cases:

Case (i): If n is of the form $n = \beta n_0 + \gamma$ where $\gamma = 0$, we get the same type of case as in (3.7). Hence the limit polynomial will be

$$\hat{\rho}_n(z) = \prod_{j=1}^I w_j^\beta = \prod_{j=1}^I (z^2 - 2x_jz + 1)^\beta = \prod_{j=1}^I ((z - e^{i\omega_j})(z - e^{-i\omega_j}))^\beta.$$

Case (ii): If n is of the form $n = \beta n_0 + \gamma$, where $\gamma \neq 0$, the case will be similar to the one in (3.8). The contribution will come from the last γ columns. In this case we will have γ “uninteresting” zeros. The limit polynomial will be of the form:

$$\hat{\rho}_n(z) = \prod_{j=1}^I ((z - e^{i\omega_j})(z - e^{-i\omega_j}))^\beta \prod_{p=1}^{\gamma} (z - z_p^{(\beta, n)}).$$

If $\alpha_0 = 0$ (3.9) this concludes the proof. If $\alpha \neq 0$ (3.1), we have to do one more simplification of the determinant.

We replace column k by

$$\text{col}(k) - \text{col}(k-1).$$

Then we get rid of terms which turn out to be of the form $2^I \prod_{j=1}^I (1 - x_j)$. For details in this part of the general situation see [8] where we use a similar type of argument. $\square$

Theorem 2. Let $n = \beta n_0 + \gamma$ where $\beta = 1, 2, \dots$; $\gamma = 0, 1, \dots, n_0 - 1$; $n_0 = 2I + L$. Furthermore let $\mu_m^{(V)} := \mu_m R^{m^2} = (\alpha_0^2 + 2 \sum_{j=1}^I |\alpha_j|^2 \cos m\omega_j) R^{m^2}$ where $R \in (0, 1)$; $d = 1 - R$. Let $\Delta_{n-1}^{(V)}$ be the Toeplitz determinant

$$\Delta_{n-1}^{(V)} = \begin{vmatrix} \mu_0^{(V)} & \mu_{-1}^{(V)} & \mu_{-2}^{(V)} & \cdots & \mu_{-n+1}^{(V)} \\ \mu_1^{(V)} & \mu_0^{(V)} & \mu_{-1}^{(V)} & \cdots & \mu_{-n}^{(V)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}^{(V)} & \mu_{n-2}^{(V)} & \mu_{n-3}^{(V)} & \cdots & \mu_0^{(V)} \end{vmatrix}.$$

Then for all $n \geq n_0$ the Toeplitz determinant $\Delta_{n-1}^{(V)}$ will be of at least the following power in d :

$$d^{(1/2)\beta(\beta-1)n_0+\beta\gamma}$$

Proof. Let us first take a look at the special case used in the proof of Theorem 1. If we consider the power of d in the columns of the Toeplitz determinant of degree n , we will see the following pattern when $n = 2\beta + \gamma$, ($\gamma = 0$ or $\gamma = 1$) (3.7, 3.8):

$$\mu_0^{(V)} \mu_{-1}^{(V)} d \ d \ d^2 \ d^2 \ \dots \ d^{\beta-1} \ d^{\beta-1} \ d^\beta$$

Then the power of d will be $d^{2(0+1+2+\dots+(\beta-1))+\beta\gamma} = d^{\beta(\beta-1)+\beta\gamma}$. In the general case when $n = \beta n_0 + \gamma$ we have the situation:

$$\begin{array}{ccccccc} \mu_0^{(V)} \dots \mu_{-1}^{(V)} & d \dots d & d^2 \dots d^2 & \dots & d^{\beta-1} \dots d^{\beta-1} & d^\beta \dots d^\beta \\ (n_0 \text{ tms}) & (n_0 \text{ tms}) & (n_0 \text{ tms}) & \dots & (n_0 \text{ tms}) & (\gamma \text{ tms}) \end{array}$$

Then the power of d will be $d^{n_0((0+1+2+\dots+(\beta-1))+\beta\gamma)} = d^{(1/2)\beta(\beta-1)n_0+\beta\gamma}$.

Remarks. In Theorem 1 we did not prove that $\Delta_n^{(V)} > 0$ for all n . Thus we do not know if the moment sequence $\{\mu_m^{(V)}\}$ is positive definite. From this fact we may conclude that we do not know if a corresponding measure $\psi_N^{(V)}$ does exist. The separation of the frequency points from the “uninteresting” zeros is different from the other processes. The property (1.3) does not exist for $(n - n_0)$ zeros. We have multiple zeros of the frequency points in the V-process. The theorem does not exclude the possibility that more than βn_0 of the zeros tend to the frequency points. In the numerical experiments we observe this situation (Fig. 1c–d).

3.1. Numerical experiments

We illustrate the result in Theorem 1 with a few numerical experiments.

Fig. 1a and b: $x(m) = 6 \cos(m\pi/4) + 2 \cos(m\pi/3)$.

Fig. 1a: 4 “double” frequency points and 2 (real) “uninteresting” zeros.

Fig. 1b: 4 “double” frequency points.

Fig. 1c and d: $x(m) = 10 \cos(m\pi/4) + 2 \cos(m\pi/3) + 4 \cos(m3\pi/8) + 0.6 \cos(m2\pi/3)$.

Fig. 1c: “Double”: $(\pm\pi/4, \pm2\pi/3)$, “simple”: $(\pm\pi/3, \pm3\pi/8)$ frequency points.

Fig. 1d: Multiplicities 5: $(\pm\pi/3)$, 3: $(\pm\pi/4)$, 2: $(\pm3\pi/8)$. 2 “uninteresting” zeros.

Acknowledgements

The author wishes to thank Stig Petersen for able assistance in computing the numerical experiments.

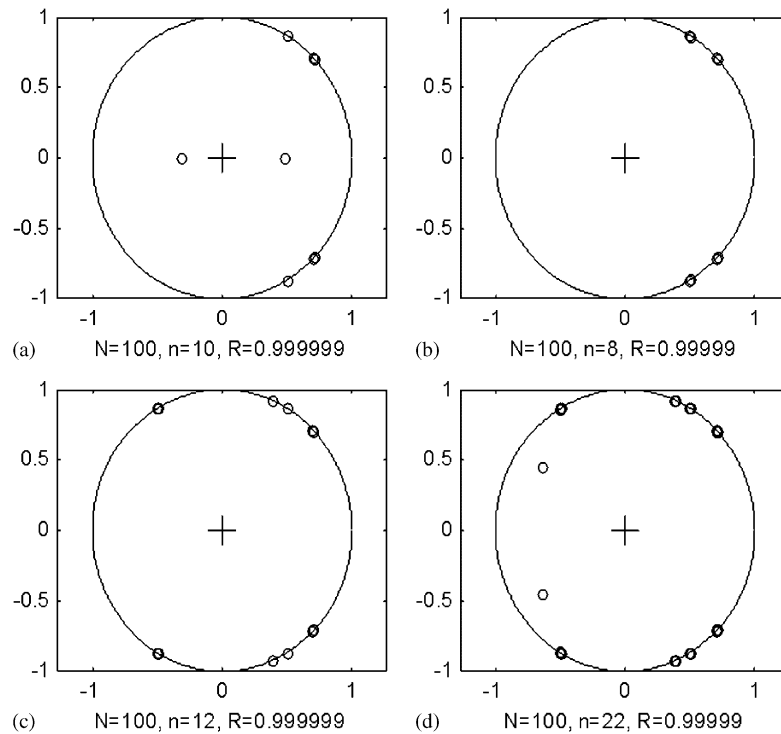


Fig. 1. Illustration of zeros.

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