Padé approximation for a multivariate Markov transform

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Abstract

Methods of Padé approximation are used to analyse a multivariate Markov transform which has been recently introduced by the authors. The first main result is a characterization of the rationality of the Markov transform via Hankel determinants. The second main result is a cubature formula for a special class of measures.
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1. Introduction

Let σ be a non-negative finite measure on a subinterval $[a, b]$ of the real line $\mathbb{R}$. Then the numbers $\int_a^b x^l \, d\sigma(x)$ are called the *moments of the measure σ*. The *Markov transform* of σ is defined for $\zeta \in \mathbb{C} \backslash [a, b]$ by the formula

$$\widehat{\sigma}(\zeta) := \int_a^b \frac{1}{\zeta - x} \, d\sigma(x). \tag{1}$$

In the theory of moments Padé approximation of the Markov transform $\widehat{\sigma}(\zeta)$ is an important tool, see [1,5,6,18] and Section 6. Here Padé approximation is performed at the point ∞, so we consider the asymptotic expansion

$$\widehat{\sigma}(\zeta) = \sum_{l=0}^{\infty} \int_a^b x^l \, d\sigma(x) \frac{1}{\zeta^{l+1}} \quad \text{for } |\zeta| > R. \tag{2}$$

Let now μ be a signed measure μ on the euclidean space $\mathbb{R}^d$ with support in the closed ball $\overline{B_R} := \{x \in \mathbb{R}^d : |x| \leqslant R\}$ where $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ is the euclidean distance for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. In [15] we introduced a *multivariate*

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Markov transform for the measure μ by the formula

$$\widehat{\mu}(\zeta, \theta) = \int_{\mathbb{R}^d} \frac{\zeta^{d-1}}{r(\zeta\theta - x)^d} \, d\mu(x) \quad \text{for } |\zeta| > R, \ \theta \in \mathbb{S}^{d-1}. \tag{3}$$

Here $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere, and ζ is a complex number with $|\zeta| > R$. In the denominator, the expression $r(\zeta\theta - x)$ is the analytic continuation of the function $\rho \longmapsto |\rho\theta - x|$ defined for $\rho \in \mathbb{R}$ with $\rho > R$, see Section 3 for details. The motivation for this definition stems from the work of *Aronszajn* about polyharmonic functions and the work of *Hua* about harmonic analysis on Lie groups, see [2,11,15]. Following the analogy with the one-dimensional case, we consider the asymptotic expansion of the multivariate Markov transform. From the growth behaviour at infinity of the kernel $\zeta^{d-1}/r(\zeta\theta - x)^d$ it is easily seen that the asymptotic expansion is of the form

$$\widehat{\mu}(\zeta, \theta) = \sum_{l=0}^{\infty} f_l(\theta) \frac{1}{\zeta^{l+1}}, \tag{4}$$

for $|\zeta| > R$ and $\theta \in \mathbb{S}^{d-1}$ where $f_l : \mathbb{S}^{d-1} \to \mathbb{C}$ are continuous functions. The aim of this paper is to show that methods from Padé approximation can be successfully used for an analysis of the multivariate Markov transform. Roughly speaking, we shall perform in (4) the classical univariate Padé approximation for each fixed $\theta \in \mathbb{S}^{d-1}$ obtaining a Padé pair $(Q_n(\zeta, \theta), P_n(\zeta, \theta))$.

Let us describe the results in the paper: In Section 2 we shall first review the basic notions from Padé approximation which are needed in the paper. In Section 3 the asymptotic expansion defined in (4) will be investigated. It turns out that each coefficient function f_l in (4) is a finite sum of spherical harmonics of degree $\leqslant l$, and each f_l is the restriction of a homogeneous polynomial $F_l(x)$ of degree l to the unit sphere. The *Hankel determinant* of the multivariate Markov transform $\widehat{\mu}$ (or a measure μ) is defined by the expression

$$H_n(\mu, \theta) := \det \begin{pmatrix} f_0(\theta) & f_1(\theta) & \cdots & f_{n-1}(\theta) \\ f_1(\theta) & f_2(\theta) & \cdots & f_n(\theta) \\ \cdots & \cdots & \cdots & \cdots \\ f_{n-1}(\theta) & f_n(\theta) & \cdots & f_{2n-2}(\theta) \end{pmatrix}. \tag{5}$$

In Section 4 we show that the Hankel determinant $H_n(\mu, \theta)$ of a measure μ is the restriction of a homogeneous polynomial of degree $n(n-1)$ to the unit sphere. In Section 5 we shall prove a Kronecker type theorem: the Hankel determinants $H_n(\mu, \theta)$ are zero for all large n if and only if the function $\zeta \longmapsto \widehat{\mu}(\zeta, \theta)$ is rational for each $\theta \in \mathbb{S}^{d-1}$. Moreover, this is equivalent to the rationality of the multivariate Markov transform $\widehat{\mu}$.

A measure μ is called *Hankel positive* if the Hankel determinants $H_n(\mu, \theta)$ are strictly positive for all natural numbers n and for all $\theta \in \mathbb{S}^{d-1}$. In Section 6 we prove that for each Hankel positive measure μ there exists a non-negative measure μ_n which is equal to μ for all polynomials of degree $\leqslant 2n - 1$ and which has support contained in an algebraic variety. Further we characterize Hankel positivity by an extension property of the multivariate Markov transform.

Finally, we need some notations from harmonic analysis. A function $Y : \mathbb{S}^{d-1} \to \mathbb{C}$ is called a *spherical harmonic* of degree $k \in \mathbb{N}_0$ if there exists a *homogeneous harmonic* polynomial $P(x)$ of degree k (in general, with complex coefficients) such that $P(\theta) = Y(\theta)$ for all $\theta \in \mathbb{S}^{d-1}$. Throughout the paper we assume that $Y_{k,m}(x), m = 1, \ldots, a_k$, is a basis of the set of all harmonic homogeneous polynomials of degree k which are orthonormal with respect to scalar product

$$\langle f, g \rangle_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(\theta)\overline{g(\theta)} \, d\theta.$$

Here a_k denotes the dimension of the space of all harmonic homogeneous polynomials of degree k. By ω_d we denote the surface area of $\mathbb{S}^{d-1}$.

2. Basic facts from Padé approximation

At first let us recall some basic facts from Padé approximation (we refer to [18] for proofs): let f be a holomorphic function for $\zeta \in \mathbb{C}$, $|\zeta| > R$, of the form

$$f(\zeta) = \sum_{l=0}^{\infty} f_l \frac{1}{\zeta^{l+1}}.$$

Let n be a natural number. Then there exists a polynomial $P_n \neq 0$ of degree $\leqslant n$ such that

$$P_n(\zeta) f(\zeta) - Q_n(\zeta) = \sum_{l=n}^{\infty} f_l \frac{1}{\zeta^{l+1}}, \tag{6}$$

where Q_n is the polynomial part of the series $P_n(z) f(\zeta)$; it is easy to see that Q_n has degree $\leqslant n - 1$. A pair (P_n, Q_n) is called an nth *Padé pair* if P_n and Q_n are polynomials, $P_n \neq 0$, $\deg P_n \leqslant n$ and $\deg Q_n \leqslant n - 1$, and they satisfy (6). An index n is called *normal* if for any nth Padé pair (P_n, Q_n) the polynomial $\zeta \mapsto P_n(\zeta)$ has degree exactly n. Proposition 3.2 in [18] shows that n is normal if and only if the *Hankel determinant*

$$H_n(f) := \det \begin{pmatrix} f_0 & f_1 & \cdots & f_{n-1} \\ f_1 & f_2 & \cdots & f_n \\ \cdots & \cdots & \cdots & \cdots \\ f_{n-1} & f_n & \cdots & f_{2n-2} \end{pmatrix} \tag{7}$$

is not zero. If n is normal then the polynomial

$$P_n(\zeta) := \det \begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ \cdots & \cdots & \cdots & \cdots \\ f_{n-1} & \cdots & \cdots & f_{2n-1} \\ 1 & \zeta & \cdots & \zeta^n \end{pmatrix}$$

has exact degree n and (P_n, Q_n) is an nth Padé pair where Q_n is the polynomial part of $P_n(z) f(z)$. For arbitrary n, the rational function

$$\pi_n(\zeta) := \frac{Q_n(\zeta)}{P_n(\zeta)}$$

is called the nth *diagonal Padé approximant* of f.

3. Asymptotic expansion of the multivariate Markov transform

Following the exposition in [2, Section 2.2] we show that the multivariate Markov transform is well-defined. Let us set $r(x) := |x|$. For $\rho > 0$ and $\theta \in \mathbb{S}^{d-1}$ and $x = (x_1, \ldots, x_d)$ we have $r^2(\rho\theta - x) = \rho^2 - 2\rho\langle\theta, x\rangle + |x|^2$ where $\langle\theta, x\rangle$ is the usual inner product in $\mathbb{R}^d$. We replace ρ by a complex number ζ and obtain

$$r^2(\zeta\theta - x) = \zeta^2 - 2\zeta\langle\theta, x\rangle + |x|^2 = (\zeta - \langle\theta, x\rangle)^2 + |x|^2 - |\langle\theta, x\rangle|^2.$$

Note that $|x|^2 - |\langle\theta, x\rangle|^2 \geqslant 0$ for each $\theta \in \mathbb{S}^{d-1}$. If we define

$$a(\theta, x) := \langle\theta, x\rangle + i\sqrt{|x|^2 - |\langle\theta, x\rangle|^2},$$

then $r^2(\zeta\theta - x) = (\zeta - a(\theta, x))(\zeta - \overline{a(\theta, x)})$. Since $|a(\theta, x)|^2 = |x|^2$ it follows that $r^2(\zeta\theta - x) \neq 0$ for all $|\zeta| > |x|$. Next we see that the function g, defined by

$$g(\zeta) := \frac{r^2(\zeta\theta - x)}{\zeta^2} = \left(1 - \frac{a(\theta, x)}{\zeta}\right)\left(1 - \frac{\overline{a(\theta, x)}}{\zeta}\right)$$

for $|\zeta| > |x|$, has the property that $g(\zeta) \notin (-\infty, 0]$: since $|a(\theta, x)/\zeta| < 1$ and $|\overline{a(\theta, x)}/\zeta| < 1$ it follows that $1 - a(\theta, x)/\zeta$ and $1 - \overline{a(\theta, x)}/\zeta$ are in the right half plane, i.e., that their real parts are strictly positive, and therefore $g(\zeta) \notin (-\infty, 0]$. Using the square root function $\sqrt{\cdot}$ defined on $\mathbb{C} \backslash (-\infty, 0]$ one can define for $|\zeta| > |x|$ the analytic function

$$\zeta \longmapsto \sqrt{\frac{r^2(\zeta\theta - x)}{\zeta^2}}.$$

It follows from these facts that the multivariate Markov transform $\widehat{\mu}(\zeta, \theta)$ is well-defined.

Further we will make use of a real version of the multivariate Markov transform which we define by (note that we use d instead of $d - 1$ as exponent in the nominator)

$$\widehat{\mu}_{\text{real}}(y) := \int_{\mathbb{R}^d} \frac{|y|^d}{r(y - x)^d} \, \mathrm{d}\mu(x) \quad \text{for } y \in \mathbb{R}^d \text{ with } |y| > R. \tag{8}$$

The *real Markov transform* $\widehat{\mu}_{\text{real}}(y)$ is related to $\widehat{\mu}(\zeta, \theta)$ in the following way: using results about harmonicity hulls and Lie norms (see [2, p. 64]) one may show that the function $y \longmapsto \widehat{\mu}_{\text{real}}(y)$ has a holomorphic extension to a natural set C_R in the complex space $\mathbb{C}^d$, and the extension will be denoted by $\widehat{\mu}_{\text{real}}(z)$ for complex $z \in C_R$. The set C_R is the set of all $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ such that

$$L_-(z) := \sqrt{|z|^2 - \sqrt{|z|^4 - |q(z)|^2}} > R, \tag{9}$$

where we have defined $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$ and $q(z) = z_1^2 + \cdots + z_d^2$. The set C_R is connected and open, and it contains all points $\zeta \cdot \theta$ with $\zeta \in \mathbb{C}$, $|\zeta| > R$ and $\theta \in \mathbb{S}^{d-1}$. The Markov transforms $\widehat{\mu}(\zeta, \theta)$ and $\widehat{\mu}_{\text{real}}(z)$ are related by the simple formula

$$\widehat{\mu}_{\text{real}}(\zeta\theta) = \zeta\widehat{\mu}(\zeta, \theta) \quad \text{for all } \zeta \in \mathbb{C}, \ |\zeta| > R \text{ and } \theta \in \mathbb{S}^{d-1}. \tag{10}$$

Next we want to describe the asymptotic expansion of the multivariate Markov transform. Using the *Gauß decomposition* of a polynomial (see [3, Theorem 5.5, 22,13]) it is easy to see that the system

$$|x|^{2s} Y_{k,m}(x), s, k \in \mathbb{N}_0, \quad m = 1, \ldots, a_k$$

is a basis of the set of all polynomials. The numbers

$$c_{s,k,m} := \int_{\mathbb{R}^d} |x|^{2s} \overline{Y_{k,m}(x)} \, \mathrm{d}\mu(x), \quad s, k \in \mathbb{N}_0, \ m = 1, \ldots, a_k \tag{11}$$

are sometimes called the *distributed moments*, see [12]. For a treatment and formulation of the *multivariate moment problem* we refer to [9,2]. From [15] we cite:

Theorem 1. *Let μ be a signed measure on $\mathbb{R}^d$ with support in the closed ball $\overline{B_R}$. Then for all $|\zeta| > R$ and for all $\theta \in \mathbb{S}^{d-1}$ the following relation holds*:

$$\widehat{\mu}(\zeta, \theta) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{Y_{k,m}(\theta)}{\zeta^{2s+k+1}} \int_{\mathbb{R}^d} |x|^{2s} \overline{Y_{k,m}(x)} \, \mathrm{d}\mu(x). \tag{12}$$

For f_l defined in (4), a rearrangement of the series (12) in powers ζ^{l+1} yields the relation

$$f_l(\theta) = \sum_{t=0}^{[l/2]} \sum_{m=1}^{\alpha_{l-2t}} c_{t,l-2t,m} Y_{l-2t,m}(\theta), \tag{13}$$

where $[x]$ denotes the largest integer n such that $n \leqslant x$.

Proposition 2. *For each $l \in \mathbb{N}_0$ the coefficient function f_l in (4) is a finite sum of spherical harmonics of degree $\leqslant l$. Moreover, there exists a homogeneous polynomial $F_l(x)$ of degree l such that*

$$F_l(\zeta\theta) = \zeta^l f_l(\theta) \quad \text{for all } \theta \in \mathbb{S}^{d-1} \text{ and } \zeta \in \mathbb{C}.$$

Proof. Formula (13) shows that f_l is a sum of spherical harmonics of degree $\leqslant l$. Define a homogeneous polynomial F_l of degree l by

$$F_l(x) := \sum_{t=0}^{[l./2]} \sum_{m=1}^{\alpha_{l-2t}} c_{t,l-2t,m} |x|^{2t} Y_{l-2t,m}(x). \tag{14}$$

By inserting $x = \rho\theta$ in (14) for positive ρ we obtain $F_l(\rho\theta) = \rho^l f_l(\theta)$. Since $\rho \longmapsto F_l(\rho\theta)$ is holomorphic we may replace ρ by a complex number ζ. The proof is finished. $\quad\square$

The coefficient function f_l can also be described by Legendre polynomials $P_k(t)$ of degree k and dimension d, for definition see [17]. Clearly (13) and (11) imply that

$$f_l(\theta) = \int_{\mathbb{R}^d} \sum_{t=0}^{[l/2]} |x|^{2t} \sum_{m=1}^{a_{l-2t}} Y_{l-2t,m}(x) \cdot Y_{l-2t,m}(\theta) \, d\mu(x).$$

The addition theorem for spherical harmonics (see [17]) says that

$$\sum_{m=1}^{a_k} Y_{k,m}(x) \cdot Y_{k,m}(\theta) = |x|^k a_k P_k\left(\left\langle \frac{x}{|x|}, \theta \right\rangle\right),$$

so one obtains the alternative description

$$f_l(\theta) = \sum_{t=0}^{[l/2]} a_{l-2t} \int_{\mathbb{R}^d} |x|^l P_{l-2t}\left(\left\langle \frac{x}{|x|}, \theta \right\rangle\right) d\mu(x).$$

We conclude this section with some examples and results illustrating the definitions.

Example 3. Let σ be a finite non-negative measure on an interval $[a, b]$ with $a \geqslant 0$ and consider the measure $\mu = \sigma \otimes d\theta$, i.e., for every continuous function f holds

$$\int f(x) \, d\mu := \int_a^b \int_{\mathbb{S}^{d-1}} f(r\theta) \, d\sigma(r) \, d\theta.$$

Then the distributed moments $c_{s,k,m}$ are zero for all $k > 0$ since $Y_{k,m}(\theta)$ is orthogonal to the constant function with respect to the measure $d\theta$. Hence (12) shows that

$$\widehat{\mu}(\zeta, \theta) = \sum_{s=0}^{\infty} \frac{1}{\zeta^{2s+1}} \int_a^b r^{2s} \, d\sigma(r) = \int_a^b \frac{\zeta}{\zeta^2 - r^2} \, d\sigma(r).$$

From this we conclude that for all $l \in \mathbb{N}_0$ and $\theta \in \mathbb{S}^{d-1}$

$$f_{2l}(\theta) = \int_a^b r^{2l} \, d\sigma(r) \quad \text{and} \quad f_{2l+1}(\theta) = 0.$$

A measure μ on $\mathbb{R}^d$ is called *rotation invariant* if $\mu(T^{-1}(B)) = \mu(B)$ for all Borel sets B and for all orthogonal linear maps $T: \mathbb{R}^d \to \mathbb{R}^d$. The following result shows that a rotation invariant measure has a Markov transform $\widehat{\mu}(\zeta, \theta)$ which does not depend on $\theta \in \mathbb{S}^{d-1}$. Since the result is not needed later we omit the proof.

Theorem 4. *Let μ be a measure on $\mathbb{R}^d$ with support in $\overline{B_R}$. Then $\widehat{\mu}(\zeta, \theta)$ is independent of θ if and only if μ is rotation invariant. In that case the multivariate Markov transform possesses an analytic continuation to the upper half plane, namely*

$$\widehat{\mu}(\zeta, \theta) = \int \frac{\zeta}{\zeta^2 - |x|^2} \, d\mu(x) = \sum_{l=0}^{\infty} \int |x|^{2l} \, d\mu \frac{1}{\zeta^{2l+1}}$$

for all $\operatorname{Im} \zeta > 0$ *and* $\theta \in \mathbb{S}^{d-1}$.

4. Multivariate Padé approximation and Hankel determinants

We start with the following observation:

Proposition 5. *Let $H_n(\mu, \theta)$ be the Hankel determinant defined in (5). Then there exists a homogeneous polynomial $\widetilde{H}_n(x)$ of degree $n(n-1)$ such*

$$\widetilde{H}_n(\zeta \theta) = \zeta^{n(n-1)} H_n(\mu, \theta) \quad \text{for all } \theta \in \mathbb{S}^{d-1}.$$

Proof. By Proposition 2 there exists a homogeneous polynomial $F_l(x)$ of degree l such that $F_l(\zeta\theta) = \zeta^l f_l(\theta)$. Let us define

$$\widetilde{H}_n(x) := \det \begin{pmatrix} F_0(x) & F_1(x) & \cdots & F_{n-1}(x) \\ F_1(x) & F_2(x) & \cdots & F_n(x) \\ \cdots & \cdots & \cdots & \cdots \\ F_{n-1}(x) & F_n(x) & \cdots & F_{2n-2}(x) \end{pmatrix}.$$

Now we replace x by $\zeta\theta$ and we apply the Leibniz formula for determinants to the matrix $A = (a_{i,j})_{i,j=1,\ldots,n}$ defined by $a_{i,j} = F_{i+j}(\zeta\theta)$ for $i, j = 0, \ldots, n-1$. Then

$$\widetilde{H}_n(\zeta\theta) = \sum_{\sigma \text{ permutation}} \operatorname{sign}(\sigma) F_{0+\sigma(0)}(\zeta\theta) \ldots F_{n-1+\sigma(n-1)}(\zeta\theta).$$

Note that

$$0 + \sigma(0) + 1 + \sigma(1) + \cdots + (n-1) + \sigma(n-1) = n(n-1).$$

It is obvious that $\widetilde{H}_n(x)$ is a homogeneous polynomial of degree $n(n-1)$. We can factor out $\zeta^{n(n-1)}$ and we see that $\widetilde{H}_n(\zeta\theta) = \zeta^{n(n-1)} H_n(\mu, \theta)$. $\square$

In the following it is convenient to introduce the following notation: for a natural number n define a polynomial $\widetilde{P}_n(\zeta, \theta)$ of a univariate variable ζ of degree $\leqslant n$ by

$$\widetilde{P}_n(\zeta, \theta) := \det \begin{pmatrix} f_0(\theta) & f_1(\theta) & \cdots & f_n(\theta) \\ \cdots & \cdots & \cdots & \cdots \\ f_{n-1}(\theta) & \cdots & \cdots & f_{2n-1}(\theta) \\ 1 & \zeta & \cdots & \zeta^n \end{pmatrix}. \tag{15}$$

We shall also write

$$\widetilde{P}_n(\zeta, \theta) = p_0(\theta) + p_1(\theta)\zeta + \cdots + p_n(\theta)\zeta^n. \tag{16}$$

We define $\widetilde{Q}_n(\zeta, \theta)$ as the polynomial part of $\widetilde{P}_n(\zeta, \theta)\widehat{\mu}(\zeta, \theta)$, so

$$\widetilde{Q}_n(\zeta, \theta) = p_n f_0 \zeta^{n-1} + (p_{n-1} f_0 + p_n f_1)\zeta^{n-2} + \cdots + (p_1 f_0 + p_2 f_1 + \cdots + p_n f_{n-1}). \tag{17}$$

From the results in Section 2 the following is clear:

Theorem 6. *If $H_n(\mu, \theta) \neq 0$ then $(\widetilde{P}_n(\zeta, \theta), \widetilde{Q}_n(\zeta, \theta))$ is an nth Padé pair for the function*

$$\zeta \longmapsto \widehat{\mu}(\zeta, \theta) = \sum_{l=0}^{\infty} f_l(\theta) \frac{1}{\zeta^{l+1}}$$

for $|\zeta| > R$ where $\theta \in \mathbb{S}^{d-1}$ acts as a parameter.

In Example 9 below we shall show that $\zeta \longmapsto \widetilde{P}_n(\zeta, \theta)$ may be the zero polynomial for certain $\theta \in \mathbb{S}^{d-1}$, so $(\widetilde{P}_n(\zeta, \theta), \widetilde{Q}_n(\zeta, \theta))$ is *not always* an nth Padé pair.

The advantage of working with $\widetilde{P}_n(\zeta, \theta)$ is seen from the following result:

Theorem 7. *Let $\widetilde{P}_n(\zeta, \theta)$ and $\widetilde{Q}_n(\zeta, \theta)$ be defined in (15) and (17). Then there exists a polynomial A_n of degree $\leqslant n^2 + n$ and a polynomial B_n of degree $\leqslant n^2 + n - 2$ such that*

$$\zeta^{n^2} \widetilde{P}_n(\zeta, \theta) = A_n(\zeta\theta) \quad and \quad \zeta^{n^2} \widetilde{Q}_n(\zeta, \theta) = \zeta B_n(\zeta\theta). \tag{18}$$

Proof. By Proposition 2 there exists for each $l \in \mathbb{N}_0$ a homogeneous polynomial F_l of degree l such that $\zeta^l f_l(\theta) = F_l(\zeta\theta)$. Let us multiply each jth column in (15), $j = 0, \ldots, n$, with ζ^{n-1+j}. Let us define d_n to be the sum of $n - 1 + j$ for $j = 0, \ldots, n$. It follows that $\zeta^{d_n} P_n(\zeta, \theta)$ is equal to

$$\det \begin{pmatrix} \zeta^{n-1} f_0(\theta) & \zeta^n f_1(\theta) & \cdots & \zeta^{2n-1} f_n(\theta) \\ \cdots & \cdots & \cdots & \cdots \\ \zeta^{n-1} f_{n-1}(\theta) & \cdots & \cdots & \zeta^{2n-1} f_{2n-1}(\theta) \\ \zeta^{n-1} 1 & \zeta^n \zeta & \cdots & \zeta^{2n-1} \zeta^n \end{pmatrix}.$$

Since $F_l(\zeta\theta) = \zeta^l f_l(\theta)$ we obtain that $\zeta^{d_n} P_n(\zeta, \theta)$ is equal to

$$\det \begin{pmatrix} \zeta^{n-1} F_0(\zeta\theta) & \zeta^{n-1} F_1(\zeta\theta) & \cdots & \zeta^{n-1} F_n(\zeta\theta) \\ \cdots & \cdots & \cdots & \cdots \\ F_{n-1}(\zeta\theta) & \cdots & \cdots & F_{2n-1}(\zeta\theta) \\ \zeta^{n-1} 1 & \zeta^{n-1} \zeta^2 & \cdots & \zeta^{n-1} \zeta^{2n} \end{pmatrix}.$$

From the jth row factor out ζ^{n-1-j} for $j = 0, \ldots, n - 1$ and from the last one ζ^{n-1}. Then $d_n - (n - 1) - \sum_{j=0}^{n-1} j$ is equal to n^2. Hence we have proved that

$$\zeta^{n^2} \widetilde{P}_n(\zeta, \theta) = \det \begin{pmatrix} F_0(\zeta\theta) & F_1(\zeta\theta) & \cdots & F_n(\zeta\theta) \\ \cdots & \cdots & \cdots & \cdots \\ F_{n-1}(\zeta\theta) & \cdots & \cdots & F_{2n-1}(\zeta\theta) \\ 1 & \zeta^2 & \cdots & \zeta^{2n} \end{pmatrix}.$$

It follows that $\zeta^{n^2} P_n(\zeta, \theta) = A_n(\zeta\theta)$ where $A_n(x)$ is defined as

$$A_n(x) := \det \begin{pmatrix} F_0(x) & F_1(x) & \cdots & F_n(x) \\ \cdots & \cdots & \cdots & \cdots \\ F_{n-1}(x) & \cdots & \cdots & F_{2n-1}(x) \\ 1 & |x|^2 & \cdots & |x|^{2n} \end{pmatrix}. \tag{19}$$

This formula shows that the degree of $A_n(x)$ is lower than or equal to $n^2 + n$.

Let us discuss the polynomial part $\widetilde{Q}_n(\zeta, \theta)$. Let us write $\widetilde{P}_n(\zeta, \theta) = p_0(\theta) + p_1(\theta)\zeta + \cdots + p_n(\theta)\zeta^n$. By formula (15) it is clear that $p_j(\theta)$ can be defined by the determinant of the matrix in (15) where we have deleted the j column and the

last row. An analysis analog to the above shows that there exists a polynomial $R_j(x)$ such that $R_j(\zeta\theta) = \zeta^{n^2-j} p_j(\theta)$. Now formula (17) shows that

$$\zeta^{n^2}\widetilde{Q}_n(\zeta, \theta) = \zeta^{n^2} \sum_{k=0}^{n-1} \zeta^k \sum_{l=0}^{n-1-k} f_l(\theta) p_{k+1+l}(\theta).$$

Since $\zeta^{n^2}\zeta^k f_l(\theta) p_{k+1+l}(\theta) = \zeta^{2k+1} R_{k+1+l}(\zeta\theta) F_l(\zeta\theta)$ one can conclude that $(1/\zeta)\zeta^{n^2}\widetilde{Q}_n(\zeta, \theta)$ is a polynomial.   □

We want to relate the Padé approximation in Theorem 6 to Padé approximation in the context of polynomials in several real variables. Let F_l be the homogeneous polynomial of degree l defined in Proposition 2. The asymptotic expansion

$$\widehat{\mu}(\zeta, \theta) = \sum_{l=0}^{\infty} f_l(\theta) \frac{1}{\zeta^{l+1}} = \sum_{l=0}^{\infty} F_l(\zeta\theta) \frac{1}{\zeta^{2l+1}}$$

and the identity $\widehat{\mu}_{\mathrm{real}}(\zeta\theta) = \zeta\widehat{\mu}(\zeta, \theta)$, see (10), yield the asymptotic expansion of the real Markov transform $\widehat{\mu}_{\mathrm{real}}(y)$, namely

$$\widehat{\mu}_{\mathrm{real}}(y) = \sum_{l=0}^{\infty} F_l(y) \frac{1}{|y|^{2l}}.$$

By formula (6), Theorems 6 and 7 we can find polynomials $A_n(y)$ and $B_n(y)$ such that

$$A_n(\zeta\theta)\widehat{\mu}(\zeta, \theta) - \zeta B_n(\zeta\theta) = \zeta^{n^2} \sum_{l=n}^{\infty} f_l(\theta) \frac{1}{\zeta^{l+1}},$$

for all $\theta \in \mathbb{S}^{d-1}$ such that the index n is normal, i.e., $H_n(\mu, \theta) \neq 0$. We multiply this equation by ζ and write

$$A_n(\zeta\theta)\zeta\widehat{\mu}(\zeta, \theta) - \zeta^2 B_n(\zeta\theta) = \zeta^{n^2} \sum_{l=n}^{\infty} f_l(\zeta\theta) \frac{1}{\zeta^{2l}}.$$

Further for the polynomial $h(y) = |y|^2$ we have $h(\zeta\theta) = \zeta^2$, so the last equation implies for real $y = \rho\theta$ with $|y| > R$ and $H_n(\mu, \theta) \neq 0$

$$A_n(y)\widehat{\mu}_{\mathrm{real}}(y) - |y|^2 B_n(y) = |y|^{n^2} \sum_{l=n}^{\infty} F_l(y) \frac{1}{|y|^{2l}}.$$

Here $A_n(y)$ and $B_n(y)$ are subject to the conditions expressed in (18), and it seems to be rather technical to convert this in direct conditions for A_n, B_n.

We refer to [7,8] for multivariate Padé approximation based on polynomials in several variables.

Now we want to give an example of a measure μ such that the polynomial $\zeta \longmapsto \widetilde{P}_n(\zeta, \theta)$ (defined in (15)) is the zero polynomial. We recall at first the following result from [15].

Proposition 8. *Let σ be a measure on $\mathbb{R}$ with compact support, δ_0 be the Dirac measure on $\mathbb{R}$ at the point 0 and let $\mu = \sigma \otimes \delta_0$ be the product measure. Then the multivariate Markov transform $\widehat{\mu}$ is given by*

$$\widehat{\sigma \otimes \delta_0}(\zeta, \mathrm{e}^{\mathrm{i}t}) = \frac{1}{\omega_2} \sum_{l=0}^{\infty} \int x^l \mathrm{d}\sigma(x) \frac{\sin(l+1)t}{\sin t} \frac{1}{\zeta^{l+1}}, \tag{20}$$

where ω_2 is the area measure of $\mathbb{S}^2$.

The last proposition has been used to show that there exists a measure μ with a support contained in an algebraic set such that $\zeta\widehat{\mu}(\zeta, \theta)$ is not a rational function.

Example 9. Let σ be the Lebesgue measure on $[0, 1]$, so $\int_0^1 x^l \, d\sigma(x) = 1/(l+1)$ and let $\mu = \sigma \otimes \delta_0$ as in Proposition 8. Then $f_l(1) = 1$ for all $l \in \mathbb{N}_0$, and this implies that $\widetilde{P}_n(\zeta, 1) = 0$ for all $n \geqslant 2$.

5. Rationality of the multivariate Markov transform

Recall that a function $f: \mathbb{R}^d \to \mathbb{C}$ is *rational* if there exist polynomials $p(x)$ and $q(x) \neq 0$ with $f(x) = p(x)/q(x)$ for all x with $q(x) \neq 0$.

A theorem of Kronecker [18, Theorem 3.1] says that a necessary and sufficient condition for a series $f(\zeta)$ of a single variable ζ to be the Laurent expansion of a rational function is that the Hankel determinants $H_m(f)$ are zero for all sufficiently large m.

We have now the following analogue for the multivariate Markov transform:

Theorem 10. *Let μ be a measure with support in $\overline{B_R}$. Then the following statements are equivalent*:

(a) *For each $\theta \in \mathbb{S}^{d-1}$ the function $\zeta \mapsto \widehat{\mu}(\zeta, \theta)$ is rational.*
(b) *There exists $n \in \mathbb{N}$ such that $H_m(\mu, \theta) \equiv 0$ for all $m \geqslant n$ and $\theta \in \mathbb{S}^{d-1}$.*
(c) *There exists $n \in \mathbb{N}$ such that for each $\theta \in \mathbb{S}^{d-1}$ the function $\zeta \mapsto \widehat{\mu}(\zeta, \theta)$ is rational of degree $\leqslant n$.*
(d) *There exist polynomials $P(x)$ and $Q(x)$ such that for all $\theta \in \mathbb{S}^{d-1}$ and for all $|\zeta| > R$*

$$P(\zeta\theta) \neq 0 \quad and \quad \widehat{\mu}(\zeta, \theta) = \zeta \frac{Q(\zeta\theta)}{P(\zeta\theta)}. \tag{21}$$

Proof. Assume (a) and let $d(\theta)$ be the degree of the rational function $\zeta \mapsto \widehat{\mu}(\zeta, \theta)$ for $\theta \in \mathbb{S}^{d-1}$ (recall that the degree of a rational function $f = p/q$ with relatively prime polynomials is defined as $\max\{\deg p, \deg q\}$, see [18, p. 38]). By Kronecker's theorem (cf. [18, p. 46]) it follows that the Padé approximant $\pi_n(\zeta, \theta)$ is equal to $\widehat{\mu}(\zeta, \theta)$ for all indices $n > d(\theta)$ and $H_n(\mu, \theta) = 0$ for all $n > d(\theta)$. It follows that $\mathbb{S}^{d-1}$ is the union of the following sets:

$$A_n := \bigcap_{m=n}^{\infty} \{\theta \in \mathbb{S}^{d-1}: H_m(\mu, \theta) = 0\}$$

for $n \in \mathbb{N}$. Moreover, A_n is closed by continuity of $\theta \mapsto H_n(\mu, \theta)$. By Baire's category theorem there exists an index n such that A_n contains an interior point. Hence there exists $\theta_0 \in \mathbb{S}^{d-1}$ and a neighbourhood U of θ_0 such that $H_m(\mu, \theta) = 0$ for all $\theta \in U$ and for all $m \geqslant n$. Since $\zeta^{n(n-1)} H_m(\mu, \theta) = \widetilde{H}_n(\zeta\theta)$ by Proposition 5 we see that the polynomial $\widetilde{H}_n$ vanishes in a neighbourhood of $\theta_0 \in \mathbb{R}^d$. Thus, $\widetilde{H}_n(x)$ is the zero polynomial and $H_m(\mu, \theta) = 0$ for $\theta \in \mathbb{S}^{d-1}$ and for all $m \geqslant n$. Hence we have proved (b).

The implication (b) $\to$ (c) follows from Kronecker's theorem. The implication is (c) $\to$ (a) is trivial.

Clearly (d) implies (a), and it suffices to show that (b) implies (d). Now suppose that $H_m(\mu, \theta) \equiv 0$ for all $m > n$ and that $H_n(\mu, \theta)$ is not the zero function. Let N_n be the set of all $\theta \in \mathbb{S}^{d-1}$ such that $H_n(\mu, \theta) \neq 0$, so n is a normal index for each $\theta \in N_n$. Clearly N_n is an open non-empty set. Let $m > n$ be arbitrary. By [18, Proposition 3.3, p. 45] $\pi_m(\zeta, \theta) = \pi_n(\zeta, \theta)$ for all $\theta \in N_n$ and this implies $\pi_n(\zeta, \theta) = \widehat{\mu}(\zeta, \theta)$ for all $|\zeta| > R$ and $\theta \in N_n$. Let $\widetilde{P}_n(\zeta, \theta)$ and $\widetilde{Q}_n(\zeta, \theta)$ as in the last section. Since the index n is normal for $\theta \in N_n$ it is clear that $(\widetilde{P}_n(\zeta, \theta), \widetilde{Q}_n(\zeta, \theta))$ is an nth Padé pair and we infer

$$\widetilde{P}_n(\zeta, \theta)\widehat{\mu}(\zeta, \theta) = \widetilde{Q}_n(\zeta, \theta). \tag{22}$$

By Theorem 7 there exist polynomials $A(x)$, $B(x)$ such that

$$A(\zeta\theta) = \zeta^{n^2} \widetilde{P}_n(\zeta, \theta) \quad and \quad \zeta B(\zeta\theta) = \zeta^{n^2} \widetilde{Q}_n(\zeta, \theta)$$

for all ζ and $\theta \in \mathbb{S}^{d-1}$. So we have for all $y = \rho\theta$ with $\rho > R$ and $\theta \in N_n$ that

$$A(y) \cdot \widehat{\mu}_{\text{real}}(y) = |y|^2 B(y), \tag{23}$$

where $\widehat{\mu}_{\text{real}}$ is defined in (8). Since $\widehat{\mu}_{\text{real}}(y)$, and obviously $A(y)$ and $B(y)$, are real-analytic for all $y \in \mathbb{R}^d$, $|y| > R$, the equality (23), valid on an open subset of $\mathbb{R}^d \setminus \overline{B_R}$, holds for all $y \in \mathbb{R}^d$, $|y| > R$ as well. Moreover, $\widehat{\mu}_{\text{real}}$ has a holomorphic continuation for all $z \in \mathbb{C}^d$ with $L_-(z) > R$ where L_- is defined in (9). So we obtain

$$A(z)\widehat{\mu}_{\text{real}}(z) = (z_1^2 + \cdots + z_d^2)B(z) \quad \text{for all } z \in \mathbb{C}^d, \quad L_-(z) > R. \tag{24}$$

Suppose now that A contains an irreducible factor g which has a zero $z_0 \in \mathbb{C}^d$ with $L_-(z_0) > R$. By continuity of L_- there exists a neighbourhood U of z_0 with $L_-(z) > R$ for all $z \in U$. Eq. (24) shows that $U \cap g^{-1}\{0\} \subset B^{-1}\{0\}$ (recalling that $z_1^2 + \cdots + z_d^2 \neq 0$ for all $z \in \mathbb{C}^d$ with $L_-(z) > R$). It follows that g must divide B, see [23, p. 26]. Inductively, we can factor out each irreducible factor of A which has zero $z_0 \in \mathbb{C}^d$ with $L_-(z_0) > R$. Finally we obtain polynomials $A_1(z)$ and $B_1(z)$ such that

$$A_1(z)\widehat{\mu}_{\text{real}}(z) = (z_1^2 + \cdots + z_d^2)B_1(z) \quad \text{for all } z \in \mathbb{C}^d \text{ with } L_-(z) > R,$$

and $A_1(z) \neq 0$ for all $L_-(z_0) > R$. The proof is accomplished. $\quad\square$

6. A cubature formula

Let σ be a measure with finite moments on $\mathbb{R}$ and support in $[-R, R]$ and consider the functional

$$T(u) := \frac{1}{2\pi i} \int_{\Gamma_{R_1}} u(\zeta)\widehat{\sigma}(\zeta)\,d\zeta, \tag{25}$$

where $\Gamma_{R_1}(t) = R_1 e^{it}$ for $t \in [0, 2\pi]$ for any $R_1 > R$. For a polynomial $u(\zeta) = u_0 + u_1\zeta + \cdots + u_m\zeta^m$ we have

$$T(u) = \sum_{l=0}^{m} u_l \cdot f_l = \int_{-R}^{R} u(x)\,d\sigma(x), \tag{26}$$

where $f_l := \int_a^b x^l\,d\sigma(x)$ are the coefficients in the asymptotic expansion of $\widehat{\sigma}$ given in (2).

We shall make use of the following classical fact (see e.g. [18]): Let (Q_n, P_n) be the nth Padé pair of the Markov transform $\widehat{\sigma}(\zeta)$ of a non-negative measure σ with support in the interval $[a, b]$. If n is normal (so the Hankel determinant $H_n(\widehat{\sigma})$ is not zero) then the zeros $x_1, \ldots, x_n$ of P_n are real and simple and lie in the interval (a, b); moreover there exist positive coefficients $\alpha_1, \ldots, \alpha_n$ such that the discrete measure

$$\sigma_n = \alpha_1 \delta_{x_1} + \cdots + \alpha_n \delta_{x_n}$$

is identical to σ on the subspace of all polynomials $p(x)$ of degree $\leqslant 2n - 1$ and we have $\alpha_k = Q_n(x_k)/P_n'(x_k)$ for $k = 1, \ldots, n$. For any polynomial $u(x)$ Cauchy's theorem yields

$$\frac{1}{2\pi i} \int_{\Gamma_{R_1}} u(\zeta)\frac{Q_n(\zeta)}{P_n(\zeta)}\,d\zeta = \sum_{k=1}^{n} \alpha_k u(x_k) = \int_a^b u(x)\,d\sigma_n. \tag{27}$$

Combining this with (26) we obtain the following formula:

$$\frac{1}{2\pi i} \int_{\Gamma_{R_1}} u(\zeta)\frac{Q_n(\zeta)}{P_n(\zeta)}\,d\zeta = \sum_{l=0}^{2n-1} u_l \cdot f_l \tag{28}$$

valid for any polynomial u of degree $\leqslant 2n - 1$.

Let $\mathscr{P}(\mathbb{R}^d)$ be the set of all polynomials in d variables.

Definition 11. A functional $T: \mathscr{P}(\mathbb{R}^d) \to \mathbb{C}$ is positive definite if

$$T(u^*u) \geqslant 0$$

for all $u \in \mathscr{P}(\mathbb{R}^d)$ where u^* is the polynomial obtained from u by conjugating the coefficients.

Definition 12. A measure μ on $\mathbb{R}^d$ with support in the closed ball $\overline{B_R}$ is called Hankel-positive if the Hankel determinants are strictly positive, i.e.,

$$H_n(\mu, \theta) > 0 \quad \text{for all } n \in \mathbb{N}, \ \ \theta \in \mathbb{S}^{d-1}.$$

Obviously, an equivalent formulation for Hankel positivity is the requirement that

$$(f_l(\theta))_{l=0,1,\ldots} \text{ is strictly positive definite}$$

for each $\theta \in \mathbb{S}^{d-1}$. This means that for each $\theta \in \mathbb{S}^{d-1}$ and for all $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, $(x_0, \ldots, x_n) \neq 0$

$$\sum_{i=0}^{n} \sum_{j=0}^{n} f_{i+j}(\theta) x_i x_j > 0.$$

The following result is needed in the next theorem:

Proposition 13. *Let $\widetilde{P}_n(\zeta, \theta)$ be defined in* (15). *If the Hankel determinant $\theta \mapsto H_n(\mu, \theta)$ has no zeros then there exists $R_1 > 0$ such that*

$$\widetilde{P}_n(\zeta, \theta) \neq 0 \quad \text{for all } \theta \in \mathbb{S}^{d-1}, \ \ \zeta \in \mathbb{C} \text{ with } |\zeta| \geqslant R_1. \tag{29}$$

Proof. By assumption $H_n(\mu, \theta) \neq 0$ for all $\theta \in \mathbb{S}^{d-1}$, so it follows that $\zeta \longmapsto \widetilde{P}_n(\zeta, \theta)$ defined in (15) is a polynomial of degree exactly n. Let us write

$$\widetilde{P}_n(\zeta, \theta) = p_0(\theta) + p_1(\theta)\zeta + \cdots + p_n(\theta)\zeta^n.$$

Then $p_n(\theta) \neq 0$ for all $\theta \in \mathbb{S}^{d-1}$ and p_n is continuous. A straightforward estimate now shows that there exists $R_1 > 0$ such that $\widetilde{P}_n(\zeta, \theta) \neq 0$ for all $|\zeta| > R_1$ and for all $\theta \in \mathbb{S}^{d-1}$. $\square$

The following is an analogue of (26) for the multivariate Markov transform, for the proof we refer to [15]:

Proposition 14. *Let μ be a measure on $\mathbb{R}^d$ with support in $\overline{B_R}$ and let $R_1 > R$. Then for any $u \in \mathscr{P}(\mathbb{R}^d)$*

$$M(u) := \frac{1}{2\pi i \omega_d} \int_{\Gamma_{R_1}} \int_{\mathbb{S}^{d-1}} u(\zeta\theta)\widehat{\mu}(\zeta, \theta) \, d\zeta \, d\theta = \int_{\mathbb{R}^d} u(x) \, d\mu(x). \tag{30}$$

The following result is an extension of the Gauß quadrature formula to the multivariate setting. It can be seen as a solution of the *truncated moment problem* for the class of Hankel-positive measures. We refer to [4,9] for a description of the multivariate moment problem.

Theorem 15. *Let μ be a Hankel-positive measure with support in $\overline{B_R}$. Let $\widetilde{P}_n(\zeta, \theta)$ and $\widetilde{Q}_n(\zeta, \theta)$ be defined in* (15) *and* (17), *and let $R_1 > R$ so large such that* (29) *holds. Then the functional $T_n : \mathscr{P}(\mathbb{R}^d) \to \mathbb{C}$, defined by*

$$T_n(u) := \frac{1}{2\pi i \omega_d} \int_{\Gamma_{R_1}} \int_{\mathbb{S}^{d-1}} u(\zeta\theta) \frac{\widetilde{Q}_n(\zeta, \theta)}{\widetilde{P}_n(\zeta, \theta)} \, d\zeta \, d\theta \tag{31}$$

for all $u \in \mathscr{P}(\mathbb{R}^d)$, is positive definite and for each polynomial $u(x)$ of degree $\leqslant 2n - 1$

$$T_n(u) = \int u(x) \, d\mu(x).$$

Moreover, there exists a non-negative measure μ_n with support in an algebraic bounded set in $\mathbb{R}^d$ such that

$$T_n(u) = \int u(x) \, d\mu_n(x)$$

for any polynomial u.

Proof. (1) Since (29) holds for $R_1 > R$ expression (31) is well defined. Moreover $(\widetilde{Q}_n(\zeta, \theta), \widetilde{P}_n(\zeta, \theta))$ is an nth Padé pair since $H_n(\mu, \theta) \neq 0$ for all $\theta \in \mathbb{S}^{d-1}$ by Hankel positivity.

(2) Suppose that u is a function of the form

$$u(\zeta\theta) = u_0(\theta) + \cdots + u_{2n-1}(\theta)\zeta^{2n-1} \tag{32}$$

with continuous functions $u_0, \ldots, u_{2n-1}$. By (28) we have

$$\frac{1}{2\pi i} \int_{\Gamma_{R_1}} u(\zeta\theta) \frac{Q_n(\zeta, \theta)}{P_n(\zeta, \theta)} \, d\zeta = \sum_{l=0}^{2n-1} u_l(\theta) f_l(\theta). \tag{33}$$

Integration over $\mathbb{S}^{d-1}$ gives

$$T_n(u) = \frac{1}{\omega_d} \sum_{l=0}^{2n-1} \int_{\mathbb{S}^{d-1}} u_l(\theta) f_l(\theta) \, d\theta. \tag{34}$$

(3) Let u be a polynomial. Proposition 14 shows that

$$\int u(x) \, d\mu(x) = \frac{1}{2\pi i \omega_d} \int_{\mathbb{S}^{d-1}} \int_{\Gamma_{R_1}} u(\zeta\theta) \widehat{\mu}(\zeta, \theta) \, d\zeta \, d\theta. \tag{35}$$

If u has degree $\leqslant 2n - 1$ then for each $\theta \in \mathbb{S}^{d-1}$ the function $\zeta \mapsto u(\zeta\theta)$ is a polynomial of degree $\leqslant 2n - 1$, so u is of the form (32). Insert (4) in (35) and integrate over Γ_{R_1} to obtain

$$\int u(x) \, d\mu(x) = \frac{1}{\omega_d} \sum_{l=0}^{2n-1} \int_{\mathbb{S}^{d-1}} u_l(\theta) f_l(\theta) \, d\theta. \tag{36}$$

Comparing (34) with (36) we conclude that $T_n(u)$ is equal to $\int u(x) \, d\mu(x)$ for any polynomial of degree $\leqslant 2n - 1$.

(4) Let us discuss the question of positive definiteness of T_n. Let $R(x)$ be a real-valued polynomial. We have to show that $T_n(R^2) \geqslant 0$. By the euclidean algorithm applied to the polynomials $\zeta \mapsto R(\zeta\theta)$ and $\zeta \mapsto \widetilde{P}_n(\zeta\theta)$ for each fixed θ, there exist a polynomial $\zeta \mapsto d(\zeta, \theta)$, and a polynomial $\zeta \mapsto e(\zeta, \theta)$ of degree $< n$, such that

$$R(\zeta\theta) = d(\zeta, \theta)\widetilde{P}_n(\zeta, \theta) + e(\zeta, \theta).$$

Write $e(\zeta, \theta) = e_0(\theta) + \cdots + e_{n-1}(\theta)\zeta^{n-1}$. Then

$$(R(\zeta\theta))^2 = d^2(\zeta, \theta)(\widetilde{P}_n(\zeta, \theta))^2 + 2d(\zeta, \theta)e(\zeta, \theta)\widetilde{P}_n(\zeta, \theta) + e^2(\zeta, \theta).$$

Multiply the last equation with $\widetilde{Q}_n(\zeta, \theta)/\widetilde{P}_n(\zeta, \theta)$ and integrate with respect to ζ over Γ_{R_1}. Then

$$b(\theta) := \frac{1}{2\pi i} \int_{\Gamma_{R_1}} R^2(\zeta\theta) \frac{\widetilde{Q}_n(\zeta, \theta)}{\widetilde{P}_n(\zeta, \theta)} \, d\zeta = \frac{1}{2\pi i} \int_{\Gamma_{R_1}} e^2(\zeta, \theta) \frac{\widetilde{Q}_n(\zeta, \theta)}{\widetilde{P}_n(\zeta, \theta)} \, d\zeta. \tag{37}$$

Since $\zeta \to e^2(\zeta, \theta)$ is a polynomial of degree $\leqslant 2n - 1$, (33) yields

$$b(\theta) = \frac{1}{2\pi i} \int_{\Gamma_{R_1}} e^2(\zeta, \theta) \frac{\widetilde{Q}_n(\zeta, \theta)}{\widetilde{P}_n(\zeta, \theta)} \, d\zeta = \sum_{k,l=0}^{n-1} e_k(\theta)e_l(\theta) f_{k+l}(\theta). \tag{38}$$

Integrate the last equation with respect to $\mathbb{S}^{d-1}$ and use the definition of T_n in order to obtain

$$T_n(R^2) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \left(\sum_{k,l=0}^{n-1} e_k(\theta)e_l(\theta) f_{k+l}(\theta) \right) d\theta.$$

Since $(f_l)_l$ is strictly positive definite we know that $\sum_{k,l=0}^{n-1} e_k(\theta)e_l(\theta) f_{k+l}(\theta) \geqslant 0$ for each $\theta \in \mathbb{S}^{d-1}$, in particular $T_n(R^2) \geqslant 0$.

(5) Let A_n be the polynomial defined in Theorem 7 such that $A_n(\zeta\theta) = \zeta^{n^2}\widetilde{P}_n(\zeta\theta)$. Note that A_n has real coefficients. It follows that for any polynomial u

$$T_n(A_n u) = \frac{1}{2\pi i \omega_d} \int_{\mathbb{S}^{d-1}} \int_{\Gamma_{R_1}} \zeta^{n^2} u(\zeta\theta)\widetilde{Q}_n(\zeta,\theta)\, d\zeta\, d\theta = 0$$

since $\zeta \longmapsto \zeta^{n^2} u(\zeta\theta)\widetilde{Q}_n(\zeta,\theta)$ is a polynomial. The polynomial $\zeta \longmapsto \widetilde{P}_n(\zeta\theta)$ has only zeros in the interval $(-R, R)$, so it follows that $\widetilde{P}_n(\rho\theta) \neq 0$ for all $\rho > R$. Hence $A_n(\rho\theta) \neq 0$ for all $\rho > R$, so the zero set of $y \longmapsto A_n(y)$ is contained in the ball $\overline{B_R}$.

(6) The existence of a representation measure μ_n follows from Theorem 16 below, cf. [19] (applied to $m = 2$ and $f_1 = A_n$ and $f_2 = -A_n$). The proof is finished. $\square$

Theorem 16 (Schmüdgen). *Let $\mathfrak{S}: \mathscr{P}(\mathbb{R}^d) \to \mathbb{C}$ be a positive definite functional and let $f_1, \ldots, f_m$ be polynomials with real coefficients such that the set*

$$K := \{x \in \mathbb{R}^d : f_j(x) \geqslant 0 \text{ for all } j = 1, 2, \ldots, m\}$$

is compact. Then there exists a measure μ with support in K representing $\mathfrak{S}$ if and only if

$$\mathfrak{S}(f_{j_1} \ldots f_{j_s} \cdot p^* p) \geqslant 0 \tag{39}$$

for all pairwise different $j_1, \ldots, j_s \in \{1, \ldots, m\}$ and for all $p \in \mathscr{P}(\mathbb{R}^d)$.

It is not difficult to see that a rotation invariant measure with non-algebraic support is Hankel-positive, cf. Theorem 4. Now we give a different class of examples:

Proposition 17. *Let $w_0, w_1: [0, \infty) \to \mathbb{R}$ be bounded continuous functions with compact support such that $|w_1(r)| \leqslant w_0(r)$ for all $r \geqslant 0$, and $w_0 \neq 0$. Assume that the measure μ has the density $d\mu := w(r, \vartheta) r\, dr\, d\vartheta$ where*

$$w(r, \vartheta) = w_0(r) + w_1(r)\cos\vartheta$$

for all $r > 0$, $\vartheta \in [0, 2\pi]$. Then μ is Hankel-positive.

Proof. Note that the assumption $|w_1(r)| \leqslant w_0(r)$ for all $r \geqslant 0$ assures that

$$w(r, \vartheta) = w_0(r) + w_1(r)\cos\vartheta \geqslant 0 \tag{40}$$

for all $r > 0$, $\vartheta \in [0, 2\pi]$. Let us write $\theta = e^{i\vartheta}$ with $\vartheta \in [0, 2\pi]$. Note that $Y_0(\theta) = 1/\sqrt{2\pi}$ and $Y_{k,1}(\theta) = (1/\sqrt{\pi})\cos k\vartheta$ and $Y_{k,2}(\theta) = (1/\sqrt{\pi})\sin k\vartheta$, $k \in \mathbb{N}_0$, provides an orthonormal basis of spherical harmonics. Then

$$\int_{\mathbb{R}^d} |x|^{2s} Y_0(x)\, d\mu = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^{2\pi} r^{2s} w(r, \vartheta) r\, dr\, d\vartheta = \sqrt{2\pi} \int_0^\infty r^{2s+1} w_0(r)\, dr$$

and

$$\int_{\mathbb{R}^d} |x|^{2s} Y_{1,1}(x)\, d\mu = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^{2\pi} r^{2s} \cdot r\cos\vartheta \cdot w(r, \vartheta) r\, dr\, d\vartheta$$

$$= \sqrt{\pi} \int_0^\infty r^{2s+2} w_1(r)\, dr,$$

while all other distributed moments $c_{s,k,m}$ are zero. By Theorem 1 we obtain the Markov transform:

$$\widehat{\mu}(\zeta, e^{i\vartheta}) = \sum_{s=0}^\infty \frac{1}{\zeta^{2s+1}} \int_0^\infty r^{2s+1} w_0(r)\, dr + \sum_{s=0}^\infty \frac{\cos\vartheta}{\zeta^{2s+2}} \int_0^\infty r^{2s+2} w_1(r)\, dr.$$

So $f_{2s}(e^{i\vartheta}) = \int_0^\infty r^{2s+1} w_0(r)\, dr$ and $f_{2s+1}(e^{i\vartheta}) = \cos\vartheta \int_0^\infty r^{2s+2} w_1(r)\, dr$.

Extend the function w_0 to an odd function w_0^{odd} on $\mathbb{R}\backslash\{0\}$, so define $w_0^{\text{odd}}(-r) := -w_0(r)$ for $r > 0$, and extend w_1 to an even function w_1^{ev}, so $w_1^{\text{ev}}(-r) = w_1(r)$ for $r > 0$. Define a function $G_\vartheta \colon \mathbb{R} \to \mathbb{R}$ by

$$G_\vartheta(r) := r \cdot [w_0^{\text{odd}}(r) + w_1^{\text{ev}}(r) \cos \vartheta].$$

Note that $G_\vartheta(r) \geqslant 0$ for all $r \geqslant 0$ by (40). Moreover $G_\vartheta(-r) = r w_0(r) - r w_1(r) \cos \vartheta \geqslant 0$ for $r > 0$ again by (40). Hence $G_\vartheta(r) \geqslant 0$ for all $r \in \mathbb{R}$. A straightforward calculation shows that

$$f_l(\mathrm{e}^{\mathrm{i}\vartheta}) = \frac{1}{2} \int_{-\infty}^{\infty} r^l \ G_\vartheta(r) r \, \mathrm{d}r$$

for all $l \in \mathbb{N}_0$. This shows that $(f_l(\mathrm{e}^{\mathrm{i}\vartheta}))_{l \in \mathbb{N}_0}$ is a positive definite sequence. If the sequence is not strictly positive definite then there exists a polynomial $p(r) \neq 0$ such that

$$\frac{1}{2} \int_{-\infty}^{\infty} (p(r))^2 G_\vartheta(r) r \, \mathrm{d}r = 0.$$

Since $G_\vartheta(r)$ is continuous on $\mathbb{R}\backslash\{0\}$ this implies that $G_\vartheta(r) = 0$ for all $r \neq 0$. Then $0 = G_\vartheta(r) + G_\vartheta(-r) = 2r w_0(r)$ for all $r > 0$, a contradiction to our assumption $w_0 \neq 0$. $\quad\square$

In Theorem 4 we have seen that the Markov transform $\widehat{\mu}(\zeta, \theta)$ of a rotation invariant measure has the property that $\zeta \longmapsto \widehat{\mu}(\zeta, \theta)$ possesses an analytic continuation to the upper half plane. Next we show that the same is true for Hankel-positive measures.

Theorem 18. *Let μ be a finite measure on $\mathbb{R}^d$ with support in $\overline{B_R}$. If μ is Hankel-positive then for each $\theta \in \mathbb{S}^{d-1}$ the function $\zeta \longmapsto \widehat{\mu}(\zeta, \theta)$ possesses an analytic continuation to the upper half plane such that*

$$\operatorname{Im} \widehat{\mu}(\zeta, \theta) \leqslant 0 \quad \text{for all } \operatorname{Im} \zeta > 0, \ \ \theta \in \mathbb{S}^{d-1}.$$

Proof. Suppose that the sequence $(f_l(\theta))_{l=0,1,\ldots}$ is strictly positive definite. By the solution of the Hamburger moment problem ([18, p. 65]) there exists a finite non-negative measure σ_θ on $\mathbb{R}$ such that $\widehat{\mu}(\zeta, \theta) = \int (1/(\zeta - t)) \, \mathrm{d}\sigma_\theta(t)$. Hence $\zeta \longmapsto \widehat{\mu}(\zeta, \theta)$ extends to the upper half plane for ζ and the condition $\operatorname{Im} \widehat{\mu}(\zeta, \theta) \leqslant 0$ for all $\operatorname{Im} \zeta > 0$ and $\theta \in \mathbb{S}^{d-1}$ follows from this integral representation. $\quad\square$

Note that the measure σ_θ in the last proof has the property that its support set is infinite since $(f_l(\theta))_{l=0,1,\ldots}$ is *strictly* positive definite. If we know that $\operatorname{Im} \widehat{\mu}(\zeta, \theta) \leqslant 0$ for all $\theta \in \mathbb{S}^{d-1}$ and for all $\operatorname{Im} \zeta > 0$ then the function g_θ defined by $g_\theta(\zeta) := \widehat{\mu}(\zeta, \theta)$ is in the Nevanlinna class (see [1]) and the coefficients of the Laurent expansion are exactly the numbers $f_l(\theta)$. Hence we can conclude that the sequence $(f_l(\theta))_l$ is positive definite for each $\theta \in \mathbb{S}^{d-1}$. Note that Hankel positivity means that the sequence $(f_l(\theta))_l$ is strictly positive definite for each $\theta \in \mathbb{S}^{d-1}$.

Let us remark that in [16] we have introduced a different method for approximating a large class of signed measures, the so-called *pseudo-positive measures*, and we have provided a multivariate generalization of the Gauss–Jacobi quadrature formula.

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Further Reading

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