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An analogue of Euler's identity and new combinatorial properties of n -colour compositions[☆]

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Abstract

An analogue of Euler's partition identity: "The number of partitions of a positive integer v into odd parts equals the number of its partitions into distinct parts" is obtained for ordered partitions. The ideas developed are then used in obtaining several new combinatorial properties of the n -colour compositions introduced recently by the author.

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1. Introduction

The first partition identity which states "The number of partitions of a positive integer into odd parts equals the number of its partitions into distinct parts" is due to Euler (see [6, p. 277]). In Section 2 we shall prove analytically as well as combinatorially an analogue of Euler's identity for ordered partitions (also called compositions in [7]). The ideas developed in Section 2 will be used in Section 3 in obtaining several new combinatorial properties of n -colour compositions introduced recently [1]. First, we recall the following definitions which will be used in the sequel:

Definition 1 (Andrews [3]). An "odd–even" partition of a positive integer v is a partition in which the parts (arranged in ascending order) alternate in parity starting with the smallest part odd. Thus, for example, the "odd–even" partitions of 7 are: $7, 1 + 6, 3 + 4$.

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Definition 2 (Frobenius [5]). A two-rowed array of nonnegative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$ and $v = r + \sum_{i=1}^r a_i + \sum_{i=1}^r b_i$ is called a Frobenius partition of v . We note that each partition can be represented by a Frobenius notation since if we delete from the Ferrers graph of the partition the main diagonal which suppose possesses r dots then the remaining rows of dots to the right of the diagonal are enumerated to provide one strictly decreasing sequence of r nonnegative integers (the r th such row might be empty thus producing 0). The remaining dots below the diagonal are enumerated by columns to provide a second strictly decreasing sequence of r nonnegative integers (the r th such column might be empty thus producing 0). The resulting two sequences are then presented in the Frobenius notation. For example, the Frobenius notation for $7 + 7 + 5 + 4 + 4 + 1$ is

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}.$$

Definition 3 (Agarwal and Andrews [2]). An n -colour partition (or, a partition with “ n copies of n ”) is a partition in which a part of size n , $n \geq 1$ can come in n different colours denoted by subscripts: n_1, n_2, \dots, n_n and parts satisfy the order

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < 5_2 \cdots.$$

For example, there are six n -colour partitions of 3, viz., $3_1, 3_2, 3_3, 2_1 + 1_1, 2_2 + 1_1, 1_1 + 1_1 + 1_1$.

Definition 4 (Agarwal [1]). An n -colour ordered partition is called an n -colour composition. For example, there are eight n -colour compositions of 3, viz., $3_1, 3_2, 3_3, 2_1 1_1$,

$$1_1 2_2, 2_1 1_1, 1_1 2_1, 1_1 1_1 1_1.$$

Let $C(v)$ denote the number of n -colour compositions of v , $C(m, v)$ denote the number of n -colour compositions of v into m parts and $C(m; q)$ and $C(q)$ denote the enumerative generating functions for $C(m, v)$ and $C(v)$, respectively. The following basic formulas were proved in [1]:

$$C(m; q) = \frac{q^m}{(1 - q)^{2m}}, \quad (1.1)$$

$$C(q) = \frac{q}{1 - 3q + q^2}, \quad (1.2)$$

$$C(m, v) = \binom{v + m - 1}{2m - 1} \quad (1.3)$$

and

$$C(v) = F_{2v}, \quad (1.4)$$

where F_{2v} is the $(2v)$ th Fibonacci number.

2. An analogue of the Euler's partition identity

We shall prove the following result:

Theorem 2.1. *Let $c(O, v)$ denote the number of compositions of v into odd parts and $c^m(O, v)$ denote the number of compositions of v into exactly m odd parts. Let OE_v denote the number of “odd–even” partitions with largest part v and OE_v^m denote the number of “odd–even” partitions with largest part v into exactly m parts. Then*

$$c^m(O, v) = OE_v^m \quad (2.1)$$

and

$$c(O, v) = OE_v. \quad (2.2)$$

Example. For $v = 6$, $m = 2$, we see that $c^2(O, 6) = 3$, since there are three compositions of 6 into two odd parts, viz., $5 + 1$, $1 + 5$, $3 + 3$. Also, $OE_6^2 = 3$, since there are three “odd–even” partitions into two parts with largest part 6, viz., $6 + 5$, $6 + 3$, $6 + 1$. Further, we see that $c(O, 6) = 8$, since there are eight compositions of 6 into odd parts viz., $5 + 1$, $1 + 5$, $3 + 3$, $3 + 1 + 1 + 1$, $1 + 3 + 1 + 1$, $1 + 1 + 3 + 1$, $1 + 1 + 1 + 3$, $1 + 1 + 1 + 1 + 1 + 1$. Also, $OE_6 = 8$, since the relevant “odd–even” partitions are $6 + 5 + 4 + 3 + 2 + 1$, $6 + 5 + 4 + 3$, $6 + 5 + 4 + 1$, $6 + 5 + 2 + 1$, $6 + 3 + 2 + 1$, $6 + 5$, $6 + 3$, $6 + 1$.

Remark 1. We see that (2.2) is very similar in structure to Euler's identity. We call (2.2) an analogue of Euler's identity for compositions.

Remark 2. Obviously (2.2) is an immediate consequence of (2.1). We shall provide two proofs for (2.1).

First proof of (2.1) (analytical). Let $OE_k^m(v)$ denote the number of “odd–even” partitions of v into m parts with largest part k . Then by using MacMahon's partition analysis [4, Chapter 11] we have

$$\begin{aligned} \sum_{v=0}^{\infty} OE_k^m(v) z^k q^v &= \Omega_{\geq} \sum_{n_1 \geq n_2 \geq \dots \geq n_m \geq 0} z^{2n_1+m} q^{(2n_1+m)+(2n_2+m-1)+\dots+(2n_m+1)} \\ &\quad \times \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \dots \lambda_{m-1}^{n_{m-1}-n_m}, \end{aligned} \quad (2.3)$$

where the variables $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ handle the inequalities satisfied by n_j while the n_j themselves become free. The linear operator Ω_{\geq} when applied to the Laurent series in $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ annihilates

terms with any negative exponents and in the remaining terms sets $\lambda_i = 1$. Hence (2.3) becomes

$$\begin{aligned} \sum_{k,v=0}^{\infty} OE_k^{(v)} z^k q^v &= z^m q^{m(m+1)/2} \Omega_{\geq} \sum_{n_1, n_2, \dots, n_m \geq 0} (z^2 q^2 \lambda_1)^{n_1} \left(q^2 \frac{\lambda_2}{\lambda_1} \right)^{n_2} \cdots \left(\frac{q^2}{\lambda_{m-1}} \right)^{n_m} \\ &= z^m q^{m(m+1)/2} \frac{1}{(1 - z^2 q^2 \lambda_1)(1 - q^2 \frac{\lambda_2}{\lambda_1}) \cdots (1 - q^2 \frac{\lambda_{m-1}}{\lambda_{m-2}})(1 - \frac{q^2}{\lambda_{m-1}})}. \end{aligned} \quad (2.4)$$

Now on applying to each of $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ in Eq. (2.4) the following result [4, Lemma 11.2.1, p. 556]

$$\Omega_{\geq} \frac{1}{(1 - \lambda x)(1 - y/\lambda)} = \frac{1}{(1 - x)(1 - xy)}, \quad (2.5)$$

we obtain

$$\sum_{k,v=0}^{\infty} OE_k^{(v)} z^k q^v = \frac{z^m q^{m(m+1)/2}}{(1 - z^2 q^2)(1 - z^2 q^4) \cdots (1 - z^2 q^{2m})}. \quad (2.6)$$

By setting $q = 1$ in (2.6), we obtain

$$\sum_{k=0}^{\infty} OE_k z^k = \frac{z^m}{(1 - z^2)^m}. \quad (2.7)$$

On the other hand, we see that

$$\sum_{v=0}^{\infty} c^m(O, v) z^v = (z + z^3 + z^5 + \cdots)^m = \frac{z^m}{(1 - z^2)^m}. \quad (2.8)$$

A comparison of (2.7) and (2.8) leads to (2.1). \square

Remark 3. Using (2.7) and (2.8) one can easily show that

$$\sum_{n=0}^{\infty} OE_n q^n = \sum_{n=0}^{\infty} c(O, n) q^n = \frac{q}{1 - q - q^2}. \quad (2.9)$$

Since the extreme right-hand side of (2.9) also generates Fibonacci numbers, we conclude that

$$OE_n = c(O, n) = F_n. \quad (2.10)$$

Second proof of (2.1) (combinatorial). Let Graph A be the graph of an “odd–even” partition π into m parts with largest part n . We represent each part ‘ a ’ by a row of ‘ a ’ dots. (In Graph A, $n = 6$, $m = 4$ and $\pi = 6 + 5 + 2 + 1$, also note that the x -axis is drawn one unit of length below the last row and the y -axis one unit of length to the left of the first column.)

We draw vertical lines from the corner point of each row and measure the distance of each line from its preceding one taking y -axis also into consideration. Since π is an “odd–even” partition (that is, its parts alternate in parity starting with the smallest part odd), these distances are all of odd

lengths and sum upto the largest part n . Consequently, these distances give rise to a composition of n into exactly m odd parts. Since the correspondence is one-to-one (2.1) is proved. \square

The following is easily verified:

$$6 + 5 + 4 + 3 \rightarrow 3 + 1 + 1 + 1$$

$$6 + 5 + 4 + 1 \rightarrow 1 + 3 + 1 + 1$$

$$6 + 5 + 2 + 1 \rightarrow 1 + 1 + 3 + 1$$

$$6 + 3 + 2 + 1 \rightarrow 1 + 1 + 1 + 3.$$

3. New combinatorial properties of n -colour compositions

We shall prove the following results:

Theorem 3.1. *The number of n -colour compositions of v equals the number of “odd–even” partitions with largest part $2v$.*

Theorem 3.2. *The number of n -colour compositions of v equals the number of compositions of $2v$ into odd parts.*

Theorem 3.3. *Let Δ_v denote the number of “odd–even” partitions with largest part odd and $\leq 2v - 1$. Then Δ_v equals the number of n -colour compositions of v .*

Example. $\Delta_v = 8$, since the relevant “odd–even” partitions are: $5, 5 + 4 + 3 + 2 + 1, 5 + 4 + 3, 5 + 4 + 1, 5 + 2 + 1, 3, 3 + 2 + 1, 1$. We have earlier seen that there are eight n -colour compositions of 3.

Theorem 3.4. *The number of n -colour compositions of v equals the number of self-conjugate partitions with largest part $2v$ such that in the Frobenius notation*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}, a_i$$

alternate in parity.

Example. For $v=3$, there are eight relevant self-conjugate partitions viz., $6^2 2^4, 6 \ 4 \ 2^2 1^2, 6 \ 2 \ 1^4, 6^4 4^2, 6^3 4 \ 3^2, 6^2 4^2 2^2, 6 \ 4^3 1^2, 6^6$.

Theorem 3.5. *The number of n -colour compositions of v equals the number of partitions into an even number of odd parts with largest part $4v - 1$ such that the parts are alternately $\equiv 3$ and $1 \pmod{4}$.*

Example. For $v=3$, there are eight relevant partitions, viz., $11 + 9, 11 + 5, 11 + 1, 11 + 9 + 7 + 5, 11 + 9 + 7 + 1, 11 + 9 + 3 + 1, 11 + 5 + 3 + 1, 11 + 9 + 7 + 5 + 3 + 1$.

Proofs of Theorems 3.1–3.5. Theorems 3.1 and 3.2 follow from (1.4) and (2.10). Theorem 3.3 follows from (1.4), (2.8) and the identity

$$F_1 + F_3 + \cdots + F_{2v-1} = F_{2v}. \quad (3.1)$$

To prove Theorem 3.4 we establish a bijection between the “odd–even” partitions with largest part $2v$ and the self-conjugate partitions with largest part $2v$ such that in the Frobenius notation

$$\begin{pmatrix} a_1 & \cdots & a_r \\ a_1 & \cdots & a_r \end{pmatrix}, a_i$$

alternate in parity and then use Theorem 3.1. We do it as follows:

Let $\pi = a_1 + a_2 + \cdots + a_r$ ($2v = a_1 > a_2 > \cdots > a_r$) be an “odd–even” partition with largest part $2v$. We consider a graph which consists of r successive bends viz., a_1 -bend, a_2 -bend, \dots , a_r -bend. Here by a k -bend we mean a right-bend containing k dots in the first row as well as in the first column. For example, a 3-bend means

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & & . \\ 0 & & \end{array}$$

We see immediately that this graph represents a self-conjugate partition with largest part equal to $2v$ such that in the Frobenius notation

$$\begin{pmatrix} a_1 & \cdots & a_r \\ a_1 & \cdots & a_r \end{pmatrix}, a_i$$

alternate in parity. The correspondence being one-to-one, the Theorem 3.4 is proved. \square

Example. For $v = 3$, let us consider $\pi = 6 + 5 + 2 + 1$. Then

$$\begin{array}{ccccccc} 6 + 5 + 2 + 1 & \rightarrow & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & \\ & & 0 & 0 & 0 & 0 & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \end{array}$$

which is a graph of the self-conjugate partition $6^2 4^2 2^2$ with largest part 6 such that in the Frobenius notation

$$\begin{pmatrix} 5 & 4 & 1 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}, a_i$$

alternate in parity.

Theorem 3.5 follows from Theorem 3.4 once we observe that if the right-bends in the graph of a self-conjugate partition of Theorem 3.4 are straightened then we get a partition of Theorem 3.5. For example, the self-conjugate partition $6^2 4^2 2^2$ of the previous example corresponds to the partition $11 + 9 + 3 + 1$ which is a partition of the type described in Theorem 3.5.

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