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Dissipative/conservative Galerkin method using discrete partial derivatives for nonlinear evolution equations

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Abstract

A new method is proposed for designing Galerkin schemes that retain the energy dissipation or conservation properties of nonlinear evolution equations such as the Cahn–Hilliard equation, the Korteweg–de Vries equation, or the nonlinear Schrödinger equation. In particular, as a special case, dissipative or conservative finite-element schemes can be derived. The key device there is the new concept of discrete partial derivatives. As examples of the application of the present method, dissipative or conservative Galerkin schemes are presented for the three equations with some numerical experiments.

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1. Introduction

In this paper, the numerical integration of partial differential equations (PDEs for short) which have some "energy" conservation or dissipation properties is considered. For example, the Cahn–Hilliard (CH) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(pu + ru^3 + q\frac{\partial^2 u}{\partial x^2}\right), \quad 0 < x < L, \quad t > 0, \tag{1}$$

where $p < 0$, $q < 0$, $r > 0$, has the "energy" dissipation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\frac{p}{2}u^2 + \frac{r}{4}u^4 - \frac{q}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x \leqslant 0, \quad t > 0,$$

when appropriate boundary conditions are imposed. The Korteweg–de Vries (KdV) equation,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{\partial^2 u}{\partial x^2}\right), \quad 0 < x < L, \quad t > 0, \tag{2}$$

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has the energy conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\frac{1}{6}u^3 - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x = 0,$$

again, when appropriate boundary conditions are imposed. The nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x} - \gamma |u|^{p-1}, \quad 0 < x < L, \quad t > 0, \tag{3}$$

where $\mathrm{i} = \sqrt{-1}$, $p = 3, 4, \ldots$, and $\gamma \in \mathbf{R}$, has the "energy" conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(-\left|\frac{\partial u}{\partial x}\right|^2 + \frac{2\gamma}{p+1}|u|^{p+1}\right)\mathrm{d}x = 0, \quad t > 0,$$

under appropriate boundary conditions.

It is widely accepted that numerical schemes which retain the dissipation or conservation properties of the PDEs are advantageous in that they often yield physically correct results and numerical stability [3]. We call such schemes "dissipative/conservative schemes" in this paper. In the literature, this area was first approached by the development of a number of specific schemes corresponding to specific problems; the interested reader may refer to [1,4,5,10,11] among others (see also [6,8]). A more unified method was then given in [6–9], by which dissipative or conservative *finite-difference* schemes can be constructed automatically for certain classes of dissipative/conservative PDEs. More specifically, this method targets dissipative/conservative PDEs which are defined using a variational derivative. In Furihata [6], real-valued equations of the form

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\left(\frac{\partial}{\partial x}\right)^{2s}\frac{\delta G}{\delta u}, \quad s = 0, 1, 2, \ldots \tag{4}$$

were considered, where $\delta G/\delta u$ is the variational derivative of $G(u, u_x)$ with respect to $u(x,t)$. Under appropriate boundary conditions, these PDEs become dissipative. For example, the CH equation belongs to this class with $s = 1$ and $G(u, u_x) = pu^2/2 + ru^4/4 - qu_x^2/2$ (where $u_x = \partial u/\partial x$). Furihata also targeted real-valued conservative PDEs of the form

$$\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{2s-1}\frac{\delta G}{\delta u}, \quad s = 1, 2, 3, \ldots \ . \tag{5}$$

The KdV equation is an example of this class with $s = 1$ and $G(u, u_x) = u^3/6 - u_x^2/2$. Later, Matsuo and Furihata [8] considered complex-valued conservative equations of the form

$$\mathrm{i}\frac{\partial u}{\partial t} = -\frac{\delta G}{\delta \overline{u}}, \tag{6}$$

where $\delta G/\delta\overline{u}$ is a complex variational derivative, and $\bar{u}$ is the complex conjugate of u. An example of this class is the NLS equation. Dissipative PDEs of the form $\partial u/\partial t = -\delta G/\delta\overline{u}$, were also treated. The key step for the above studies was the introduction of the "discrete variational derivative," which is a rigorous discretization of the variational derivative. With the discrete variational derivative, a finite-difference scheme is defined analogously to the original equation, so that the dissipation/conservation property is automatically retained. Due to this underlying idea, the method is now called the "discrete variational derivative method" (DVDM). The method does, however, suffer from drawbacks due to being based on the finite-difference method. Specifically, the use of non-uniform grids and application to two- or three-dimensional problems with complex domain structures are not straightforward.

Motivated by this situation, the aim of this paper is to extend the finite-difference DVDM to the Galerkin framework in spatial dimension. Although we have in mind the finite-element method as its most important special case, the discussions are done in abstract Galerkin framework in order to keep the generality of the proposed method. In the present study we limit ourselves to spatially one-dimensional cases for brevity, which is still enough to illustrate our essential idea.

There are mainly two difficulties in the attempt of the extension. First, if we simply try to translate the finite-difference DVDM into the Galerkin framework, the "variational derivative," which has been exactly the key to the dissipation/conservation properties, can cause a trouble. Since it essentially includes second-derivative u_{xx}, we would be *always* forced to employ C^1-elements (basis functions) in the resulting schemes, which is clearly unfavorable. To circumvent this issue, in the present study we newly introduce the concept of "discrete *partial* derivatives," and construct schemes in appropriate weak forms. In spite of this modification, the resulting schemes can successfully retain the dissipation/conservation properties. Sufficient conditions for these dissipation/conservation are then given. The second difficulty arises in high-order problem such as (4) with $s \geqslant 1$. An standard way to unfold higher-order derivatives is to introduce intermediate variables and consider mixed formulations (see, for example [2]). In our project for dissipative/conservative schemes, however, things are not so simple in that mixed formulations corresponding to a given problem is generally not unique, and only a few of them correctly lead to dissipative/conservative schemes (see Remark 12). Furthermore, the conditions for the dissipation/conservation properties become more complicated. Answers to these issues are also given in the present paper.

This paper is organized as follows: In Section 2 the target equations are defined. Section 3 is devoted to the proposed Galerkin method, while in Section 4 several application examples are shown. Finally, Section 5 offers some concluding remarks.

2. Target equations

Target PDEs and their dissipation or conservation properties are summarized. The first class is that given by all real-valued PDEs of the form of Eq. (4):

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\left(\frac{\partial}{\partial x}\right)^{2s}\frac{\delta G}{\delta u}, \quad s = 0, 1, 2, \ldots . \tag{4}$$

As mentioned above, these PDEs are dissipative.

Proposition 1 (*Dissipation property of (4)*). *Let us assume that boundary conditions satisfy*

$$\left[\frac{\partial G}{\partial u_x}\frac{\partial u}{\partial t}\right]_0^L = 0, \quad t > 0. \tag{7}$$

Let us also assume when $s \geqslant 1$ that

$$\left[\left(\frac{\partial^{j-1}}{\partial x^{j-1}}\frac{\delta G}{\delta u}\right)\left(\frac{\partial^{2s-j}}{\partial x^{2s-j}}\frac{\delta G}{\delta u}\right)\right]_0^L = 0, \quad t > 0, \quad j = 1, \ldots, s. \tag{8}$$

Then solutions to the PDEs (4) *satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u, u_x)\,\mathrm{d}x \leqslant 0, \quad t > 0.$$

That is, the PDEs are dissipative.

A proof can be found in [6]. Throughout this paper we call $G(u, u_x)$ the "local energy," and $\int_0^L G(u, u_x)\,\mathrm{d}x$ the "global energy." As stated above, the CH equation (1) is a member of this class with $s=1$ and $G(u, u_x) = pu^2/2 + ru^4/4 - qu_x^2/2$.

The second class is the real-valued conservative PDEs of the form of Eq. (5):

$$\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{2s-1}\frac{\delta G}{\delta u}, \quad s = 1, 2, 3, \ldots . \tag{5}$$

Proposition 2 (*Conservation property of (5)*). *Let us assume that boundary conditions satisfy* (7) *and*

$$\left[\left(\frac{\partial^{j-1}}{\partial x^{j-1}}\frac{\delta G}{\delta u}\right)\left(\frac{\partial^{2s-1-j}}{\partial x^{2s-1-j}}\frac{\delta G}{\delta u}\right)\right]_0^L = 0, \quad t > 0, \quad j = 1, \ldots, s. \tag{9}$$

Then solutions to the PDEs (5) *satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\,\mathrm{d}x = 0, \qquad t>0.$$

That is, *the PDEs are conservative*.

The KdV equation (2) is an example of this class with $s=1$ and $G(u,u_x)=u^3/6-u_x^2/2$.
The third class of PDEs considered in this study are the complex-valued PDEs (6):

$$\mathrm{i}\frac{\partial u}{\partial t} = -\frac{\delta G}{\delta \overline{u}}. \tag{6}$$

Proposition 3 (*Conservation property of (6)*). *Let us assume that boundary conditions satisfy*

$$\left[\frac{\partial G}{\partial u_x}\frac{\partial u}{\partial t} + \frac{\partial G}{\partial \overline{u_x}}\frac{\partial \overline{u}}{\partial t}\right]_0^L = 0. \tag{10}$$

Then solutions to the PDEs (6) *satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\,\mathrm{d}x = 0, \qquad t>0.$$

That is, *these PDEs are conservative*.

A proof is given in [8]. Upon setting $G(u,u_x)=-|u_x|^2+2\gamma|u|^{p+1}/(p+1)$, it can be seen that the NLS equation (3) is an example of this class of equations.

3. The new Galerkin method

In this section the new method for designing dissipative or conservative Galerkin schemes is presented for the PDEs (4)–(6) separately.

3.1. Dissipative schemes for the real-valued PDEs (4)

We commence by introducing the concept of "discrete partial derivatives." Suppose that local energy is of the form

$$G(u,u_x) = \sum_{l=1}^{M} f_l(u)g_l(u_x), \tag{11}$$

where $M\in\{1,2,\ldots\}$, and f_l, g_l are real-valued functions. For example, the local energy of the CH equation (1) can be expressed in this form with $M=3$, $f_1(u)=pu^2/2$, $g_1(u_x)=1$, $f_2(u)=ru^4/4$, $g_2(u_x)=1$, $f_3(u)=1$, $g_3(u_x)=-qu_x^2/2$. Let us denote the Galerkin approximate solution by $u^{(m)}\simeq u(x,m\Delta t)$ (Δt is the time mesh size). Then "discrete partial derivatives" are defined as follows.

Definition 4 (*Discrete partial derivatives*). We call the discrete quantities

$$\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)},u^{(m)}\right)} :\equiv \sum_{l=1}^{M}\left(\frac{f_l(u^{(m+1)})-f_l(u^{(m)})}{u^{(m+1)}-u^{(m)}}\right)\left(\frac{g_l(u_x^{(m+1)})+g_l(u_x^{(m)})}{2}\right), \tag{12}$$

$$\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)} :\equiv \sum_{l=1}^{M}\left(\frac{f_l(u^{(m+1)})+f_l(u^{(m)})}{2}\right)\left(\frac{g_l(u_x^{(m+1)})-g_l(u_x^{(m)})}{u_x^{(m+1)}-u_x^{(m)}}\right), \tag{13}$$

the "discrete partial derivatives," which corresponds to $\partial G/\partial u$ and $\partial G/\partial u_x$, respectively.[1]

[1] Expressions similar to $(f(a)-f(b))/(a-b)$ should be interpreted as $f'(a)$ when $a=b$.

It can be easily verified that, corresponding to the continuous chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\,\mathrm{d}x = \int_0^L \left(\frac{\partial G}{\partial u}u_t + \frac{\partial G}{\partial u_x}u_{xt}\right)\mathrm{d}x,$$

the following discrete chain rule holds (hereafter $G(u^{(m)}, u_x^{(m)})$ is abbreviated as $G(u^{(m)})$ to save space).

Theorem 5 (*Discrete chain rule (real-valued case)*). *Concerning the discrete partial derivatives* (12) *and* (13), *the following identity holds*:

$$\begin{aligned}&\frac{1}{\Delta t}\int_0^L \left(G(u^{(m+1)}) - G(u^{(m)})\right)\mathrm{d}x\\ &\quad= \int_0^L \left\{ \frac{\partial G_\mathrm{d}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right) + \frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}\left(\frac{u_x^{(m+1)}-u_x^{(m)}}{\Delta t}\right)\right\}\mathrm{d}x. \qquad (14)\end{aligned}$$

Now we are in a position to describe our schemes for Eq. (4). The simplest case $s=0$ and general cases $s=1,2,\ldots$ are treated separately. Let us denote the trial space by S_1, and the test space by W_1. We also use the notation $(f,g)=\int_0^L fg\,\mathrm{d}x$, and its associated norm $\|\cdot\|_2$.

Scheme 1 (*Galerkin scheme for* $s=0$). *Suppose* $u^{(0)}(x)$ *is given in* S_1. *Find* $u^{(m)} \in S_1$ $(m=1,2,\ldots)$ *such that, for any* $v \in W_1$,

$$\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t}, v\right) = -\left(\frac{\partial G_\mathrm{d}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}, v\right) - \left(\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}, v_x\right) + \left[\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}v\right]_0^L. \qquad (15)$$

Because the discrete partial derivatives (12) and (13) do not include second derivatives, the scheme can be implemented using only H^1-elements, such as the standard piecewise linear function space. Note that the existence, uniqueness, stability, and convergence of the solution of the scheme above. The scheme is dissipative in the following sense.

Theorem 6 (*Dissipation property of Scheme 1*). *Assume that boundary conditions and the trial and test spaces are set such that*

$$\left[\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right)\right]_0^L = 0 \qquad (16)$$

and $\left(u^{(m+1)}-u^{(m)}\right)/\Delta t \in W_1$ *hold. Then Scheme* 1 *is dissipative in the sense that*

$$\frac{1}{\Delta t}\int_0^L \left(G(u^{(m+1)}) - G(u^{(m)})\right)\mathrm{d}x \leqslant 0, \quad m=0,1,2,\ldots .$$

Proof.

$$\frac{1}{\Delta t}\int_0^L \left(G(u^{(m+1)}) - G(u^{(m)})\right) \mathrm{d}x$$

$$= \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}, \frac{u^{(m+1)} - u^{(m)}}{\Delta t}\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}, \frac{u_x^{(m+1)} - u_x^{(m)}}{\Delta t}\right)$$

$$= -\left\|\frac{u^{(m+1)} - u^{(m)}}{\Delta t}\right\|_2^2 + \left[\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}\right)\right]_0^L \leqslant 0.$$

The first equality is by Theorem 5. The second one is shown by making use of expression (15) and the assumption $(u^{(m+1)} - u^{(m)})/\Delta t \in W_1$. The inequality is shown by assumption (16). □

The assumption (16) corresponds to the condition (7). The assumption $(u^{(m+1)} - u^{(m)})/\Delta t \in W_1$ is an additional condition for the dissipation property, which can be usually satisfied with natural choices of S_1 and W_1. For example, when the Dirichlet boundary conditions $u(0)=a,\ u(L)=b$ are imposed, it is natural to take $S_1=\{u|u(0)=a,\ u(L)=b\}$ and $W_1 = \{v|v(0) = 0,\ v(L) = 0\}$. In this setting the assumption is satisfied.

Note that any mathematical properties of the resulting schemes except for the targeted dissipation property are rather left open here; the existence, uniqueness, stability, and convergence of the solutions of Scheme 1 should be investigated in each resulting scheme. This note applies to all the proposed scheme in the present paper.

Next we proceed to the general case $s \geqslant 1$. We first observe that by recursively introducing intermediate variables: $p_1=-(p_2)_{xx}, \ldots, p_{s-1}=-(p_s)_{xx}$, and $p_s=\delta G/\delta u$, the original equation (4) can be rewritten as a system of equations $u_t = (p_s)_{xx},\ p_{j-1} = -(p_j)_{xx}\ (j \in J)$, and $p_s = \delta G/\delta u$, where the set $J = \{2, \ldots, s\}$ when $s \geqslant 2$ or $J = \emptyset$ when $s = 1$. This leads us to the following scheme. We assume the trial spaces $S_1, \ldots, S_{s+1}$, and test spaces $W_1, \ldots, W_{s+1}$ accordingly.

Scheme 2 (*Galerkin scheme for* $s \geqslant 1$). *Suppose that* $u^{(0)}(x)$ *is given in* S_{s+1}. *Find* $u^{(m+1)} \in S_{s+1}, p_1^{(m+1/2)} \in S_1, \ldots, p_s^{(m+1/2)} \in S_s$ $(m = 0, 1, \ldots)$ *such that, for any* $v_1 \in W_1, \ldots, v_{s+1} \in W_{s+1}$,

$$\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}, v_1\right) = -\left((p_1^{(m+1/2)})_x, (v_1)_x\right) + \left[(p_1^{(m+1/2)})_x v_1\right]_0^L, \tag{17}$$

$$\left(p_{j-1}^{(m+1/2)}, v_j\right) = \left((p_j^{(m+1/2)})_x, (v_j)_x\right) - \left[(p_j^{(m+1/2)})_x v_j\right]_0^L \quad (j \in J), \tag{18}$$

$$\left(p_s^{(m+1/2)}, v_{s+1}\right) = \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}, v_{s+1}\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}, (v_{s+1})_x\right) - \left[\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)} v_{s+1}\right]_0^L. \tag{19}$$

Eq. (18) is dropped when $J = \emptyset$. This scheme can be also implemented only with H^1-elements. The dissipation property is summarized in the next theorem.

Theorem 7 (*Dissipation property of Scheme 2*). *Assume that boundary conditions and the trial and test spaces are set such that* (i) *the condition* (16) *is satisfied*; (ii) $\left[(p_j^{(m+1/2)})_x \cdot p_{s+1-j}^{(m+1/2)}\right]_0^L = 0\ (j=1, 2, \ldots, s)$; (iii) $(u^{(m+1)} - u^{(m)})/\Delta t \in W_{s+1}$; *and* (iv) $W_j \supseteq S_{s+1-j}\ (j = 1, 2, \ldots, s)$. *Then Scheme 2 is dissipative in the sense that*

$$\frac{1}{\Delta t}\int_0^L \left(G(u^{(m+1)}) - G(u^{(m)})\right) \mathrm{d}x \leqslant 0, \quad m = 0, 1, 2, \ldots .$$

Proof.

$$
\begin{aligned}
&\frac{1}{\Delta t}\int_0^L \left(G(u^{(m+1)}) - G(u^{(m)})\right) \mathrm{d}x \\
&= \left(\frac{\partial G_\mathrm{d}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}, \frac{u^{(m+1)} - u^{(m)}}{\Delta t}\right) + \left(\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}, \frac{u_x^{(m+1)} - u_x^{(m)}}{\Delta t}\right) \\
&= \left(p_s^{(m+1/2)}, \frac{u^{(m+1)} - u^{(m)}}{\Delta t}\right) + \left[\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}\right)\right]_0^L \\
&= -\left((p_1^{(m+1/2)})_x, (p_s^{(m+1/2)})_x\right) + \left[(p_1^{(m+1/2)})_x p_s^{(m+1/2)}\right]_0^L.
\end{aligned}
$$

The second equality is shown by using Eq. (19) with $v_{s+1} = (u^{(m+1)} - u^{(m)})/\Delta t$. The third equality is given by using Eq. (17) with $v_1 = p_s^{(m+1/2)}$ and the assumption $S_s \subseteq W_1$. By repeatedly making use of Eq. (18) with $j = s, 2, s-1, 3, \ldots$ in this order, which is allowed by the assumption (iv), it can be seen that the right-hand side is equal to $-\|(p_{(s+1)/2}^{(m+1/2)})_x\|_2^2$ when s is odd, or $-\|p_{s/2}^{(m+1/2)}\|_2^2$ otherwise, and so the proof is complete. All the boundary terms vanish as a result of the boundary-condition assumptions. □

Remark 8. We can see the perfect matching between Proposition 1 and Theorem 7. As noted before, the assumption (16) corresponds to the condition (7). It can be also checked that the assumption (ii) in Theorem 7 exactly corresponds to the condition (8), since the latter can be rewritten as $[(p_j)_x \cdot p_{s+1-j}]_0^L = 0$ $(j = 1, \ldots, s)$ with the intermediate variables p_j. The assumptions (iii) and (iv) are additional conditions for the dissipation property.

3.2. Conservative schemes for the real-valued PDEs (5)

Conservative schemes for the PDEs (5) are proposed using the discrete partial derivatives introduced in the previous section. The simplest case $s = 1$ and general cases $s = 2, 3, \ldots$ are treated separately. Let $S_1, \ldots, S_{s+1}$ be trial spaces, and $W_1, \ldots, W_{s+1}$ be test spaces.

Scheme 3 (*Galerkin scheme for s = 1*). *Suppose that $u^{(0)}(x)$ is given in S_2. Find $u^{(m+1)} \in S_2$, $p_1^{(m+1/2)} \in S_1$ such that, for any $v_1 \in W_1$, $v_2 \in W_2$,*

$$\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}, v_1\right) = \left((p_1^{(m+1/2)})_x, v_1\right), \tag{20}$$

$$\left(p_1^{(m+1/2)}, v_2\right) = \left(\frac{\partial G_\mathrm{d}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}, v_2\right) + \left(\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}, (v_2)_x\right) - \left[\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)} v_2\right]_0^L. \tag{21}$$

Theorem 9 (*Conservation property of Scheme 3*). *Assume that boundary conditions and the trial and test spaces are set such that* (i) *the condition* (16) *is satisfied*; (ii) $\left[(p_1^{(m+1/2)})^2\right]_0^L = 0$; (iii) $(u^{(m+1)} - u^{(m)})/\Delta t \in W_2$; *and* (iv) $S_1 \subseteq W_1$. *Then Scheme 3 is conservative in the sense that*

$$\frac{1}{\Delta t}\int_0^L \left(G(u^{(m+1)}) - G(u^{(m)})\right) \mathrm{d}x = 0, \quad m = 0, 1, 2, \ldots .$$

Proof.

$$\frac{1}{\Delta t}\int_0^L\left(G(u^{(m+1)})-G(u^{(m)})\right)\,\mathrm{d}x=\left(p_1^{(m+1/2)},\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right)+\left[\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)}\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right)\right]_0^L$$

$$=\left((p_1^{(m+1/2)})_x,p_1^{(m+1/2)}\right)=0.$$

The first equality is shown by using Eq. (21) with $v_2=(u^{(m+1)}-u^{(m)})/\Delta t$, while the second equality is given by using Eq. (20) with $v_1=p_1^{(m+1/2)}$ and the assumption $S_1\subseteq W_1$. The last equality is from the assumption (ii). □

In order to describe the scheme for $s\geqslant 2$, let us define the set $J=\{2,\ldots,s\}\backslash\{n+1\}$ when $s=2n$ $(n=1,2,\ldots)$, or $J=\{2,\ldots,s\}\backslash\{n\}$ when $s=2n-1$ $(n=2,3,\ldots)$.

Scheme 4 (*Galerkin scheme for* $s\geqslant 2$). *Suppose that* $u^{(0)}(x)$ *is given in* S_{s+1}. *Find* $u^{(m+1)}\in S_{s+1}$, $p_1^{(m+1/2)}\in S_1,\ldots,p_s^{(m+1/2)}\in S_s$ $(m=0,1,\ldots)$ *such that, for any* $v_1\in W_1,\ldots,v_{s+1}\in W_{s+1}$,

$$\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t},v_1\right)=-\left((p_1^{(m+1/2)})_x,(v_1)_x\right)+\left[(p_1^{(m+1/2)})_xv_1\right]_0^L, \tag{22}$$

$$\left(p_{j-1}^{(m+1/2)},v_j\right)=-\left((p_j^{(m+1/2)})_x,(v_j)_x\right)+\left[(p_j^{(m+1/2)})_xv_j\right]_0^L\quad(j\in J), \tag{23}$$

$$\left(p_n^{(m+1/2)},(v_{n+1})_x\right)=\left((p_{n+1}^{(m+1/2)})_x,(v_{n+1})_x\right)\quad(\textit{when } s=2n), \tag{24}$$

$$\left(p_{n-1}^{(m+1/2)},v_n\right)=\left((p_n^{(m+1/2)})_x,v_n\right)\quad(\textit{when } s=2n-1), \tag{25}$$

$$\left(p_s^{(m+1/2)},v_{s+1}\right)=\left(\frac{\partial G_\mathrm{d}}{\partial\left(u^{(m+1)},u^{(m)}\right)},v_{s+1}\right)+\left(\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)},(v_{s+1})_x\right)-\left[\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)}v_{s+1}\right]_0^L. \tag{26}$$

Eq. (23) is dropped when $J=\emptyset$. The conservation property is summarized in the next theorem.

Theorem 10 (*Conservation property of Scheme 4*). *Assume that boundary conditions and trial and test spaces are set such that* (i) *the condition* (16) *is satisfied*; (ii) $\left[(p_n^{(m+1/2)})^2\right]_0^L=0$ *and* $\left[(p_j^{(m+1/2)})_xp_{s+1-j}^{(m+1/2)}\right]_0^L=0$ $(j\in J)$; (iii) $(u^{(m+1)}-u^{(m)})/\Delta t\in W_{s+1}$; *and* (iv) $W_j\supseteq S_{s+1-j}$ $(j=1,\ldots,s)$. *Then Scheme* 3 *is conservative in the sense that*

$$\frac{1}{\Delta t}\int_0^L\left(G(u^{(m+1)})-G(u^{(m)})\right)\,\mathrm{d}x=0,\quad m=0,1,2,\ldots\,.$$

Proof. The proof is similar to Theorem 7.

$$\begin{aligned}\frac{1}{\Delta t}\int_0^L\left(G(u^{(m+1)})-G(u^{(m)})\right)\,\mathrm{d}x&=-\left((p_1^{(m+1/2)})_x,(p_s^{(m+1/2)})_x\right)\\&=\begin{cases}-\left((p_n^{(m+1/2)})_x,(p_{n+1}^{(m+1/2)})_x\right) & (\text{when } s=2n),\\ \left(p_{n-1}^{(m+1/2)},p_n^{(m+1/2)}\right) & (\text{when } s=2n-1),\end{cases}\\&=(-1)^{s+1}\left(p_n^{(m+1/2)},(p_n^{(m+1/2)})_x\right)=0.\end{aligned}$$

In the second equality Eq. (23) is repeatedly used. The third equality is either from (24) or (25). □

Remark 11. The assumptions in Theorem 10 exactly correspond to those in Proposition 2, which can be checked similarly to Remark 8.

Remark 12. The proposed schemes in this paper are more or less in mixed formulation (see, for example [2]). The underlying weak forms are, however, carefully chosen for the targeted conservation/dissipation properties. Below we illustrate this using as an example the linear third-order dispersive equation $u_t = u_{xxx}$ under the periodic boundary condition. It is a special case of (5) with $s = 2$, $G = u^2/2$, and thus $\int_0^L (u^2/2)\,\mathrm{d}x$ is an invariant. Suppose that a grid and accordingly a periodic piecewise linear function space S_p over the grid is appropriately given. Then the most straightforward mixed formulations of the problem might be: find $u(\cdot, t), p(\cdot, t) \in S_\mathrm{p}$ such that

$$(u_t, v) = (p_x, v), \quad (p, w) = -(u_x, w_x), \quad \forall v, w \in S_\mathrm{p} \tag{27}$$

or

$$(u_t, v) = -(p_x, v_x), \quad (p, w) = (u_x, w), \quad \forall v, w \in S_\mathrm{p}. \tag{28}$$

On the other hand, Scheme 4 suggests a slightly different form:

$$(u_t, v) = -(p_x, v_x), \quad (p, w_x) = (u_x, w_x), \quad \forall v, w \in S_\mathrm{p}. \tag{29}$$

(Actually Scheme 4 literally suggests $(u_t, v) = -(p_x, v_x)$, $(p_1, w_x) = ((p_2)_x, w_x)$, $(p_2, z) = (u, z)$, $\forall v, w, z$, which immediately shrinks to the above.) The conservation property of (29) is guaranteed by Theorem 10, but it can be also directly viewed as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{u^2}{2}\,\mathrm{d}x = (u_t, u) = -(p_x, u_x) = -(p, p_x) = 0,$$

where (29) with $v := u, w := p$ is used. The similar calculation with the straightforward schemes, (27) and (28), turns out to fail. Actually, unless the grid is completely uniform, the straightforward schemes are not conservative in general. Accordingly any full discrete schemes based on them cannot be conservative. This example illustrates that the conservation property is so "fragile" that it can easily be lost unless correct weak forms are carefully chosen. Only when these "correct" weak forms are integrated with the "correct" time-stepping using discrete partial derivatives, the conservation property is rigorously kept. Similar notice also applies to dissipative cases.

Finally, it is interesting to point out that (27) corresponds to Scheme 3 if we regard the original problem as the special case of (5) with $s = 1$ and $G = u_x^2/2$. Thus (27) conserves another invariant $\int_0^L (u_x^2/2)\,\mathrm{d}x$, but not $\int_0^L (u^2/2)\,\mathrm{d}x$. The other scheme (28) completely fails in preserving either of the invariants.

3.3. Conservative schemes for the complex-valued PDEs (6)

Before defining schemes, we first introduce complex versions of the discrete partial derivatives. Suppose that the local energy is again of the form of Eq. (11), but that f_l and g_l are real-valued functions of a *complex-valued* function $u(x, t)$, which satisfy $f_l(u) = f_l(\bar{u})$, and $g_l(u_x) = g_l(\bar{u}_x)$. Throughout this section, we use the notation $(f, g) = \int_0^L \overline{f} g\,\mathrm{d}x$. Then the complex discrete partial derivatives are defined as follows:

Definition 13 (*Complex discrete partial derivatives*). We call the discrete quantities

$$\frac{\partial G_\mathrm{d}}{\partial\left(u^{(m+1)}, u^{(m)}\right)} :\equiv \sum_{l=1}^{M}\left(\frac{f_l(u^{(m+1)}) - f_l(u^{(m)})}{|u^{(m+1)}|^2 - |u^{(m)}|^2}\right)\left(\overline{\frac{u^{(m+1)} + u^{(m)}}{2}}\right)\left(\frac{g_l(u_x^{(m+1)}) + g_l(u_x^{(m)})}{2}\right), \tag{30}$$

$$\frac{\partial G_\mathrm{d}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)} :\equiv \sum_{l=1}^{M}\left(\frac{f_l(u^{(m+1)}) + f_l(u^{(m)})}{2}\right)\left(\frac{g_l(u_x^{(m+1)}) - g_l(u_x^{(m)})}{|u_x^{(m+1)}|^2 - |u_x^{(m)}|^2}\right)\left(\overline{\frac{u_x^{(m+1)} + u_x^{(m)}}{2}}\right), \tag{31}$$

which correspond to $\partial G/\partial u$ and $\partial G/\partial u_x$, respectively, "complex discrete partial derivatives."

Note that the complex discrete partial derivatives satisfy

$$\overline{\left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)},u^{(m)}\right)}\right)}=\frac{\partial G_{\mathrm{d}}}{\partial\left(\overline{u^{(m+1)}},\overline{u^{(m)}}\right)},\quad \overline{\left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)}\right)}=\frac{\partial G_{\mathrm{d}}}{\partial\left(\overline{u_x^{(m+1)}},\overline{u_x^{(m)}}\right)}.$$

The following identity holds concerning the complex partial derivatives.

Theorem 14 (*Discrete chain rule (complex-valued case)*).

$$\frac{1}{\Delta t}\int_0^L\left(G(u^{(m+1)})-G(u^{(m)})\right)\mathrm{d}x=\int_0^L\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)},u^{(m)}\right)}\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right)\mathrm{d}x$$
$$+\int_0^L\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)}\left(\frac{u_x^{(m+1)}-u_x^{(m)}}{\Delta t}\right)\mathrm{d}x+(\mathrm{c.c.}),$$

where (c.c.) *denotes the complex conjugates of the preceding terms.*

Making use of the complex discrete partial derivatives, a conservative scheme for the PDEs (5) is proposed as follows:

Scheme 5 (*Galerkin scheme for the PDEs (5)*). *Suppose that $u^{(0)}(x)$ is given in S_1. Find $u^{(m)}\in S_1$ $(m=1,2,\ldots)$ such that, for any $v\in W_1$,*

$$\mathrm{i}\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t},v\right)=-\left(\frac{\partial G_{\mathrm{d}}}{\partial\left(\overline{u^{(m+1)}},\overline{u^{(m)}}\right)},v\right)-\left(\frac{\partial G_{\mathrm{d}}}{\partial\left(\overline{u_x^{(m+1)}},\overline{u_x^{(m)}}\right)},v_x\right)+\left[\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)}v\right]_0^L.$$

Theorem 15 (*Conservation property of Scheme 5*). *Assume that boundary conditions are imposed so that*

$$\left[\left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)}\right)\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right)+(\mathrm{c.c.})\right]_0^L=0,$$

and $(u^{(m+1)}-u^{(m)})/\Delta t\in W_1$. Then Scheme 5 is conservative in the sense that

$$\frac{1}{\Delta t}\int_0^L\left(G(u^{(m+1)})-G(u^{(m)})\right)\mathrm{d}x=0,\quad m=0,1,2,\ldots.$$

Proof.

$$\frac{1}{\Delta t}\int_0^L\left(G(u^{(m+1)})-G(u^{(m)})\right)\mathrm{d}x$$
$$=\left(\frac{\partial G_{\mathrm{d}}}{\partial\left(\overline{u^{(m+1)}},\overline{u^{(m)}}\right)},\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right)+\left(\frac{\partial G_{\mathrm{d}}}{\partial\left(\overline{u_x^{(m+1)}},\overline{u_x^{(m)}}\right)},\frac{u_x^{(m+1)}-u_x^{(m)}}{\Delta t}\right)+(\mathrm{c.c.})$$
$$=-\mathrm{i}\left\|\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right\|_2^2+\left[\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)},u_x^{(m)}\right)}\left(\frac{u^{(m+1)}-u^{(m)}}{\Delta t}\right)\right]_0^L+(\mathrm{c.c.})$$
$$=0.\qquad\square$$

4. Application examples

Application examples for the CH equation (1), the KdV equation (2), and the NLS equation (3) are presented. Suppose that the interval $[0, L]$ is partitioned appropriately, and let $S_h \in H^1(0, L)$ be, for example, the piecewise linear function space over the grid.

4.1. The CH equation

The CH equation (1) is an example of Eq. (4) with $s = 1$ and $G(u, u_x) = pu^2/2 + ru^4/4 - qu_x^2/2$, which is usually solved subject to the boundary conditions

$$u_x = 0 \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta u}\right) = 0 \ \text{ at } x = 0, L. \tag{32}$$

Motivated by nature of the boundary conditions, let us set the trial spaces as $S_1, S_2 = \{v | v \in S_h, v_x(0) = v_x(L) = 0\}$, and the test spaces as $W_1, W_2 = S_h$. Then Scheme 2 reads as follows: find $u^{(m)} \in S_2$ and $p_1^{(m+1/2)} \in S_1$ such that, for all $v_1 \in W_1$ and $v_2 \in W_2$,

$$\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}, v_1\right) = -\left((p_1^{(m+1/2)})_x, (v_1)_x\right), \tag{33}$$

$$\left(p_1^{(m+1/2)}, v_2\right) = \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}, v_2\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}, (v_2)_x\right) \tag{34}$$

hold, where the terms

$$\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)}, u^{(m)}\right)} = p\left(\frac{u^{(m+1)} + u^{(m)}}{2}\right) + r\left(\frac{(u^{(m+1)})^2 + (u^{(m)})^2}{2}\right)\left(\frac{u^{(m+1)} + u^{(m)}}{2}\right), \tag{35}$$

$$\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)} = q\left(\frac{u_x^{(m+1)} + u_x^{(m)}}{2}\right), \tag{36}$$

are obtained from (12) and (13). Note that the boundary term $[(p_1^{(m+1/2)})_x v_1]_0^L$ which should appear in Eq. (33) and also the $[\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)} v_2]_0^L$ term in (34) vanish, because $(p_1^{(m+1/2)})_x = u_x^{(m+1)} = u_x^{(m)} = 0$ at $x = 0, L$. It is easily checked that all the assumptions in Theorem 7 are satisfied, and thus the scheme is dissipative. This scheme coincides with the Du–Nicolaides scheme [5], except in the fact that Du and Nicolaides discussed this scheme only with (unphysical) zero Dirichlet boundary conditions. Under such boundary conditions, Du–Nicolaides has shown the convergence of the scheme.

Remark 16. In practice, the trial spaces can be taken as $S_1 = S_2 = S_h$ as in the standard elliptic problems. Then the boundary conditions (32) are automatically recovered as the natural boundary conditions from Eqs. (33) and (34).

Remark 17. The scheme has an additional conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{u^{(m+1)} - u^{(m)}}{\Delta t}\,\mathrm{d}x = 0, \qquad m = 0, 1, 2, \ldots, \tag{37}$$

which can be easily seen from Eq. (17) with $v_1 = 1$.

4.2. The KdV equation

The KdV equation (2) is an example of Eq. (5) with $s = 1$ and $G(u, u_x) = u^3/6 - u_x^2/2$. The periodic boundary conditions are assumed:

$$u(0,t) = u(L,t), \quad u_x(0,t) = u_x(0,L), \quad t > 0. \tag{38}$$

Let us select the trial and test spaces $S_1 = S_2 = W_1 = W_2 = \{v \mid v \in S_h,\ v(0) = v(L), v_x(0) = v_x(L)\}$. (Strictly speaking, we consider the L-periodic problem on $x \in (-\infty, \infty)$, and slightly staggered L-periodic grid which does not have nodes on $x = 0, L$, in order to avoid the ambiguity of u_x at $x = 0, L$.) Then Scheme 3 reads as follows: find $u^{(m)} \in S_2$ and $p_1^{(m+1/2)} \in S_1$ such that, for all $v_1 \in W_1$ and $v_2 \in W_2$,

$$\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}, v_1\right) = \left((p_1^{(m+1/2)})_x, v_1\right), \tag{39}$$

$$\left(p_1^{(m+1/2)}, v_2\right) = \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)}, u^{(m)}\right)}, v_2\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)}, (v_2)_x\right) \tag{40}$$

hold, where

$$\frac{\partial G_{\mathrm{d}}}{\partial\left(u^{(m+1)}, u^{(m)}\right)} = \frac{\left(u^{(m+1)}\right)^2 + u^{(m+1)}u^{(m)} + \left(u^{(m)}\right)^2}{6}, \tag{41}$$

$$\frac{\partial G_{\mathrm{d}}}{\partial\left(u_x^{(m+1)}, u_x^{(m)}\right)} = -\frac{u_x^{(m+1)} + u_x^{(m)}}{2}, \tag{42}$$

are obtained from definitions (12) and (13). The boundary term appearing in (21) vanishes due to the periodicity of S_1 and W_1. Due to the periodicity of S_1, the assumption $[(p_1^{(m+1/2)})^2]_0^L = 0$ is satisfied. The periodicity also implies that condition (16) is satisfied, thus all the assumptions in Theorem 9 are satisfied, and hence the scheme is conservative. To the best of our knowledge, this scheme seems new.

Remark 18. The scheme also has the additional conservation law (37). Set $v_1 = 1$ in Eq. (20).

The new conservative scheme is tested numerically. The length of the spatial period is set to $L = 20$, and the period is divided into a non-uniform grid consisting of N points which concentrate at the center (see Fig. 1 for an example of $N = 201$). The approximation space $S_h \in H^1(0, L)$ is set to the standard piecewise linear function space over this grid. The initial condition is set to $u(x, 0) = 48\,\mathrm{sech}^2(2(x - 14)) + 24\,\mathrm{sech}^2(x - 10)$ (soliton-like pulses). For comparison, a standard Crank–Nicolson type scheme:

$$\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}, v_1\right) = \left((p_1^{(m+1/2)})_x, v_1\right), \tag{43}$$

$$\left(p_1^{(m+1/2)}, v_2\right) = \left(\frac{1}{2}\left(\frac{u^{(m+1)} + u^{(m)}}{2}\right)^2, v_2\right) - \left(\frac{u_x^{(m+1)} + u_x^{(m)}}{2}, (v_2)_x\right), \tag{44}$$

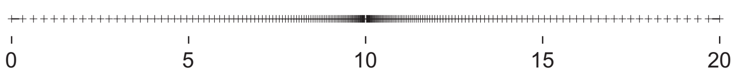


Fig. 1. The non-uniform mesh ($N = 201$).

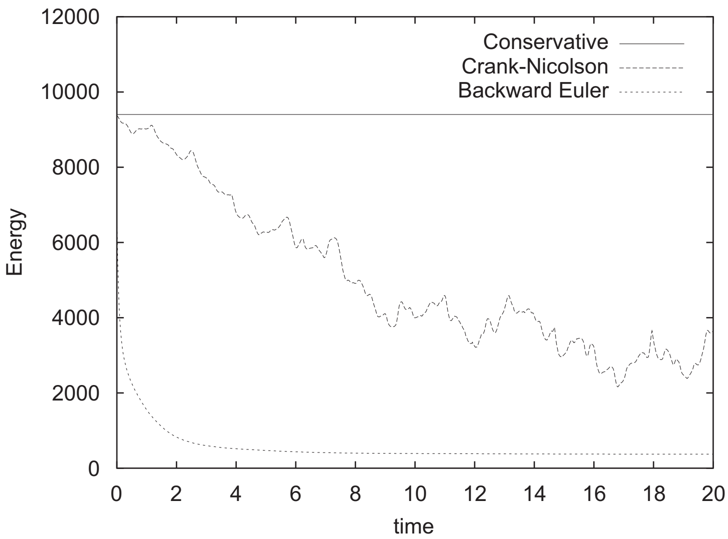


Fig. 2. Evolution of the global energies ($N = 201$).

and an backward Euler scheme:

$$\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}, v_1\right) = \left((p_1^{(m+1/2)})_x, v_1\right), \tag{45}$$

$$\left(p_1^{(m+1/2)}, v_2\right) = \left(\frac{(u^{(m+1)})^2}{2}, v_2\right) - \left(u_x^{(m+1)}, (v_2)_x\right), \tag{46}$$

are also tested.

First, the number of spatial mesh points is set to $N = 201$, the temporal mesh size $\Delta t = 0.025$, and the problem is integrated for $0 \leqslant t \leqslant 20$. Fig. 2 shows the evolution of the global energies. The conservative scheme strictly conserves the energy as constructed. In the Crank–Nicolson scheme the energy is not conserved but goes down. This energy decrease is even more drastic in the backward Euler scheme, whereas the reason for the decrease is not quite the same as the Crank–Nicolson case. This can be understood with Fig. 3, which shows the initial evolution of numerical solutions (for $0 \leqslant t \leqslant 3$). The solution by the backward Euler scheme collapses and eventually becomes flat; thus the energy decrease should be understood as the energy dissipation. On the other hand, the solution by the Crank–Nicolson scheme is not flattened; instead it develops undesired oscillations. The energy decrease should be attributed to this oscillations, which increases the term $\int_0^L u_x^2 \,\mathrm{d}x$ in the global energy. The oscillations can be also observed in the conservative scheme. The intensity is, however, smaller than the Crank–Nicolson case, and the solution is the best obtained among the three.

The undesired oscillations come out of the insufficiency in spatial accuracy. Next the conservative and Crank–Nicolson scheme are tested with the finer mesh $N = 401$, which is again non-uniform similar to the case of $N = 201$. The problem is then integrated for $0 \leqslant t \leqslant 100$ with $\Delta t = 0.01$. Fig. 4 shows the evolution of the energies, and Fig. 5 the solutions. The solutions are improved in both schemes. In the Crank–Nicolson scheme, however, the drifting of the energy persists. On the other hand, the conservative scheme successfully conserves the energy, and thus is better.

4.3. The NLS equation

Let us consider the NLS equation (3) under the periodic boundary condition (38). This is an example of Eq. (5) with $G(u, u_x) = -|u_x|^2 + 2\gamma|u|^{p+1}/(p+1)$. Let us select the trial and test spaces $S_1 = W_1 = \{v \mid v \in S_h,\ v(0) = v(L),\ v_x(0) = v_x(L)\}$.

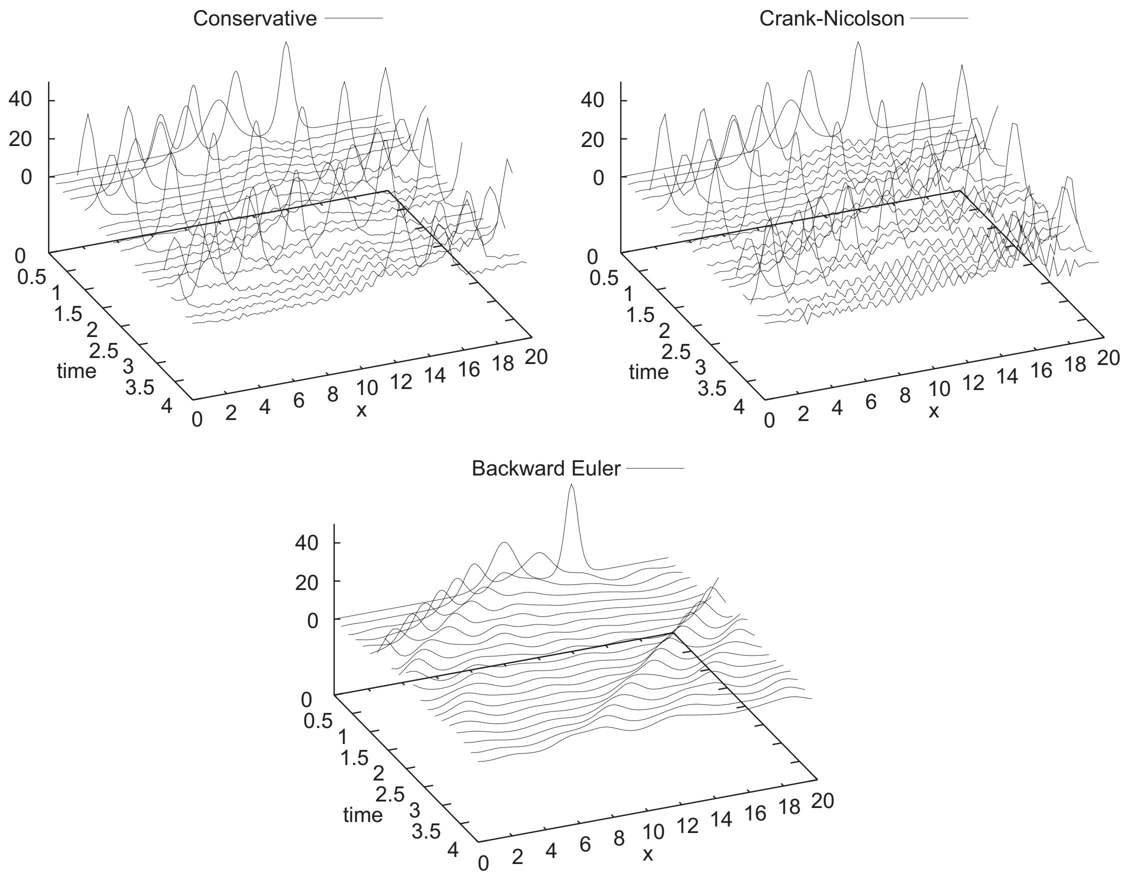


Fig. 3. The numerical solutions ($N = 201$): (top-left) the conservative scheme; (top-right) the Crank–Nicolson scheme; (bottom) the backward Euler scheme.

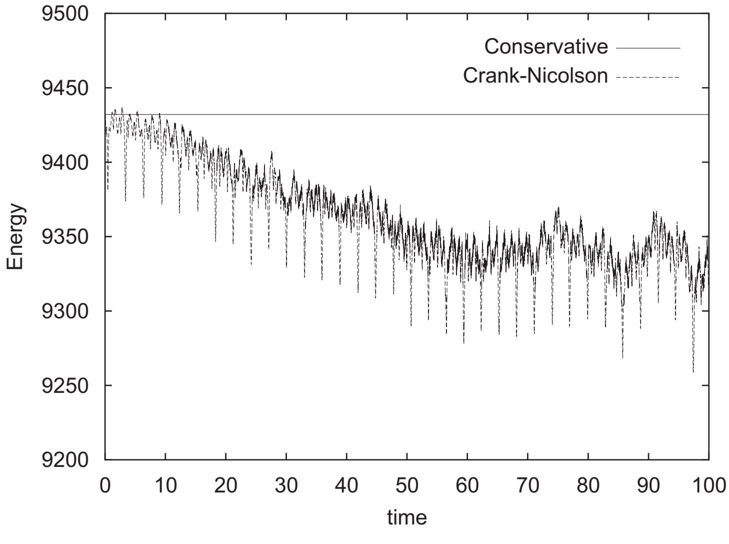
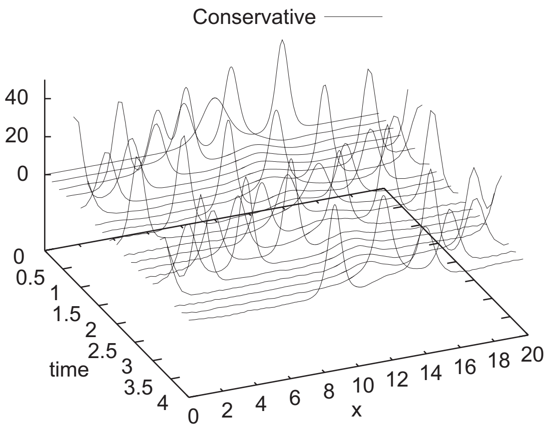


Fig. 4. Evolution of the global energies ($N = 401$).



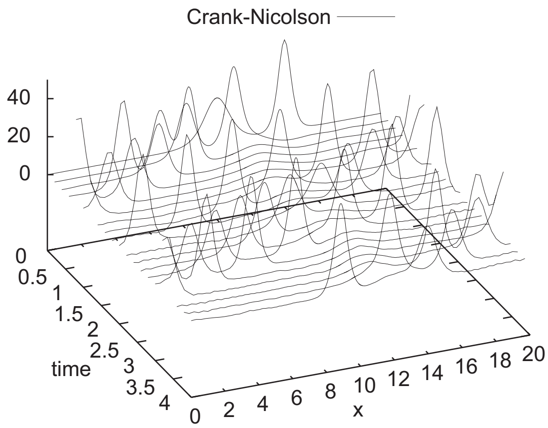


Fig. 5. The numerical solutions ($N = 401$): (left) the conservative scheme; (right) the Crank–Nicolson scheme.

Then Scheme 5 becomes: find $u \in S_1$ such that, for all $v \in W_1$,

$$\mathrm{i}\left(\frac{u^{(m+1)} - u^{(m)}}{\Delta t}, v\right) = -\left(\frac{\partial G_\mathrm{d}}{\partial\left(\overline{u^{(m+1)}}, \overline{u^{(m)}}\right)}, v\right) - \left(\frac{\partial G_\mathrm{d}}{\partial\left(\overline{u_x^{(m+1)}}, \overline{u_x^{(m)}}\right)}, v_x\right),$$

where the terms

$$\frac{\partial G_\mathrm{d}}{\partial\left(\overline{u^{(m+1)}}, \overline{u^{(m)}}\right)} = \gamma\left(\frac{|u^{(m+1)}|^{p+1} - |u^{(m)}|^{p+1}}{|u^{(m+1)}|^2 - |u^{(m)}|^2}\right)\left(\frac{u^{(m+1)} + u^{(m)}}{2}\right), \tag{47}$$

$$\frac{\partial G_\mathrm{d}}{\partial\left(\overline{u_x^{(m+1)}}, \overline{u_x^{(m)}}\right)} = -\frac{u_x^{(m+1)} + u_x^{(m)}}{2}, \tag{48}$$

are obtained from definitions (30) and (31). The boundary term appearing in Scheme 5 vanishes due to the periodicity of S_1 and W_1. The periodicity also implies that condition (16) is satisfied, and thus the conservation property follows from Theorem 15. It may be noted that this scheme is simply the Akrivis–Dougalis–Karakashian scheme [1], whose stability and convergence is already guarranteed.

5. Concluding remarks

In this paper, a new method for designing dissipative/conservative Galerkin (or finite-element) schemes has been proposed. The resulting schemes by the method can be implemented only with cheap H^1 elements. Though we limited ourselves to spatially one-dimensional cases in this paper, the essential idea must be also valid in two- or three-dimensional cases. In such circumstances, however, more careful considerations on boundary and spatial integrations are required (note that all the spatial integrations should be done in machine accuracy in order for the strict dissipation or conservation). These issues will be discussed elsewhere in the near future.

As application examples, three schemes for the CH, KdV, and NLS equations have been presented. The schemes for the CH and NLS coincided with the novel schemes in the literature whose theoretical aspects are well-known, which illustrates the generality and effectiveness of the proposed method. The scheme for the KdV seems new, and its superiority over several standard schemes has been shown through numerical experiments.

Finally, it is worth mentioning that the time mesh size can be changed adaptively in actual computation without destroying the strict dissipation or conservation property. It can be easily seen in each dissipation or conservation theorem. This feature can help reducing overall computational costs. This property can be also utilized for increasing the temporal accuracy of the resulting *conservative* schemes by the so-called "composition" techniques, in which the base scheme is called repeatedly with different time step sizes.

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