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Uniform convergent monotone iterates for semilinear singularly perturbed parabolic problems

Igor Boglaev

Institute of Fundamental Sciences, Massey University, Private Bag 11-222, Palmerston North, New Zealand

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ABSTRACT

This paper deals with discrete monotone iterative methods for solving semilinear singularly perturbed parabolic problems. Monotone sequences, based on the accelerated monotone iterative method, are constructed for a nonlinear difference scheme which approximates the semilinear parabolic problem. This monotone convergence leads to the existence-uniqueness theorem. An analysis of uniform convergence of the monotone iterative method to the solutions of the nonlinear difference scheme and continuous problem is given. Numerical experiments are presented.

1. Introduction

We are interested in monotone iterative methods for solving semilinear singularly perturbed parabolic problems in the form

$$\begin{aligned} &u_t - Lu + f(x,y,t,u) = 0, \quad (x,y,t) \in \omega \times (0,T], \\ &u(x,y,t) = g(x,y,t), \quad (x,y,t) \in \partial\omega \times (0,T], \\ &u(x,y,0) = \psi(x,y), \quad (x,y) \in \overline{\omega}, \end{aligned} \tag{1}$$

where ω is a connected bounded domain in $\mathbb{R}^2$ with boundary $\partial\omega$. The differential operator L is given by

$$Lu = \varepsilon(u_{xx} + u_{yy}) + b_1(x,y,t)u_x + b_2(x,y,t)u_y,$$

where ε is a small positive parameter, the functions b_1, b_2, f, g and ψ are smooth in their respective domains, and f satisfies the constraint

$$f_u \geq 0, \qquad (x,y,t,u) \in \overline{\omega} \times [0,T] \times (-\infty,\infty), \qquad (f_u \equiv \partial f/\partial u). \tag{2}$$

This assumption can always be obtained by a change of variables. Indeed, introduce $z(x,y,t) = \exp(-\lambda t)u(x,y,t)$, where λ is a constant. Now, $z(x,y,t)$ satisfies (1) with $\varphi = \lambda z + \exp(-\lambda t) f(x,y,t,\exp(\lambda t)z)$, instead of f, and we have $\varphi_z = \lambda + f_u$. Thus, if $\lambda \geq -\min f_u$, where a minimum is taken over the domain from (2), we conclude $\varphi_z \geq 0$.

For $\varepsilon \ll 1$, the problem is singularly perturbed and characterized by boundary layers (regions with rapid change of solutions) near boundary $\partial\omega$.

In the study of numerical methods for nonlinear singularly perturbed problems, the two major points to be developed are: (i) constructing robust difference schemes (this means that unlike classical schemes, the error does not increase to infinity, but rather remains bounded, as the small parameter approaches zero); (ii) obtaining reliable and efficient computing algorithms for solving nonlinear discrete problems.

E-mail address: I.Boglaev@massey.ac.nz.

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For solving these nonlinear discrete systems, the iterative approach presented in this paper is based on the method of upper and lower solutions and associated monotone iterates. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. The above monotone iterative method is well known and has been widely used for continuous and discrete elliptic and parabolic boundary value problems. Most of the publications on this topic involve monotone iterative schemes whose rate of convergence is of linear rate (cf. [1–5]). Some accelerated monotone iterative schemes for solving discrete elliptic boundary value problems are given in [6,7]. An advantage of this accelerated approach is that it leads to sequences which converge either quadratically or nearly quadratically. In [8], an accelerated monotone iterative method for solving discrete parabolic boundary value problems is presented. In the recent paper [9], a combination of the accelerated monotone iterative method from [8] with monotone Picard iterates is constructed. In [8,9], the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were omitted.

In [2], we investigate uniform convergence properties of the monotone iterative methods from [10,11] applied to solving the semilinear problems (1) of reaction–diffusion and convection–reaction–diffusion types. These monotone methods possess linear convergence rate. In this paper, we extend the accelerated monotone iterative method from [8] to the case when on each time level nonlinear difference schemes are solved inexactly, and investigate uniform convergence properties of the proposed monotone iterative method.

The structure of the paper as follows. In Section 2, we introduce a nonlinear difference scheme for the numerical solution of (1). The monotone iterative method is presented in Section 3. Section 4 deals with existence and uniqueness of the solution to the nonlinear difference scheme. An analysis of uniform convergence of the proposed monotone iterative method to the solution of the nonlinear difference scheme is given in Section 5. In Section 6, we investigate uniform convergence of the monotone iterative method applied to the exact solution of (1) for the reaction–diffusion and convection–reaction–diffusion problems. The final Section 7 presents results of numerical experiments where iteration counts are compared between the proposed monotone iterative method and the monotone iterative methods from [10,11], whose rate of convergence is linear.

2. The nonlinear difference scheme

On $\overline{\omega}$, we introduce a mesh $\overline{\omega}^h = \omega^h \cup \partial\omega^h$, where ω^h is a set of interior mesh points $\omega^h \subset \omega$ and $\partial\omega^h$ is a set of boundary mesh points $\partial\omega^h \subset \partial\omega$. On $[0, T]$, a mesh $\overline{\omega}^\tau$ is chosen in the form

$$\overline{\omega}^\tau = \{t_0 = 0 < t_1 < \cdots < t_{N_\tau - 1} < t_{N_\tau} = T\}, \qquad \tau_k = t_k - t_{k-1}.$$

For solving (1), consider the nonlinear two-level implicit difference scheme in the canonical form [12]

$$\mathcal{L}U(p, t_k) + f(p, t_k, U) - \tau_k^{-1} U(p, t_{k-1}) = 0, \quad (p, t_k) \in \omega^h \times (\overline{\omega}^\tau \setminus \{0\}), \tag{3}$$

with the boundary and initial conditions

$$U(p, t_k) = g(p, t_k), \quad (p, t_k) \in \partial\omega^h \times (\overline{\omega}^\tau \setminus \{0\}),$$
$$U(p, 0) = \psi(p), \quad p \in \overline{\omega}^h.$$

The difference operator $\mathcal{L}$ is defined by

$$\mathcal{L}U(p, t_k) = \mathcal{L}^h U(p, t_k) + \tau_k^{-1} U(p, t_k),$$
$$\mathcal{L}^h U(p, t_k) \equiv d(p, t_k) U(p, t_k) - \sum_{p' \in \sigma'(p)} a(p', t_k) U(p', t_k),$$

where $\sigma'(p) = \sigma(p) \setminus \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \omega^h$. We make the following assumptions on the coefficients of the difference operator $\mathcal{L}^h$:

$$d(p, t_k) > 0, \qquad a(p', t_k) \geq 0, \quad p' \in \sigma'(p),$$
$$d(p, t_k) - \sum_{p' \in \sigma'(p)} a(p', t_k) \geq 0, \quad (p, t_k) \in \omega^h \times (\overline{\omega}^\tau \setminus \{0\}). \tag{4}$$

We also assume that the mesh $\overline{\omega}^h$ is connected. It means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1, p_2, \ldots, p_s\}$ such that

$$p_1 \in \sigma'(\tilde{p}), \qquad p_2 \in \sigma'(p_1), \ldots, \ p_s \in \sigma'(p_{s-1}), \qquad \hat{p} \in \sigma'(p_s). \tag{5}$$

On each time level t_k, $k \geq 1$, introduce the linear problems

$$(\mathcal{L} + c(p, t_k)) W(p, t_k) = \Phi(p, t_k), \quad p \in \omega^h,$$
$$c(p, t_k) \geq 0, \qquad W(p, t_k) = g(p, t_k), \quad p \in \partial\omega^h. \tag{6}$$

We now formulate the maximum principle for the difference operator $\mathcal{L} + c$, and give an estimate to the solution of (6).

Lemma 1. *Let the coefficients of the difference operator $\mathcal{L}^h$ satisfy (4) and the mesh $\overline{\omega}^h$ be connected.*

(i) *If a mesh function $W(p, t_k)$ satisfies the conditions*

$$(\mathcal{L} + c(p, t_k))W(p, t_k) \geq 0 \ (\leq 0), \quad p \in \omega^h,$$
$$W(p, t_k) \geq 0 \ (\leq 0), \quad p \in \partial\omega^h,$$

then $W(p, t_k) \geq 0$ (≤ 0) in $\overline{\omega}^h$.

(ii) *The following estimates to the solutions to (6) hold true*

$$\|W(t_k)\|_{\overline{\omega}^h} \leq \max\left\{ \|g(t_k)\|_{\partial\omega^h}, \max_{p\in\omega^h} \frac{|\Phi(p, t_k)|}{c(p, t_k) + \tau_k^{-1}} \right\}, \tag{7}$$

where

$$\|W(t_k)\|_{\overline{\omega}^h} = \max_{p\in\overline{\omega}^h} |W(p, t_k)|, \qquad \|g(t_k)\|_{\partial\omega^h} = \max_{p\in\partial\omega^h} |g(p, t_k)|.$$

The proof of the lemma can be found in [12].

Remark 1. A difference scheme which satisfies the maximum principle from Lemma 1 is said to be monotone. The monotonicity condition guarantees that the systems of algebraic equations based on such methods are well-posed.

3. The monotone iterative method

We say that on time level t_k, $k \geq 1$, $V_1(p, t_k)$ is an upper solution with a given function $V(p, t_{k-1})$ if it satisfies

$$\mathcal{L}V_1(p, t_k) + f(p, t_k, V_1) - \tau_k^{-1}V(p, t_{k-1}) \geq 0, \quad p \in \omega^h,$$
$$V_1(p, t_k) \geq g(p, t_k), \quad p \in \partial\omega^h.$$

Similarly, $V_{-1}(p, t_k)$ is called a lower solution on a time level t_k, $k \geq 1$ with a given function $V(p, t_{k-1})$, if it satisfies the reversed inequalities.

We now construct an iterative method for solving (3) in the following way. On each time level t_k, $k \geq 1$, initial upper and lower solutions $V_\alpha^{(0)}(p, t_k)$ ($\alpha = 1$ and $\alpha = -1$ correspond to, respectively, the upper and lower cases) are calculated by solving the linear problems

$$\begin{aligned}
&\mathcal{L}W_\alpha^{(0)}(p, t_k) = \alpha|\mathcal{R}(p, t_k, S)|, \quad p \in \omega^h, \qquad W_\alpha^{(0)}(p, t_k) = 0, \quad p \in \partial\omega^h,\\
&\mathcal{R}(p, t_k, S) \equiv \mathcal{L}S(p, t_k) + f(p, t_k, S) - \tau_k^{-1}V_\alpha(p, t_{k-1}),\\
&V_\alpha^{(0)}(p, t_k) = S(p, t_k) + W_\alpha^{(0)}(p, t_k), \quad p \in \overline{\omega}^h, \ \alpha = 1, -1,
\end{aligned} \tag{8}$$

where $S(p, t_k)$ is defined on $\overline{\omega}^h$ and satisfies the boundary condition $S(p, t_k) = g(p, t_k)$ on $\partial\omega^h$. For $n \geq 1$, we calculate upper and lower solutions $\{V_\alpha^{(n)}(p, t_k)\}$, $\alpha = 1, -1$, by using the recurrence formulae

$$\begin{aligned}
&(\mathcal{L} + c^{(n-1)}(p, t_k))Z_\alpha^{(n)}(p, t_k) = -\mathcal{R}(p, t_k, V_\alpha^{(n-1)}), \quad p \in \omega^h,\\
&\mathcal{R}(p, t_k, V_\alpha^{(n-1)}) \equiv \mathcal{L}V_\alpha^{(n-1)}(p, t_k) + f(p, t_k, V_\alpha^{(n-1)}) - \tau_k^{-1}V_\alpha(p, t_{k-1}),\\
&Z_\alpha^{(n)}(p, t_k) = 0, \quad p \in \partial\omega^h,\\
&V_\alpha^{(n)}(p, t_k) = V_\alpha^{(n-1)}(p, t_k) + Z_\alpha^{(n)}(p, t_k), \quad p \in \overline{\omega}^h,\\
&V_\alpha(p, t_k) = V_\alpha^{(n_k)}(p, t_k), \quad p \in \overline{\omega}^h,\\
&V_\alpha(p, 0) = \psi(p), \quad p \in \overline{\omega}^h,
\end{aligned} \tag{9}$$

where $\mathcal{R}(p, t_k, V_\alpha^{(n)})$ is the residual of the difference scheme (3) on $V_\alpha^{(n)}$ for upper $\alpha = 1$ and lower $\alpha = -1$ sequences, respectively, and n_k is a number of iterative steps on time-level t_k. The mesh function $c^{(n-1)}(p, t_k)$ is given by

$$c^{(n-1)}(p, t_k) = \max_V\{f_u(p, t_k, V), \ V_{-1}^{(n-1)}(p, t_k) \leq V \leq V_1^{(n-1)}(p, t_k)\}, \tag{10}$$

where below in Theorem 1, we prove that $V_{-1}^{(n-1)}(p, t_k) \leq V_1^{(n-1)}(p, t_k)$, $p \in \overline{\omega}^h$.

Remark 2. The accelerated monotone iterative method from [8] is based on (9), (10), where the residual in (9) is in the form

$$\mathcal{R}(p, t_k, V_\alpha^{(n-1)}) = \mathcal{L}V_\alpha^{(n-1)}(p, t_k) + f(p, t_k, V_\alpha^{(n-1)}) - \tau_k^{-1}V^*(p, t_{k-1}),$$

where instead of $V_\alpha(p, t_{k-1})$ in (9), the exact solution $V^*(p, t_{k-1})$ of the difference scheme (3) is in use.

We assume that $V_1^{(n)}(p,t_k) \geq V_{-1}^{(n)}(p,t_k)$, $p \in \overline{\omega}^h$, and define the sector

$$\langle V_{-1}^{(n)}(t_k), V_1^{(n)}(t_k)\rangle = \{V_{-1}^{(n)}(p,t_k) \leq V(p,t_k) \leq V_1^{(n)}(p,t_k),\ p \in \overline{\omega}^h\}.$$

Introduce the notation

$$F(p,t_k,V) = c^{(n)}(p,t_k)V(p,t_k) - f(p,t_k,V), \tag{11}$$

and give a monotone property of F.

Lemma 2. *If $U, V \in \langle V_{-1}^{(n)}(t_k), V_1^{(n)}(t_k)\rangle$ such that $U(p,t_k) \geq V(p,t_k)$, and (10) holds, then*

$$F(p,t_k,U) \geq F(p,t_k,V), \quad p \in \overline{\omega}^h. \tag{12}$$

Proof. From (11), we have

$$F(p,t_k,U) - F(p,t_k,V) = c^{(n)}(p,t_k)[U(p,t_k) - V(p,t_k)] - [f(p,t_k,U) - f_1(p,t_k,V)].$$

By the mean-value theorem,

$$f(p,t_k,U) - f(p,t_k,V) = f_u(p,t_k,E)[U(p,t_k) - V(p,t_k)],$$

where $E \in \langle V(t_k), U(t_k)\rangle$. Thus, from here, (10) and the assumptions of the lemma, we conclude (12). □

In the following theorem we prove the monotone property of the iterative method (8)–(10).

Theorem 1. *Assume that the coefficients of the difference operator $\mathcal{L}$ in (3) satisfy (4) and the computational mesh $\overline{\omega}^h$ is connected (5). The sequences $\{V_1^{(n)}\}$, $\{V_{-1}^{(n)}\}$, generated by (8)–(10) converge monotonically*

$$V_{-1}^{(n-1)}(p,t_k) \leq V_{-1}^{(n)}(p,t_k) \leq V_1^{(n)}(p,t_k) \leq V_1^{(n-1)}(p,t_k), \quad p \in \overline{\omega}^h, \tag{13}$$

where $k \geq 1$ and $n \geq 1$.

Proof. We show that $V_1^{(0)}(p,t_k)$ defined by (8) is an upper solution. From the maximum principle in Lemma 1 and mean-value theorem, it follows that $W_1^{(0)}(p,t_k) \geq 0$ on $\overline{\omega}^h$. Now using the difference equation for $W_1^{(0)}(p,t_k)$, we have

$$\begin{aligned}&\mathcal{L}(S(p,t_k) + W_1^{(0)}(p,t_k)) + f(p,t_k,S + W_1^{(0)}) - \tau_k^{-1}V_1(p,t_{k-1})\\ &\quad = \mathcal{R}(p,t_k,S) + |\mathcal{R}(p,t_k,S)| + f_u^{(0)}(p,t_k,E)W_1^{(0)}(p,t_k),\end{aligned}$$

where $S \leq E \leq S + W_1^{(0)}$. Since $f_u \geq 0$ and $W_1^{(0)}$ is nonnegative, we conclude that $V_1^{(0)}(p,t_k) = S(p,t_k) + W_1^{(0)}(p,t_k)$ is an upper solution. Similarly, we can prove that $V_{-1}^{(0)}(p,t_k) = S(p,t_k) + W_{-1}^{(0)}(p,t_k)$ is a lower solution, where $W_{-1}^{(0)}$ is nonpositive. Thus, $V_\alpha^{(0)}(p,t_k)$ are upper ($\alpha = 1$) and lower ($\alpha = -1$) solutions of (3) and satisfy (13).

Since $V_1^{(0)}$ is an upper solution, then from (9) we conclude that

$$(\mathcal{L} + c^{(0)}(p,t_1))Z_1^{(1)}(p,t_1) \leq 0, \quad p \in \omega^h, \qquad Z_1^{(1)}(p,t_1) = 0, \quad p \in \partial\omega^h,$$

where $t_1 = \tau_1$. From Lemma 1, it follows that

$$Z_1^{(1)}(p,t_1) \leq 0, \quad p \in \overline{\omega}^h. \tag{14}$$

Similarly, for a lower solution $V_{-1}^{(0)}$, we conclude that

$$Z_{-1}^{(1)}(p,t_1) \geq 0, \quad p \in \overline{\omega}^h. \tag{15}$$

We now prove that

$$V_{-1}^{(1)}(p,t_1) \leq V_1^{(1)}(p,t_1), \quad p \in \overline{\omega}^h. \tag{16}$$

In the notation (11), by (9),

$$\begin{aligned}&(\mathcal{L} + c^{(0)}(p,t_1))V_\alpha^{(1)}(p,t_1) = F(p,t_1,V_\alpha^{(0)}) + \tau_1^{-1}\psi(p), \quad p \in \omega^h,\\ &V_\alpha^{(1)}(p,t_1) = g(p,t_1), \quad p \in \partial\omega^h,\ \alpha = 1,-1.\end{aligned}$$

Letting $W^{(n)} = V_1^{(n)} - V_{-1}^{(n)}$, $n \geq 0$, we have

$$\begin{aligned}&(\mathcal{L} + c^{(0)}(p,t_1))W^{(1)}(p,t_1) = F(p,t_1,V_1^{(0)}) - F(p,t_1,V_{-1}^{(0)}), \quad p \in \omega^h,\\ &W^{(1)}(p,t_1) = 0, \quad p \in \partial\omega^h.\end{aligned}$$

Since $V_1^{(0)}(p,t_1) \ge V_{-1}^{(0)}(p,t_1)$, by Lemma 2, we conclude that the right-hand side in the difference equation is nonnegative. The positivity property in Lemma 1 implies $W^{(1)}(p,t_1) \ge 0$, and this leads to (16).

We now prove that $V_1^{(1)}(p,t_1)$ and $V_{-1}^{(1)}(p,t_1)$ are upper and lower solutions, respectively. Using the mean-value theorem, from (9) we obtain

$$\mathcal{R}(p,t_1,V_1^{(1)}) = -(c^{(0)}(p,t_1) - f_u(p,t_1,E))Z_1^{(1)}(p,t_1), \tag{17}$$

where $E \in \langle V_1^{(1)}(t_1), V_1^{(0)}(t_1)\rangle$. From here, (10) and (14)–(16), it follows that

$$c^{(0)}(p,t_1) \ge f_u(p,t_1,E), \quad p \in \omega^h.$$

From here and (14), we conclude that

$$\mathcal{R}(p,t_1,V_1^{(1)}) \ge 0, \quad p \in \omega^h, \qquad V_1^{(1)}(p,t_1) = g(p,t_1), \quad p \in \partial\omega^h.$$

Thus, $V_1^{(1)}(p,t_1)$ is an upper solution. Similarly, we can prove that $V_{-1}^{(1)}(p,t_1)$ is a lower solution, that is,

$$\mathcal{R}(p,t_1,V_{-1}^{(1)}) \le 0, \quad p \in \omega^h, \qquad V_{-1}^{(1)}(p,t_1) = g(p,t_1), \quad p \in \partial\omega^h.$$

By induction on n, we can prove that $\{V_1^{(n)}(p,t_1)\}$ is a monotonically decreasing sequence of upper solutions and $\{V_{-1}^{(n)}(p,t_1)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (13) for t_1.

In the notation (11), by (9) with t_2,

$$(\mathcal{L} + c^{(0)}(p,t_2))V_\alpha^{(1)}(p,t_2) = F(p,t_2,V_\alpha^{(0)}) + \tau_2^{-1}V_\alpha^{(n_1)}(p,t_1), \quad p \in \omega^h,$$
$$V_\alpha^{(1)}(p,t_2) = g(p,t_2), \quad p \in \partial\omega^h, \ \alpha = 1, -1.$$

From here, we conclude that $W^{(1)}(p,t_2) = V_1^{(1)}(p,t_2) - V_{-1}^{(1)}(p,t_2)$ satisfies the difference problem

$$(\mathcal{L} + c^{(0)}(p,t_2))W^{(1)}(p,t_2) = F(p,t_2,V_1^{(0)}) - F(p,t_2,V_{-1}^{(0)}) + \tau_2^{-1}[V_1^{(n_1)}(p,t_1) - V_{-1}^{(n_1)}(p,t_1)],$$
$$p \in \omega^h, \qquad W^{(1)}(p,t_2) = 0, \quad p \in \partial\omega^h.$$

Since $V_1^{(0)}(p,t_2) \ge V_{-1}^{(0)}(p,t_2)$ and taking into account (13) with $t = t_1$ and $n = n_1$, by Lemma 2, we conclude that the right-hand side in the difference equation is nonnegative. The positivity property in Lemma 1 implies $W^{(1)}(p,t_2) \ge 0$, and this leads to

$$V_{-1}^{(1)}(p,t_2) \le V_1^{(1)}(p,t_2), \quad p \in \overline{\omega}^h.$$

The proof that $V_1^{(1)}(p,t_2)$ and $V_{-1}^{(1)}(p,t_2)$ are, respectively, upper and lower solutions is similar to the proof of this result on the time level t_1. By induction on n, we can prove that $\{V_1^{(n)}(p,t_2)\}$ is monotonically decreasing sequence of upper solutions and $\{V_{-1}^{(n)}(p,t_2)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (13) for t_2.

By induction on k, $k \ge 1$, we can prove that $\{V_1^{(n)}(p,t_k)\}$ is a monotonically decreasing sequence of upper solutions and $\{V_{-1}^{(n)}(p,t_k)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (13). Thus, we prove the theorem. □

4. Existence and uniqueness of a solution to the nonlinear difference scheme

Applying Theorem 1, we investigate existence and uniqueness of a solution to the nonlinear difference scheme (3).

Theorem 2. *Let the assumptions in Theorem 1 hold. Then the nonlinear difference scheme (3) has a unique solution.*

Proof. From (13), it follows that $\lim V_1^{(n)}(p,t_1) = V_1(p,t_1)$, $p \in \overline{\omega}^h$ as $n \to \infty$ exists, and

$$V_1(p,t_1) \le V_1^{(n)}(p,t_1), \qquad \lim_{n\to\infty} Z_1^{(n)}(p,t_1) = 0, \quad p \in \overline{\omega}^h. \tag{18}$$

Similar to (17), we can prove that

$$\mathcal{R}(p,t_1,V_1^{(n)}) = -(c^{(n-1)}(p,t_1) - f_u(p,t_1,E))Z_1^{(n)}(p,t_1), \quad n \ge 1, \tag{19}$$

where $E \in \langle V_1^{(n)}(t_1), V_1^{(n-1)}(t_1)\rangle$. From here, (18) and taking into account that the sequence $\{c^{(n)}(p,t_1)\}$ is bounded with respect to n, we conclude that $V_1(p,t_1)$ solves (3) at t_1. Using a similar argument, we can prove that the following limit

$$\lim_{n\to\infty} V_1^{(n)}(p,t_2) = V_1(p,t_2), \quad p \in \overline{\omega}^h,$$

exists and solves (3) at t_2, where according to Theorem 1, $\{V_1^{(n)}(p, t_2)\}$ is a sequence of upper solutions with respect to $V_1(p, t_1)$.

By induction on k, $k \geq 1$, we can prove that

$$V_1(p, t_k) = \lim_{n\to\infty} V_1^{(n)}(p, t_k), \quad p \in \overline{\omega}^h,\ k \geq 1,$$

is a solution of the nonlinear difference scheme (3). Similarly, we can prove that the mesh function $V_{-1}(p, t_k)$ defined by

$$V_{-1}(p, t_k) = \lim_{n\to\infty} V_{-1}^{(n)}(p, t_k), \quad p \in \overline{\omega}^h,\ k \geq 1,$$

is a solution of the nonlinear difference scheme (3).

We now show that

$$V_1(p, t_k) = V_{-1}(p, t_k), \quad p \in \overline{\omega}^h,\ k \geq 1,$$

where $V_1(p, t_k)$ and $V_{-1}(p, t_k)$ are solutions to the difference scheme (3), which are defined above. Letting $W(p, t_k) = V_1(p, t_k) - V_{-1}(p, t_k)$, from (3) we have

$$\mathcal{L}W(p, t_1) + f(p, t_1, V_1) - f(p, t_1, V_{-1}) = 0, \quad p \in \omega^h,$$
$$W(p, t_1) = 0, \quad p \in \partial\omega^h.$$

Using the mean-value theorem, we obtain

$$(\mathcal{L} + f_u(p, t_1, E))W(p, t_1) = 0, \quad p \in \omega^h, \qquad W(p, t_1) = 0, \quad p \in \partial\omega^h,$$

where $E \in \langle V_{-1}(t_1), V_1(t_1)\rangle$. Since $f_u(p, t_1, E) \geq 0$, by Lemma 1, we conclude that $W(p, t_1) = 0$, $p \in \overline{\omega}^h$. By induction on k, $k \geq 1$, we can prove that $W(p, t_k) = 0$, $p \in \overline{\omega}^h$, $k \geq 1$, and prove the theorem. □

5. Convergence analysis of the monotone iterative method

In this section, we investigate convergence properties of the monotone iterative method (8)–(10).

5.1. Convergence to the solution of the nonlinear difference scheme

5.1.1. Stopping criterion based on residual

We now choose the stopping criterion of the iterative method (8)–(10) in the form

$$\|\mathcal{R}(t_k, V_\alpha^{(n)})\|_{\omega^h} \leq \delta, \quad \alpha = 1, -1, \tag{20}$$

where δ is a prescribed accuracy, and set up $V_\alpha(p, t_k) = V_\alpha^{(n_k)}(p, t_k)$, $p \in \overline{\omega}^h$, such that n_k is minimal subject to (20).

We prove the following convergence result for the iterative method (8)–(10).

Theorem 3. *Assume that the coefficients of the difference operator $\mathcal{L}$ in (3) satisfy (4), the mesh $\overline{\omega}^h$ is connected (5). The sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, generated by (8)–(10), (20), converge uniformly in the perturbation parameter ε:*

$$\max_{t_k\in\overline{\omega}^\tau} \|V_\alpha(t_k) - V^*(t_k)\|_{\overline{\omega}^h} \leq T\delta, \quad \alpha = 1, -1, \tag{21}$$

where $V^*(p, t_k)$ is the unique solution to (3). Furthermore, on each time level the sequences converge monotonically (13).

Proof. The monotone convergence of the sequence $\{V_\alpha^{(n)}(p, t_k)\}$, $\alpha = 1, -1$, follows from Theorem 1. The existence and uniqueness of the solution to (3) have been proved in Theorem 2.

The difference problem for $V_\alpha(p, t_k) = V_\alpha^{(n_k)}(p, t_k)$, $k \geq 1$, $\alpha = 1, -1$, can be represented in the form

$$\mathcal{L}V_\alpha(p, t_k) + f(p, t_k, V_\alpha) - \tau_k^{-1}V_\alpha(p, t_{k-1}) = \mathcal{R}(p, t_k, V_\alpha), \quad p \in \omega^h,$$
$$V_\alpha(p, t_k) = g(p, t_k), \quad p \in \partial\omega^h,\ \alpha = 1, -1.$$

From here, (3) and using the mean-value theorem, we get the difference problem for $W_\alpha(p, t_k) = V_\alpha(p, t_k) - V^*(p, t_k)$, $\alpha = 1, -1$,

$$(\mathcal{L} + f_u(p, t_k, E_\alpha))W_\alpha(p, t_k) = \mathcal{R}(p, t_k, V_\alpha) + \tau_k^{-1}W_\alpha(p, t_{k-1}), \quad p \in \omega^h, \tag{22}$$
$$W_\alpha(p, t_k) = 0, \quad p \in \partial\omega^h,\ \alpha = 1, -1,$$

where $E_1 \in \langle V^*(t_k), V_1(t_k)\rangle$ for upper solutions and $E_{-1} \in \langle V_{-1}(t_k), V^*(t_k)\rangle$ for lower solutions. From here, (7) and taking into account that according to Theorem 1 the stopping criterion (20) can always be satisfied, we have

$$\|W_\alpha(t_k)\|_{\overline{\omega}^h} \leq \delta\tau_k + \|W_\alpha(t_{k-1})\|_{\overline{\omega}^h}, \quad \alpha = 1, -1.$$

Taking into account that $\|W_\alpha(t_0)\|_{\overline{\omega}^h}=0$, $\alpha = 1, -1$, by induction on k, we conclude that

$$\|W_\alpha(t_k)\|_{\overline{\omega}^h} \le \delta \sum_{l=1}^{k} \tau_l \le T\delta, \quad k \ge 1, \ \alpha = 1, -1,$$

and prove the theorem. □

5.1.2. Stopping criterion based on upper and lower solutions

We now modify the stopping criterion (20) as follows

$$\|V_1^{(n)}(t_k) - V_{-1}^{(n)}(t_k)\|_{\overline{\omega}^h} \le \sigma, \tag{23}$$

where σ is a prescribed accuracy, and set up $V_\alpha(p, t_k) = V_\alpha^{(n_k)}(p, t_k)$, $p \in \overline{\omega}^h$, such that n_k is minimal subject to (23). In view of the monotone property (13) and the uniqueness of the solution $V^*(p, t_k)$ of the nonlinear difference scheme (3), it follows that

$$V_{-1}^{(n)}(p, t_k) \le V^*(p, t_k) \le V_1^{(n)}(p, t_k), \quad p \in \overline{\omega}^h.$$

This implies that with the stopping criterion (23), we have

$$\max_{t_k \in \overline{\omega}^\tau} \|V_\alpha(t_k) - V^*(t_k)\|_{\overline{\omega}^h} \le \sigma, \quad \alpha = 1, -1.$$

5.1.3. Stopping criterion with fixed number of iterates on each time level

Without loss of generality, we assume that the boundary condition $g = 0$ in (1). This assumption can always be obtained via a change of variables. From (8) with $S = 0$, it follows that $V_\alpha^{(0)}$, $\alpha = 1, -1$, are solutions of the linear problems

$$\begin{aligned} &\mathcal{L}V_\alpha^{(0)}(p, t_k) = \alpha|f(p, t_k, 0) - \tau_k^{-1}V_\alpha(p, t_{k-1})|, \quad p \in \omega^h, \\ &V_\alpha^{(0)}(p, t_k) = 0, \quad p \in \partial\omega^h, \ \alpha = 1, -1. \end{aligned} \tag{24}$$

For the iterative method (9), (10), (24), we now choose the stopping criterion with the fixed number of iterative steps n_* on each time level (that is, n_* is independent of k), and assume that time step τ_k satisfies the inequality

$$\begin{aligned} &\tau_k < \frac{1}{c_k}, \quad k \ge 1, \\ &c_k = \max_{p \in \overline{\omega}^h} \left[\max_V \{f_u(p, t_k, V), V_{-1}^{(0)}(p, t_k) \le V \le V_1^{(0)}(p, t_k)\} \right]. \end{aligned} \tag{25}$$

where constants c_k, $k \ge 1$ are independent of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$.

Lemma 3. *Let the assumptions in Theorem 3 and (25) be satisfied. Then for the sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, generated by (9), (10), (24), the following estimates hold:*

$$\|Z_\alpha^{(n)}(t_k)\|_{\overline{\omega}^h} \le q_k^{n-1}\|Z_\alpha^{(1)}(t_k)\|_{\overline{\omega}^h}, \quad q_k = \tau_k c_k < 1, \ \alpha = 1, -1. \tag{26}$$

Proof. Using (7), from (9) we have

$$\|Z_\alpha^{(n)}(t_k)\|_{\overline{\omega}^h} \le \tau_k \|\mathcal{R}(t_k, V_\alpha^{(n-1)})\|_{\omega^h}.$$

From (19), $f_u \ge 0$ and (10), we conclude that

$$\|\mathcal{R}(t_k, V_\alpha^{(n-1)})\|_{\omega^h} \le c_k \|Z_\alpha^{(n-1)}(t_k)\|_{\overline{\omega}^h}, \quad \alpha = 1, -1.$$

Thus,

$$\|Z_\alpha^{(n)}(t_k)\|_{\overline{\omega}^h} \le \tau_k c_k \|Z_\alpha^{(n-1)}(t_k)\|_{\overline{\omega}^h}, \quad \alpha = 1, -1,$$

and, by induction on n, we prove (26). □

Theorem 4. *Let the assumptions in Theorem 3 and (25) be satisfied. Then the sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, generated by (9), (10), (24) with the fixed number of iterative steps n_* on each time level, converge uniformly in the perturbation parameter ε:*

$$\max_{t_k \in \overline{\omega}^\tau} \|V_\alpha(t_k) - V^*(t_k)\|_{\overline{\omega}^h} \le Cq^{n_*-1}, \quad q = \max_{k \ge 1} q_k < 1, \ \alpha = 1, -1, \tag{27}$$

where constant C is independent of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$, q_k is defined in (26), and $V^*(p, t_k)$ is the unique solution to (3). Furthermore, on each time level the sequences converge monotonically (13).

Proof. We consider only the case of the upper sequence, since the case of the lower sequence can be proved in a similar manner.

First, we show that constants c_k, $k \geq 1$ in (25) are independent of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$. From (7) and (24) with $k = 1$, we have

$$\|V_\alpha^{(0)}(t_1)\|_{\overline{\omega}^h} \leq \tau_1 \|f(t_1, 0)\|_{\overline{\omega}^h} + \|\psi\|_{\overline{\omega}^h} = K_1, \tag{28}$$

where for sufficiently small τ_1, constant K_1 is bounded independently of ε, τ_l, $l \geq 1$ and $\overline{\omega}^h$. Thus, constant c_1 in (25) is independent of ε, τ_l, $l \geq 1$ and $\overline{\omega}^h$. From the last estimate and (13), we conclude that $V_\alpha(p, t_1)$, $\alpha = 1, -1$, are bounded independently of ε, τ_l, $l \geq 1$ and $\overline{\omega}^h$. From here, (7) and (24) with $k = 2$, it follows that

$$\|V_\alpha^{(0)}(t_2)\|_{\overline{\omega}^h} \leq \tau_2 \|f(t_2, 0)\|_{\overline{\omega}^h} + \|V_\alpha(t_1)\|_{\overline{\omega}^h} = K_2,$$

where for sufficiently small τ_2, constant K_2 is bounded independently of ε, τ_l, $l \geq 1$ and $\overline{\omega}^h$. Thus, constant c_2 in (25) is independent of ε, τ_l, $l \geq 1$ and $\overline{\omega}^h$. By induction on k, we prove the required result.

Similarly to (17), using the mean-value theorem, from (9) we obtain

$$\mathcal{R}(p, t_k, V_1^{(n)}) = -(c^{(n-1)}(p, t_k) - f_u(p, t_k, E))Z_1^{(n)}(p, t_1),$$

where $E \in \langle V_1^{(n)}(t_k), V_1^{(n-1)}(t_k)\rangle$. From here and (22), we get the difference problem for $W(p, t_k) = V_1(p, t_k) - V^*(p, t_k)$, $V_1(p, t_k) = V_1^{(n_*)}(p, t_k)$,

$$\begin{aligned}&(\mathcal{L} + f_u(p, t_k, E))W(p, t_k) = -(c^{(n_*-1)}(p, t_k) - f_u(p, t_k, H))Z_1^{(n_*)}(p, t_k) + \tau_k^{-1} W(p, t_{k-1}), \quad p \in \omega^h,\\ &W(p, t_k) = 0, \quad p \in \partial\omega^h,\end{aligned} \tag{29}$$

where $E \in \langle V^*(t_k), V_1(t_k)\rangle$ and $H \in \langle V_1^{(n)}(t_k), V_1^{(n-1)}(t_k)\rangle$. From here, (7), $f_u \geq 0$, $c^{(n_*-1)}(p, t_1) \leq c_1$ and taking into account that $W(p, t_0) = 0$, we have

$$\|W(t_1)\|_{\overline{\omega}^h} \leq \tau_1 c_1 \|Z_1^{(n_*)}(t_1)\|_{\overline{\omega}^h},$$

where c_1 is defined in (25). From here and (26), we obtain the estimate

$$\|W(t_1)\|_{\overline{\omega}^h} \leq q_1^{n_*} \|Z_1^{(1)}(t_1)\|_{\overline{\omega}^h}. \tag{30}$$

Using (24) and the mean-value theorem, estimate $Z_1^{(1)}(p, t_1)$ from (9) by (7),

$$\begin{aligned}\|Z_1^{(1)}(t_1)\|_{\overline{\omega}^h} &\leq \tau_1 \|\mathcal{L}V_1^{(0)}(t_1)\|_{\overline{\omega}^h} + c_1\tau_1 \|V_1^{(0)}(t_1)\|_{\overline{\omega}^h} + \tau_1 \|f(t_1, 0) - \tau_1^{-1}\psi\|_{\overline{\omega}^h}\\ &\leq (2\tau_1 + c_1\tau_1^2)\|f(t_1, 0) - \tau_1^{-1}\psi\|_{\overline{\omega}^h}\\ &\leq (2 + c_1\tau_1)[\tau_1\|f(t_1, 0)\|_{\overline{\omega}^h} + \|\psi\|_{\overline{\omega}^h}] \leq A_1.\end{aligned}$$

Taking into account that c_1 is independent of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$, for sufficiently small τ_1, we conclude that constant A_1 is bounded independently of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$. Thus, from here, $q_1 = c_1\tau_1$ in (26) and (30), we conclude that

$$\|W(t_1)\|_{\overline{\omega}^h} \leq B_1 \tau_1 q_1^{n_*-1}, \quad B_1 = c_1 A_1, \tag{31}$$

where constant B_1 is independent of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$.

From (29) with $k = 2$, $f_u \geq 0$, $c^{(n_*-1)}(p, t_2) \leq c_2$ and (26), by (7),

$$\|W(t_2)\|_{\overline{\omega}^h} \leq \|W(t_1)\|_{\overline{\omega}^h} + q_2^{n_*} \|Z_1^{(1)}(t_2)\|_{\overline{\omega}^h} \tag{32}$$

Similar to estimation of $Z_1^{(1)}(p, t_1)$, using (24) and the mean-value theorem, estimate $Z_1^{(1)}(p, t_2)$ from (9) by (7),

$$\|Z_1^{(1)}(t_2)\|_{\overline{\omega}^h} \leq (2 + c_2\tau_2)[\tau_2\|f(t_2, 0)\|_{\overline{\omega}^h} + \|V_1(t_1)\|_{\overline{\omega}^h}] \leq A_2.$$

From here and (28) and taking into account that c_2 is independent of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$, for sufficiently small τ_2, we conclude that constant A_2 is bounded independently of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$. Thus, from here, $q_2 = c_2\tau_2$ in (26), (31) and (32), we conclude that

$$\|W(t_2)\|_{\overline{\omega}^h} \leq B_1 \tau_1 q_1^{n_*-1} + B_2 \tau_2 q_2^{n_*-1}, \quad B_2 = c_2 A_2, \tag{33}$$

where constant B_2 is independent of ε, $\overline{\omega}^\tau$ and $\overline{\omega}^h$.

By induction on k, we can prove

$$\|W(t_k)\|_{\overline{\omega}^h} = \sum_{s=1}^{k} B_s \tau_s q_s^{n_*-1},$$

where all constants B_s are independent of ε, $\overline{\omega}^{\tau}$ and $\overline{\omega}^{h}$. Denoting

$$B = \max_{k\geq 1} B_k,$$

and taking into account that $\sum_{s=1}^{k} \tau_s \leq T$, we prove the estimate in the theorem with $C = BT$. □

5.2. Quadratic convergence rate

We modify the recurrence formulae in (9) such that

$$\mathcal{R}(p, t_k, V_{\alpha}^{(n)}) = \mathcal{L}V_{\alpha}^{(n)}(p, t_k) + f(p, t_k, V_{\alpha}^{(n)}) - \tau_k^{-1}V^{*}(p, t_{k-1}), \tag{34}$$

where instead of $V(p, t_{k-1})$ in (9), the exact solution $V^{*}(p, t_{k-1})$ of the difference scheme (3) is in use. As follows from Remark 2, the iterative method (9), (10), (34) is essentially the accelerated monotone iterative method from [8]. The accelerated monotone iterative method (9), (10), (34) converges quadratically, such that

$$\|W^{(n+1)}(t_k)\|_{\overline{\omega}^h} \leq \tau_k r_k \|W^{(n)}(t_k)\|^2_{\overline{\omega}^h},$$
$$r_k = \max_{p\in\overline{\omega}^h}\left[\max_{V}\{|f_{uu}(p, t_k, V)|, V_{-1}^{(0)}(p, t_k) \leq V \leq V_1^{(0)}(p, t_k)\}\right]$$

where $W^{(n)}(p, t_k) = V_1^{(n)}(p, t_k) - V_{-1}^{(n)}(p, t_k)$. The proof of this result can be found in [8].

6. Uniform convergence of the monotone iterative method to the solution of (1)

In this section we assume that ω is the rectangular domain

$$\omega = \omega^x \times \omega^y = \{0 < x < 1\} \times \{0 < y < 1\}.$$

On $\overline{\omega}$ introduce nonuniform mesh $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\overline{\omega}^{hx} = \{x_i,\ 0 \leq i \leq N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\},$$
$$\overline{\omega}^{hy} = \{y_j,\ 0 \leq j \leq N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\}.$$

6.1. The reaction–diffusion problem

Consider the reaction–diffusion problem in the form

$$u_t - \mu^2(u_{xx} + u_{yy}) + f(x, y, u) = 0, \quad (x, y, t) \in \omega \times (0, T], \tag{35}$$
$$u(x, y, t) = 0, \quad (x, y, t) \in \partial\omega \times (0, T],$$
$$u(x, y, 0) = \psi(x, y), \quad (x, y) \in \overline{\omega}.$$

Here we use μ^2 instead of ε as the diffusion coefficient. This is to simplify the notation, since the solution of such problems typically exhibits a boundary layer of width $\mathcal{O}(\mu|\ln\mu|)$ near the boundary. In (3), we use the classical difference scheme with the difference operator $\mathcal{L}^h$ in the form

$$\mathcal{L}^h U = -\mu^2(\mathcal{D}_x^2 + \mathcal{D}_y^2)U, \tag{36}$$
$$\mathcal{D}_x^2 U_{ij}^k = \frac{1}{\hbar_{xi}}\left[\frac{U_{i+1,j}^k - U_{ij}^k}{h_{xi}} - \frac{U_{ij}^k - U_{i-1,j}^k}{h_{x,i-1}}\right], \quad \hbar_{xi} = (h_{x,i-1} + h_{xi})/2,$$
$$\mathcal{D}_y^2 U_{ij}^k = \frac{1}{\hbar_{yj}}\left[\frac{U_{i,j+1}^k - U_{ij}^k}{h_{yj}} - \frac{U_{ij}^k - U_{i,j-1}^k}{h_{y,j-1}}\right], \quad \hbar_{yj} = (h_{y,j-1} + h_{yj})/2,$$

where $p = (x_i, y_j) \in \omega^h$ and $U_{ij}^k = U(x_i, y_j, t_k)$. The classical difference scheme satisfies the assumptions from (4).

The piecewise uniform meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are defined in the manner of [13] and are referred to as Shishkin meshes. The boundary layer thicknesses ς_x and ς_y are chosen as

$$\varsigma_x = \min\{0.25, m_1\mu\ln N_x\}, \qquad \varsigma_y = \min\{0.25, m_2\mu\ln N_y\}, \tag{37}$$

where m_1 and m_2 are positive constants. Mesh spacings $h_{x\mu}$, h_x, $h_{y\mu}$ and h_y are defined by

$$h_{x\mu} = \frac{4\varsigma_x}{N_x}, \quad h_x = \frac{2(1-2\varsigma_x)}{N_x}, \qquad h_{y\mu} = \frac{4\varsigma_y}{N_y}, \quad h_y = \frac{2(1-2\varsigma_y)}{N_y}. \tag{38}$$

The mesh $\overline{\omega}^{hx}$ is constructed thus: in each of the subintervals $[0, \varsigma_x]$ and $[1 - \varsigma_x, 1]$ the fine mesh spacing is $h_{x\mu}$ while in the interval $[\varsigma_x, 1 - \varsigma_x]$ the coarse mesh spacing is h_x. The mesh $\overline{\omega}^{hy}$ is defined similarly. The difference scheme (3) with the difference operator (36) on the piecewise uniform mesh (37), (38) converges μ-uniformly to the solution of the continuous problem (35):

$$\max_{t_k \in \overline{\omega}^\tau} \|U(t_k) - u(t_k)\|_{\overline{\omega}^h} \le C(N^{-1} \ln N + \tau), \quad N = \min\{N_x, N_y\}, \tau = \max_{1 \le k \le N_\tau} \tau_k, \tag{39}$$

where constant C is independent of μ, N and τ_k, $k \ge 1$ (see [14] for details). From here and Theorems 3–4, we conclude the following theorem.

Theorem 5. *Let the assumptions in Theorems 3–4 be satisfied and the nonlinear difference scheme (3) be based on the classical difference approximation (36) and the piecewise uniform mesh (37), (38). Then the sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, generated by the monotone iterative methods (8)–(10), (20); (9), (10), (23), (24) and (9), (10), (24) with n_* fixed, converge μ-uniformly to the unique solution of the semilinear singularly perturbed reaction–diffusion problem (35).*

Proof. The proof follows from Theorems 3–4 and (39). □

6.2. The convection–reaction–diffusion problem

Consider the convection–reaction–diffusion problem in the form

$$\begin{aligned} &u_t - \varepsilon(u_{xx} + u_{yy}) + b_1(x, y)u_x + b_2(x, y)u_y + f(x, y, u) = 0, \\ &(x, y, t) \in \omega \times (0, T], \qquad u(x, y, t) = 0, \quad (x, y, t) \in \partial\omega \times (0, T], \\ &u(x, y, 0) = \psi(x, y), \quad (x, y) \in \overline{\omega}, \end{aligned} \tag{40}$$

where $b_1(x, y) \ge \beta_1 = \text{const} > 0$, $b_1(x, y) \ge \beta_1 = \text{const} > 0$ on $\overline{\omega}^h$. For $\varepsilon \ll 1$, the problem is singularly perturbed and characterized by boundary layers of width $\mathcal{O}(\varepsilon|\ln \varepsilon|)$ at $x = 1$ and $y = 1$.

In (3), we use the upwind difference scheme with the difference operator $\mathcal{L}^h$ in the form

$$\begin{aligned} &\mathcal{L}^h U = -\varepsilon(\mathcal{D}_x^2 + \mathcal{D}_y^2)U + b_1 \mathcal{D}_x^- U + b_2 \mathcal{D}_y^- U, \\ &\mathcal{D}_x^- U_{ij}^k = (U_{i,j}^k - U_{i-1,j}^k)/h_{x,i-1}, \qquad \mathcal{D}_y^- U_{ij}^k = (U_{i,j}^k - U_{i,j-1}^k)/h_{y,j-1}. \end{aligned} \tag{41}$$

where $p = (x_i, y_j) \in \omega^h$, $U_{ij}^k = U(x_i, y_j, t_k)$ and the difference operators $\mathcal{D}_x^2$, $\mathcal{D}_y^2$ are defined in (36). The upwind difference scheme satisfies the assumptions from (4).

The piecewise uniform meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are defined in the manner of (37), (38) and are referred to as Shishkin meshes. The boundary layer thicknesses ς_x and ς_y are chosen as

$$\varsigma_x = \min\{0.5, m_1 \varepsilon \ln N_x\}, \qquad \varsigma_y = \min\{0.5, m_2 \varepsilon \ln N_y\}, \tag{42}$$

where m_1 and m_2 are positive constants. Mesh spacings $h_{x\varepsilon}$, h_x, $h_{y\varepsilon}$ and h_y are defined by

$$h_{x\varepsilon} = \frac{2\varsigma_x}{N_x}, \qquad h_x = \frac{2(1 - \varsigma_x)}{N_x}, \qquad h_{y\varepsilon} = \frac{2\varsigma_y}{N_y}, \qquad h_y = \frac{2(1 - \varsigma_y)}{N_y}. \tag{43}$$

The mesh $\overline{\omega}^{hx}$ is constructed such that in the subinterval $[1-\varsigma_x, 1]$ the fine mesh spacing is $h_{x\varepsilon}$ while in the interval $[0, 1-\varsigma_x]$ the coarse mesh spacing is h_x. The mesh $\overline{\omega}^{hy}$ is defined similarly. The difference scheme (3) with the difference operator (41) on the piecewise uniform mesh (42), (43) converges ε-uniformly to the solution of the continuous problem (40):

$$\max_{t_k \in \overline{\omega}^\tau} \|U(t_k) - u(t_k)\|_{\overline{\omega}^h} \le C(N^{-1} \ln N + \tau), \quad N = \min\{N_x, N_y\}, \tau = \max_{1 \le k \le N_\tau} \tau_k, \tag{44}$$

where constant C is independent of ε, N and τ_k, $k \ge 1$ (see [13] for details). Similar to Theorem 5, we have the following theorem.

Theorem 6. *Let the assumptions in Theorems 3–4 be satisfied and the nonlinear difference scheme (3) be based on the upwind difference approximation (41) and the piecewise uniform mesh (42), (43). Then the sequences $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, generated by the monotone iterative methods (8)–(10), (20); (9), (10), (23), (24) and (9), (10), (24) with n_* fixed, converge ε-uniformly to the unique solution of the semilinear singularly perturbed convection–reaction–diffusion problem (40).*

Proof. The proof follows from Theorems 3–4 and (44). □

7. Numerical experiments

In this section, we compare convergence properties of the monotone iterative method (9), (10) and monotone iterative method from [2]. The monotone iterative method from [2] is constructed in the assumption that

$$0 \le f_u \le c^*, \quad c^* = \text{const} > 0. \tag{45}$$

This method utilizes c^* in (9) instead of $c^{(n)}(p, t_k)$.

It is found that in all the numerical experiments the basic feature of monotone convergence of upper and lower sequences is observed. In fact, the monotone property of the sequences holds at every mesh point in the domain. This is, of course, to be expected from our theoretical analysis in Theorem 1.

7.1. Reaction–diffusion problem

As a test problem for (35), we consider the problem

$$\begin{aligned} &u_t - \mu^2(u_{xx} + u_{yy}) + (u-4)/(5-u) = 0, \\ &(x, y, t) \in \omega \times (0, T], \quad \omega = \{0 < x < 1\} \times \{0 < y < 1\}, \\ &u(x, y, t) = 1, \qquad (x_1, x_2, t) \in \partial\omega \times (0, T], \\ &u(x, y, 0) = 0, \quad (x, y) \in \overline{\omega}. \end{aligned} \tag{46}$$

The steady state solution to the reduced problem ($\mu = 0$) is $u_r = 4$. For $\mu \ll 1$, the steady state solution increases sharply from $u = 1$ on $\partial\omega$ to $u = 4$ on the interior, and the solution to the parabolic problem approaches this steady state with time. Since $f_u = 1/(5-u)^2 > 0$, condition (2) is satisfied.

We assume that $N_x = N_y = N$ and the time mesh $\overline{\omega}^\tau$ is uniform with $\tau_k = \tau$, $k \ge 1$.

For the model problem (46), we solve the nonlinear difference scheme (3) with the difference operator $\mathcal{L}^h$ from (36) by the monotone iterative method (9), (10), (20). The mesh function $V_1^{(0)}(p, t_1)$ defined by

$$V_1^{(0)}(\omega^h, t_1) = 4, \quad V_1^{(0)}(\partial\omega^h, t_1) = 1 \tag{47}$$

is clearly an upper solution with respect to the initial condition $g(\omega^h, 0) = 0, g(\partial\omega^h, 0) = 1$. We initiate the iterative method with $V_1^{(0)}(p, t_1)$ and thus generate a sequence of upper solutions. At the next time level, t_{k+1}, $k \ge 1$, we require an initial iterate that is an upper solution with respect to $V_1(p, t_k)$. Since the boundary condition and function $f(u) = (u-4)/(5-u)$ are independent of time, we may choose $V_1^{(0)}(p, t_{k+1}) = V_1(p, t_k)$, $p \in \overline{\omega}^h$. Now, from Theorem 1, it follows by induction on k that the mesh function $V_{-1}(p, t_{k+1})$ defined by $V_{-1}(\omega^h, t_{k+1}) = 0$, $V_{-1}(\partial\omega^h, t_{k+1}) = 1$ is a lower solution with respect to $V_1(p, t_k)$ and, thus, our computed mesh functions satisfy

$$0 \le V_1^{(n)}(p, t_k) \le 4, \quad p \in \overline{\omega}^h, \ 0 \le n \le n_*, \ 0 \le k \le N_\tau. \tag{48}$$

From here and $f_{uu} = 2/(5-u)^3$, we conclude that $f_{uu} \ge 0$. This inequality and (10) imply that $c^{(n)}(p, t_k) = f_u(p, t_k, V_1^{(n)})$.

From (48), we can also conclude that $f_u = 1/(5-u)^2$ is bounded below and above by $c_* = 1/25$ and $c^* = 1$, respectively. Thus, in the monotone iterative method from [2], $c^* = 1$ is in use.

We take as our convergence tolerance $\delta = 10^{-5}$ in (20). All the discrete linear systems are solved by the ICCG-solver [15].

In Table 1, for $\tau = 0.5, 0.1, 0.05$ and for various values of μ and N, we give the average (over ten time levels) convergence iteration counts. The results, corresponding to the monotone iterative method (9), (10), (20) and monotone iterative method from [2], are given above and below the line, respectively. From the numerical data, it follows that for all values of τ, N and μ the monotone iterative method (9), (10), (20) converges faster than the corresponding monotone iterative method from [2]. For $\mu \le 10^{-1}$, the average convergence iteration counts are not affected by the values of τ and N. Similarly, for $\tau \le 0.05$, the average convergence iteration counts are independent of μ and N. For τ and N are fixed and $\mu \le 10^{-2}$, the average convergence iteration counts are uniform with respect to μ.

7.2. Convection–reaction–diffusion problem

As a test problem for (40), we consider the problem

$$\begin{aligned} &u_t - \varepsilon(u_{xx} + u_{yy}) + u_x + u_x + (u-4)/(5-u) = 0, \\ &(x, y, t) \in \omega \times (0, T], \quad \omega = \{0 < x < 1\} \times \{0 < y < 1\}, \\ &u(x, y, t) = 1, \quad (x, y, t) \in \partial\omega \times (0, T], \\ &u(x, y, 0) = 0, \quad (x, y) \in \overline{\omega}. \end{aligned} \tag{49}$$

Since $f_u = 1/(5-u)^2 > 0$, condition (2) is satisfied.

Table 1
Average convergence iteration counts for problem (46). The results, corresponding to the monotone iterative method (9), (10), (20) and monotone iterative method from [2], are given above and below the line, respectively.

N	16	32	64	128	256
μ	$\tau = 0.5$				
1	$\frac{3.2}{5.8}$	$\frac{3.4}{5.8}$	$\frac{3.6}{5.8}$	$\frac{3.4}{5.8}$	$\frac{3.8}{5.8}$
10^{-1}	$\frac{3.4}{8}$	$\frac{3.4}{8}$	$\frac{3.4}{8}$	$\frac{3.4}{8}$	$\frac{3.4}{8}$
$\leq 10^{-2}$	$\frac{3}{5.6}$	$\frac{3.4}{6.8}$	$\frac{3.4}{8}$	$\frac{3.4}{8}$	$\frac{3.4}{8}$
μ	$\tau = 0.1$				
1	$\frac{3.4}{5.6}$	$\frac{3.6}{5.6}$	$\frac{3.6}{5.6}$	$\frac{4.2}{5.6}$	$\frac{4.2}{5.6}$
10^{-1}	$\frac{3.2}{5.8}$	$\frac{3.2}{5.8}$	$\frac{3.2}{5.8}$	$\frac{3.2}{5.8}$	$\frac{3.2}{5.8}$
$\leq 10^{-2}$	$\frac{3}{3.6}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{5.6}$	$\frac{3.2}{5.6}$
μ	$\tau = 0.05$				
1	$\frac{3.4}{5}$	$\frac{3.6}{5}$	$\frac{3.6}{5}$	$\frac{3.6}{5}$	$\frac{3.6}{5}$
10^{-1}	$\frac{3}{4.8}$	$\frac{3}{5}$	$\frac{3.2}{5}$	$\frac{3.2}{5}$	$\frac{3.2}{5}$
$\leq 10^{-2}$	$\frac{3}{3}$	$\frac{3}{3.6}$	$\frac{3}{4}$	$\frac{3}{4.8}$	$\frac{3}{5}$

Table 2
Average convergence iteration counts for problem (49). The results, corresponding to the monotone iterative method (9), (10), (20) and monotone iterative method from [2], are given above and below the line, respectively.

N	16	32	64	128	256
ε	$\tau = 0.5$				
1	$\frac{3}{5.6}$	$\frac{3.4}{5.6}$	$\frac{3.4}{5.6}$	$\frac{3.4}{5.6}$	$\frac{3.6}{5.6}$
10^{-1}	$\frac{3.6}{8.6}$	$\frac{3.6}{8.4}$	$\frac{4}{8.4}$	$\frac{4.2}{8.4}$	$\frac{4.8}{8.4}$
$\leq 10^{-2}$	$\frac{3.6}{8.6}$	$\frac{3.6}{8.6}$	$\frac{3.6}{8.6}$	$\frac{4.2}{8.6}$	$\frac{4.2}{8.6}$
ε	$\tau = 0.1$				
1	$\frac{3.2}{5.6}$	$\frac{3.6}{5.6}$	$\frac{3.6}{5.6}$	$\frac{3.6}{5.6}$	$\frac{4}{5.6}$
10^{-1}	$\frac{3.2}{6}$	$\frac{4.2}{6}$	$\frac{4.2}{6}$	$\frac{4.2}{6}$	$\frac{4.2}{6}$
$\leq 10^{-2}$	$\frac{3.2}{6}$	$\frac{4.2}{6}$	$\frac{4.2}{6}$	$\frac{4.2}{6}$	$\frac{4.2}{6}$
ε	$\tau = 0.05$				
1	$\frac{3.4}{5}$	$\frac{3.6}{5}$	$\frac{3.6}{5}$	$\frac{3.6}{5}$	$\frac{3.6}{5}$
10^{-1}	$\frac{3.2}{5}$	$\frac{4.2}{5}$	$\frac{4.2}{5}$	$\frac{4.2}{5}$	$\frac{4.2}{5}$
$\leq 10^{-2}$	$\frac{3}{5}$	$\frac{3.2}{5}$	$\frac{3.2}{5}$	$\frac{3.2}{5}$	$\frac{3.2}{5}$

We assume that $N_x = N_y = N$ and the time mesh $\overline{\omega}^\tau$ is uniform with $\tau_k = \tau$, $k \geq 1$.

For the model problem (49), we solve the nonlinear difference scheme (3) with the difference operator $\mathcal{L}^h$ from (41) by the monotone iterative method (9), (10), (20). Similar to problem (46), we can show that if we initiate the iterative method with the upper solution (47), then the sequence of upper solution, based on $V_1^{(0)}(p, t_{k+1}) = V_1(p, t_k)$, $p \in \overline{\omega}^h$, satisfies (48) and $c^{(n)}(p, t_k) = f_u(p, t_k, V_1^{(n)})$.

From (48), we can also conclude that $f_u = 1/(5-u)^2$ is bounded below and above by $c_* = 1/25$ and $c^* = 1$, respectively. Thus, in the monotone iterative method from [2], $c^* = 1$ is in use.

We take as our convergence tolerance $\delta = 10^{-5}$ in (20). All the discrete linear systems are solved by the restarted GMRES-solver [15].

In Table 2, for $\tau = 0.5, 0.1, 0.05$ and for various values of ε and N, we give the average (over ten time levels) convergence iteration counts. The results, corresponding the monotone iterative method (9), (10), (20) and monotone iterative method from [2], are given above and below the line, respectively. From the numerical data, it follows that for all values of τ, N and ε the monotone iterative method (9), (10), (20) converges faster than the corresponding monotone iterative method from [2]. For $\tau = 0.5$ and ε fixed, the average convergence iteration counts increase slightly with increasing N. For $\tau = 0.1, 0.05$ and ε fixed, the average convergence iteration counts are independent of N for $N \geq 32$. For τ and N are fixed and $\varepsilon \leq 10^{-2}$, the average convergence iteration counts are uniform with respect to ε.

We draw the following conclusions from the numerical experiments:

- The proposed monotone method converges faster than the corresponding monotone iterative method from [2].

- For τ and the diffusion coefficient μ or ε fixed, the average convergence iteration counts increase slightly with increasing N or independent of N.
- For $\mu, \varepsilon \leq 10^{-2}$, the average convergence iteration counts are uniform with respect to μ or ε.

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