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New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets

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ABSTRACT

Two new algorithms based on Haar wavelets are proposed. The first algorithm is proposed for the numerical solution of nonlinear Fredholm integral equations of the second kind, and the second for the numerical solution of nonlinear Volterra integral equations of the second kind. These methods are designed to exploit the special characteristics of Haar wavelets in both one and two dimensions. Formulae for calculating Haar coefficients without solving the system of equations have been derived. These formulae are then used in the proposed numerical methods. In contrast to other numerical methods, the advantage of our method is that it does not involve any intermediate numerical technique for evaluation of the integral present in integral equations. The methods are validated on test problems, and numerical results are compared with those from existing methods in the literature.

1. Introduction

Integral equations have several applications in Physics and Engineering. Analytical solutions of integral equations, however, either do not exist or are hard to find. It is precisely due to this fact that several numerical methods have been developed for finding solutions of integral equations. Recent contributions in this regard include Chebyshev polynomials [1], the modified homotopy perturbation method [2,3], wavelet methods [4–7], radial basis functions (RBFs) [8,9], Bernstein's approximation [10], the Toeplitz matrix method [11], the linear multistep method [12], and the triangular function method [13]. Some of these methods are applicable only to linear integral equations while others are applicable to special cases of nonlinear integral equations. The need to develop a generic algorithm which can be applied to a general type of nonlinear integral equations is felt necessary in order to have a single platform to be used for the numerical solution of these types of problem. In the present paper we propose two new algorithms based on Haar wavelets which are designed for general types of both nonlinear Fredholm and Volterra integral equations.

The use of wavelets has come to prominence during the last two decades. They have wide-ranging applications in scientific computing, and it is no surprise that they have been extensively used in numerical approximation in the recent relevant literature. This is largely due to the fact that wavelets provide a natural mechanism for decomposing the solution into a set of coefficients, which depend on scale and location. Due to this property, the suitability of wavelets for numerical approximation remains uncontested. Researchers have employed various methods in applying wavelets to numerical approximations. These include the wavelet collocation method [14], the wavelet Galerkin method [15], the wavelet-based finite element method [16], and the wavelet meshless method [17]. A survey of some of the early works can be found in [18]. Researchers have applied wavelets for numerical integration [19–21], and for numerical solutions of integral equations

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[22,4–7], integro-differential equations [23], ordinary differential equations [24,14,25], partial differential equations [26,27] and fractional partial differential equations [28,29]. These methods involve different types of wavelet. The examples include Daubechies [16], Battle–Lemarie [30], B-spline [24], Chebyshev [31,32], Coifman [33], CAS [23], Legendre [34,35] and Haar wavelets [36,37,14,20]. On account of their simplicity, Haar wavelets have received the attention of many researchers. Applications of Haar wavelets for numerical approximations can be found in [37–39,36,40,41,7].

Various types of wavelet have been applied for numerical solution of different kinds of integral equation. These include Haar [7,42–44,41,5,4,45,46], Legendre [6], trigonometric [47], CAS [48], Chebyshev [31], and Coifman [33] wavelets. Many researchers have applied Haar wavelets for the numerical solution of integral equations. Babolian and Shahsavaran [7] have applied Haar wavelets to a particular class of nonlinear Fredholm integral equations. Lepik and Tamme [46] have applied Haar wavelets to nonlinear Fredholm integral equations, but their method involves approximation of certain integrals. Our method is new and different from all the existing Haar wavelet methods applied to integral equations. The main advantage of our method is that it can be applied to general types of nonlinear Fredholm and Volterra integral equations.

In the present work we will consider two types of nonlinear integral equation. The first type is nonlinear Fredholm integral equations of the second kind, given as follows:

$$u(x) = f(x) + \int_0^1 K(x, t, u(t))\,\mathrm{d}t, \tag{1}$$

and the second type is nonlinear Volterra integral equations of the second kind, given as follows:

$$u(x) = f(x) + \int_0^x K(x, t, u(t))\,\mathrm{d}t, \tag{2}$$

where $K(x, t, u(t))$ is a nonlinear function defined on $[0, 1] \times [0, 1]$. In both cases, the known function $K(x, t, u(t))$ is called the kernel of the integral equation while the unknown function $u(x)$ represents the solution of the integral equation.

The organization of the rest of the paper is as follows. In Section 2, Haar wavelets and their integrals are described. In Section 3, formulation of the method based on Haar wavelets is defined for nonlinear Fredholm and Volterra integral equations. Numerical results are reported in Section 4, and conclusions are drawn in Section 5.

2. Haar wavelets

The scaling function for the family of Haar wavelets is defined on the interval $[0, 1)$ and is given as follows:

$$\mathrm{haar}_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

All other functions in the Haar wavelet family are defined on subintervals of $[0, 1)$, and are given as follows:

$$\mathrm{haar}_i(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta) \\ -1 & \text{for } x \in [\beta, \gamma) \\ 0 & \text{elsewhere,} \end{cases} \tag{4}$$

where

$$\alpha = \frac{k}{m}, \qquad \beta = \frac{k + 0.5}{m}, \qquad \gamma = \frac{k+1}{m}, \qquad i = 2, 3, \ldots, 2M. \tag{5}$$

The integer $m = 2^j$, where $j = 0, 1, \ldots, J$, $M = 2^J$,and integer $k = 0, 1, \ldots, m-1$. The integer j indicates the level of the wavelet and k is the translation parameter. The maximal level of resolution is the integer J. The relation between i, m and k is given by $i = m + k + 1$.

The function $\mathrm{haar}_2(x)$ is called the mother wavelet, and all other functions in the Haar wavelet family except the scaling function are generated from the mother wavelet by the operations of dilation and translation.

The Haar wavelet functions are orthogonal to each other because

$$\int_0^1 \mathrm{haar}_j(x)\mathrm{haar}_k(x)\mathrm{d}x = 0, \quad \text{whenever } j \neq k. \tag{6}$$

Any function $f(x)$ which is square integrable in the interval $(0, 1)$ can be expressed as an infinite sum of Haar wavelets:

$$f(x) = \sum_{i=1}^{\infty} a_i \mathrm{haar}_i(x). \tag{7}$$

The above series terminates at finite terms if $f(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval.

We introduce the following notation:

$$\mathrm{ihaar}_{i,1}(x) = \int_0^x \mathrm{haar}_i(x')\,\mathrm{d}x'. \tag{8}$$

This integral can be evaluated using Eq. (4), and is given as follows:

$$\mathrm{ihaar}_{i,1}(x) = \begin{cases} x-\alpha & \text{for } x \in [\alpha, \beta), \\ \gamma - x & \text{for } x \in [\beta, \gamma), \\ 0 & \text{elsewhere.} \end{cases} \tag{9}$$

3. Numerical methods

In this section, we will discuss the proposed numerical methods for two different types of integral equation. In the first subsection, we prove a theorem for efficient evaluation of one-dimensional Haar wavelet approximation, while in the second subsection theoretical results are extended to efficient evaluation of two-dimensional Haar wavelet approximation. In the third and fourth subsections we apply these results for finding numerical solutions of Fredholm and Volterra integral equations. For Haar wavelet approximations, the following collocation points are considered:

$$x_p = \frac{p-0.5}{2M}, \quad p = 1, 2, \ldots, 2M, \tag{10}$$

$$t_q = \frac{q-0.5}{2N}, \quad q = 1, 2, \ldots, 2N. \tag{11}$$

3.1. One-dimensional Haar wavelet system

Any square integrable function $f(x)$ can be approximated using Haar wavelets as follows:

$$f(x) = \sum_{i=1}^{2M} a_i \mathrm{haar}_i(x). \tag{12}$$

Substituting the collocation points given in Eq. (10), we obtain the following linear system of equations:

$$f(x_p) = \sum_{i=1}^{2M} a_i \mathrm{haar}_i(x_p), \quad p = 1, 2, \ldots, 2M. \tag{13}$$

This is a $2M \times 2M$ linear system of equations whose solution for the unknown coefficients a_i can be calculated using the following theorem.

Theorem 1. *The solution of system (13) is given as follows:*

$$a_1 = \frac{1}{2M}\sum_{j=1}^{2M} f(x_j), \tag{14}$$

$$a_i = \frac{1}{\rho}\left(\sum_{p=\alpha}^{\beta} f(x_p) - \sum_{p=\beta+1}^{\gamma} f(x_p)\right), \quad i = 2, 3, \ldots, 2M, \tag{15}$$

where

$$\begin{aligned} \alpha &= \rho(\sigma-1)+1, \\ \beta &= \rho(\sigma-1)+\frac{\rho}{2}, \\ \gamma &= \rho\sigma, \\ \rho &= \frac{2M}{\tau}, \\ \sigma &= i-\tau, \\ \tau &= 2^{\lfloor \log_2(i-1) \rfloor}. \end{aligned}$$

Proof. For proof of Eq. (14) see [20]. For the second part of the proof we use induction on J. We will prove Eq. (15) for $i = 2, 3, \ldots, 2M$, where $M = 2^J$. For $J = 0$, we have $M = 1$, and the linear system in this case is given as follows:

$$\begin{aligned} f(x_1) &= a_1 + a_2 \\ f(x_2) &= a_1 - a_2. \end{aligned} \tag{16}$$

This system has the following solution:

$$a_2 = \frac{1}{2}[f(x_1) - f(x_2)]. \tag{17}$$

Now let us substitute $i = 2$ and $M = 1$ in Eq. (15) so that $\tau = 1, \sigma = 1, \rho = 2, \alpha = 1, \beta = 1, \gamma = 2$ and $a_2 = \frac{1}{2}[f(x_1) - f(x_2)]$, which agrees with Eq. (17), and hence the formula is true for $J = 0$.

Next assume that the formula is true for $J = n - 1, n = 1, 2, \ldots$, and consider the linear system with $J = n$. For $J = n$, we have $M = 2^n$, and we obtain a $2M \times 2M$ linear system. From this system we obtain a new system by adding equations corresponding to $p = 2k - 1$ and $p = 2k$ for $k = 1, 2, \ldots, M$. This new system is an $M \times M$ linear system, and it can be expressed as follows:

$$\frac{1}{2}\left(f(x_{2p-1}) + f(x_{2p})\right) = \sum_{i=1}^{M} a_i \mathrm{haar}_i(x_p), \quad p = 1, 2, \ldots, M. \tag{18}$$

Using the induction hypothesis, the solution of this system is given as follows:

$$a_i = \frac{1}{\rho'}\left(\sum_{p=\alpha'}^{\beta'} \frac{1}{2}(f(x_{2p-1}) + f(x_{2p})) - \sum_{p=\beta'+1}^{\gamma'} \frac{1}{2}(f(x_{2p-1}) + f(x_{2p}))\right), \quad i = 2, 3, \ldots, M, \tag{19}$$

where $\tau' = 2^{\lfloor \log_2(i-1) \rfloor}$, $\sigma' = i - \tau'$, $\rho' = \frac{2^n}{\tau'}$, $\alpha' = \rho'(\sigma' - 1) + 1$, $\beta' = \rho'(\sigma' - 1) + \frac{\rho'}{2}$ and $\gamma' = \rho'\sigma'$.

Eq. (19) can be written in a more compact form as follows:

$$a_i = \frac{1}{2\rho'}\left(\sum_{p=2\alpha'-1}^{2\beta'} f(x_p) - \sum_{p=2\beta'+1}^{2\gamma'} f(x_p)\right), \quad i = 2, 3, \ldots, M. \tag{20}$$

Now let us suppose that

$$\begin{aligned} &\tau = \tau' = 2^{\lfloor \log_2(i-1) \rfloor}, \\ &\sigma = \sigma' = i - \tau, \\ &\rho = 2\rho' = \frac{2^{n+1}}{\tau}, \\ &\alpha = 2\alpha' - 1 = \rho(\sigma - 1) + 1, \\ &\beta = 2\beta' = \rho(\sigma - 1) + \frac{\rho}{2}, \\ &\gamma = 2\gamma' = \rho\sigma, \end{aligned}$$

then Eq. (20) becomes exactly the same as Eq. (15) for $p = 2, 3, \ldots, M$. This proves that the formula is true for $p = 2, 3, \ldots, M$. Next we will prove that it is also true for $p = M + 1, M + 2, \ldots, 2M$. Let us assume that $p = M + j$, where $j = 1, 2, \ldots, M$. Subtracting the equation corresponding to $p = 2j$ from the equation corresponding to $p = 2j - 1$ for $j = 1, 2, \ldots, M$ in the system given in Eq. (13), we obtain the following:

$$a_{M+j} = \frac{1}{2}(f(x_{2j-1}) - f(x_{2j})), \quad j = 1, 2, \ldots, M. \tag{21}$$

Now, for $p = M + j$, we have $\tau = M, \sigma = j, \rho = 2, \alpha = 2j - 1, \beta = 2j - 1, \gamma = 2j$, and the formula given in Eq. (15) matches Eq. (21) for $j = 1, 2, \ldots, M$. This proves the theorem. □

3.2. Two-dimensional Haar wavelet system

A real-valued function $F(x, t)$ of two real variables x and t can be approximated using a two-dimensional Haar wavelet basis as

$$F(x, t) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j} \mathrm{haar}_i(x) \mathrm{haar}_j(t). \tag{22}$$

In order to calculate the unknown coefficients $b_{i,j}$, the collocation points defined in Eqs. (10) and (11) are substituted in Eq. (22). Hence, we obtain the following $4MN \times 4MN$ linear system with unknown $b_{i,j}$:

$$F_{p,q} = \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j} \mathrm{haar}_i(x_p) \mathrm{haar}_j(t_q), \quad p = 1, 2, \ldots, 2M,\ q = 1, 2, \ldots, 2N, \tag{23}$$

where for simplicity we have introduced the notation $F_{p,q}$ for the value of $F(x_p, t_q)$. The solution of this system can be calculated using the following theorem.

Theorem 2. *The solution of system (23) is given below:*

$$b_{1,1} = \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} F_{p,q}, \tag{24}$$

$$b_{i,1} = \frac{1}{\rho_1 \times 2N} \left(\sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2N} F_{p,q} - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2N} F_{p,q} \right), \quad i = 2, 3, \ldots, 2M, \tag{25}$$

$$b_{1,j} = \frac{1}{2M \times \rho_2} \left(\sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} F_{p,q} - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2} F_{p,q} \right), \quad j = 2, 3, \ldots, 2N, \tag{26}$$

$$\begin{aligned} b_{i,j} \ = \ & \frac{1}{\rho 1 \rho 2} \left(\sum_{p=\alpha_1}^{\beta_1} \sum_{q=\alpha_2}^{\beta_2} F_{p,q} - \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\beta_2+1}^{\gamma_2} F_{p,q} - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\alpha_2}^{\beta_2} F_{p,q} \right. \\ & \left. + \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\beta_2+1}^{\gamma_2} F_{p,q} \right), \quad i = 2, 3, \ldots, 2M,\ j = 2, 3, \ldots, 2N, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \alpha_1 &= \rho_1(\sigma_1 - 1) + 1, \\ \beta_1 &= \rho_1(\sigma_1 - 1) + \frac{\rho_1}{2}, \\ \gamma_1 &= \rho_1 \sigma_1, \\ \rho_1 &= \frac{2M}{\tau_1}, \\ \sigma_1 &= i - \tau_1, \\ \tau_1 &= 2^{\lfloor \log_2(i-1) \rfloor}, \end{aligned} \tag{28}$$

and similarly,

$$\begin{aligned} \alpha_2 &= \rho_2(\sigma_2 - 1) + 1, \\ \beta_2 &= \rho_2(\sigma_2 - 1) + \frac{\rho_2}{2}, \\ \gamma_2 &= \rho_2 \sigma_2, \\ \rho_2 &= \frac{2N}{\tau_2}, \\ \sigma_2 &= j - \tau_2, \\ \tau_2 &= 2^{\lfloor \log_2(j-1) \rfloor}. \end{aligned} \tag{29}$$

Proof. We will prove the theorem by using induction on $J = J_1 + J_2$, where $M = 2^{J_1}$ and $N = 2^{J_2}$. For $J = 0$, we have $M = 1$, $N = 1$, and by solving system (23), we obtain the following values of the unknown $b_{i,j}$:

$$b_{1,1} = \frac{1}{4} \left(F_{1,1} + F_{1,2} + F_{2,1} + F_{2,2} \right)$$

$$b_{1,2} = \frac{1}{4} \left(F_{1,1} - F_{1,2} + F_{2,1} - F_{2,2} \right)$$

$$b_{2,1} = \frac{1}{4}\left(F_{1,1} + F_{1,2} - F_{2,1} - F_{2,2}\right)$$
$$b_{2,2} = \frac{1}{4}\left(F_{1,1} - F_{1,2} - F_{2,1} + F_{2,2}\right),$$

which agree with Eqs. (24)–(27). Thus the theorem is true for $J = 0$. Next let us assume that the theorem is true for $J = 0, 1, \ldots, m+n-1$, and suppose that $J = m+n$. Thus $M = 2^m$ and $N = 2^n$, and we have 2^{m+n+2} equations in the resulting system.

First, assume that $m = 0$ and $n > 0$. Adding equations corresponding to $p = k$, $q = 2l-1$ and $p = k$, $q = 2l$ for $k = 1, 2$ and $l = 1, 2, \ldots, 2^n$, we obtain the following system:

$$G_{p,q} = \sum_{i=1}^{2}\sum_{j=1}^{2^n} b_{i,j}\mathrm{haar}_i(x_p)\mathrm{haar}_j(t_q), \quad p = 1, 2,\ q = 1, 2, \ldots, 2^n, \tag{30}$$

where

$$G_{p,q} = \frac{1}{2}\left(F_{p,2q-1} + F_{p,2q}\right). \tag{31}$$

Applying the induction hypothesis to system (30) we obtain the following values of $b_{i,j}$:

$$b_{1,1} = \frac{1}{2\times 2^n}\sum_{p=1}^{2}\sum_{q=1}^{2^n} G_{p,q}, \tag{32}$$

$$b_{2,1} = \frac{1}{2\times 2^n}\left(\sum_{q=1}^{2^n} G_{1,q} - \sum_{q=1}^{2^n} G_{2,q}\right), \tag{33}$$

$$b_{1,j} = \frac{1}{2\times \rho_2'}\left(\sum_{p=1}^{2}\sum_{q=\alpha_2'}^{\beta_2'} G_{p,q} - \sum_{p=1}^{2}\sum_{q=\beta_2'+1}^{\gamma_2'} G_{p,q}\right), \quad j = 2, 3, \ldots, 2^n, \tag{34}$$

$$b_{2,j} = \frac{1}{2\times \rho_2'}\left(\sum_{q=\alpha_2'}^{\beta_2'} G_{1,q} - \sum_{q=\beta_2'+1}^{\gamma_2'} G_{1,q} - \sum_{q=\alpha_2'}^{\beta_2'} G_{2,q} + \sum_{q=\beta_2'+1}^{\gamma_2'} G_{2,q}\right), \quad j = 2, 3, \ldots, 2^n, \tag{35}$$

where $\tau_2' = 2^{\lfloor \log_2(j-1)\rfloor}$, $\sigma_2' = j - \tau_2'$, $\rho_2' = \frac{2^n}{\tau_2'}$, $\alpha_2' = \rho_2'(\sigma_2'-1)+1$, $\beta_2' = \rho_2'(\sigma_2'-1) + \frac{\rho_2'}{2}$ and $\gamma_2' = \rho_2'\sigma_2'$. Substituting the value of $G_{p,q}$ and simplifying, we obtain the following:

$$b_{1,1} = \frac{1}{2\times 2^{n+1}}\sum_{p=1}^{2}\sum_{q=1}^{2^{n+1}} F_{p,q}, \tag{36}$$

$$b_{2,1} = \frac{1}{2\times 2^{n+1}}\left(\sum_{q=1}^{2^{n+1}} F_{1,q} - \sum_{q=1}^{2^{n+1}} F_{2,q}\right), \tag{37}$$

$$b_{1,j} = \frac{1}{2^n\times 2\rho_2'}\left(\sum_{p=1}^{2}\sum_{q=2\alpha_2'-1}^{2\beta_2'} F_{p,q} - \sum_{p=1}^{2}\sum_{q=2\beta_2'+1}^{2\gamma_2'} G_{p,q}\right), \quad j = 2, 3, \ldots, 2^n, \tag{38}$$

$$b_{2,j} = \frac{1}{2\times 2\rho_2'}\left(\sum_{q=2\alpha_2'-1}^{2\beta_2'} F_{1,q} - \sum_{q=2\beta_2'+1}^{2\gamma_2'} F_{1,q} - \sum_{q=2\alpha_2'-1}^{2\beta_2'} F_{2,q} + \sum_{q=2\beta_2'+1}^{2\gamma_2'} F_{2,q}\right), \quad j = 2, 3, \ldots, 2^n. \tag{39}$$

Now let us assume that

$$\tau_2 = \tau_2' = 2^{\lfloor \log_2(i-1)\rfloor},$$
$$\sigma_2 = \sigma_2' = i - \tau_2,$$
$$\rho_2 = 2\rho_2' = \frac{2^{n+1}}{\tau_2},$$
$$\alpha_2 = 2\alpha_2' - 1 = \rho_2(\sigma_2 - 1) + 1,$$

$$\beta_2 = 2\beta_2' = \rho_2(\sigma_2 - 1) + \frac{\rho_2}{2},$$
$$\gamma_2 = 2\gamma_2' = \rho_2\sigma_2.$$

Then Eqs. (36)–(39) agree with Eqs. (24)–(27) for $i = 1, 2$ and $j = 1, 2, \ldots, 2^n$. Next, subtracting equations corresponding to $p = k, q = 2l$ from equations corresponding to $p = k, q = 2l - 1$, respectively, for $k = 1, 2$ and $l = 1, 2, \ldots, 2^n$, we obtain the following system:

$$\begin{aligned} F_{1,2l-1} - F_{1,2l} &= 2b_{1,2^n+l} + 2b_{2,2^n+l}, \quad l = 1, 2, \ldots, 2^n \\ F_{2,2l-1} - F_{2,2l} &= 2b_{1,2^n+l} - 2b_{2,2^n+l}, \quad l = 1, 2, \ldots, 2^n. \end{aligned} \tag{40}$$

This implies the following:

$$\begin{aligned} b_{1,2^n+l} &= \frac{1}{4}\left(F_{1,2l-1} + F_{2,2l-1} - F_{1,2l} - F_{1,2l}\right), \quad l = 1, 2, \ldots, 2^n \\ b_{2,2^n+l} &= \frac{1}{4}\left(F_{1,2l-1} - F_{1,2l} - F_{2,2l-1} + F_{2,2l}\right), \quad l = 1, 2, \ldots, 2^n. \end{aligned} \tag{41}$$

The values given in Eq. (41) agree with the values obtained from Eqs. (26) and (27). This proves the theorem when $m = 0$ and $n > 0$. A similar reasoning proves the theorem for $m > 0$ and $n = 0$.

Finally, we assume that $m > 0$ and $n > 0$. Adding equations corresponding to $p = 2k - 1, q = l$ and $p = 2k, q = l$ for $k = 1, 2, \ldots, 2^n$ and $l = 1, 2, \ldots, 2^{n+1}$, we obtain the following system:

$$G_{p,q} = \sum_{i=1}^{2^m}\sum_{j=1}^{2^{n+1}} b_{i,j}\,\mathrm{haar}_i(x_p)\,\mathrm{haar}_j(t_q), \quad p = 1, 2, \ldots, 2^m,\ q = 1, 2, \ldots, 2^{n+1}, \tag{42}$$

where

$$G_{p,q} = \frac{1}{2}\left(F_{2p-1,q} + F_{2p,q}\right). \tag{43}$$

Applying the induction hypothesis to system (42), we obtain the following values of $b_{i,j}$:

$$b_{1,1} = \frac{1}{2^m \times 2^{n+1}} \sum_{p=1}^{2^m}\sum_{q=1}^{2^{n+1}} G_{p,q}, \tag{44}$$

$$b_{i,1} = \frac{1}{\rho_1' \times 2^{n+1}} \left(\sum_{p=\alpha_1'}^{\beta_1'}\sum_{q=1}^{2^{n+1}} G_{p,q} - \sum_{p=\beta_1'+1}^{\gamma_1'}\sum_{q=1}^{2^{n+1}} G_{p,q}\right), \quad i = 2, 3, \ldots, 2^m, \tag{45}$$

$$b_{1,j} = \frac{1}{2^m \times \rho_2} \left(\sum_{p=1}^{2^m}\sum_{q=\alpha_2}^{\beta_2} G_{p,q} - \sum_{p=1}^{2^m}\sum_{q=\beta_2+1}^{\gamma_2} G_{p,q}\right), \quad j = 2, 3, \ldots, 2^n, \tag{46}$$

$$\begin{aligned} b_{i,j} \;=\; \frac{1}{\rho_1'\rho_2} &\left(\sum_{p=\alpha_1'}^{\beta_1'}\sum_{q=\alpha_2}^{\beta_2} G_{p,q} - \sum_{p=\alpha_1'}^{\beta_1'}\sum_{q=\beta_2+1}^{\gamma_2} G_{p,q} - \sum_{p=\beta_1'+1}^{\gamma_1'}\sum_{q=\alpha_2}^{\beta_2} G_{p,q}\right. \\ &\left. + \sum_{p=\beta_1'+1}^{\gamma_1'}\sum_{q=\beta_2+1}^{\gamma_2} G_{p,q}\right), \quad i = 2, 3, \ldots, 2^m,\ j = 2, 3, \ldots, 2^n, \end{aligned} \tag{47}$$

where $\tau_1' = 2^{\lfloor \log_2(i-1) \rfloor}$, $\sigma_1' = i - \tau_1'$, $\rho_1' = \frac{2^m}{\tau_1'}$, $\alpha_1' = \rho_1'(\sigma_1' - 1) + 1$, $\beta_1' = \rho_1'(\sigma_1' - 1) + \frac{\rho_1'}{2}$, $\gamma_1' = \rho_1'\sigma_1'$, $\tau_2' = 2^{\lfloor \log_2(j-1) \rfloor}$, $\sigma_2' = j - \tau_2'$, $\rho_2' = \frac{2^{n+1}}{\tau_2'}$, $\alpha_2' = \rho_2'(\sigma_2' - 1) + 1$, $\beta_2' = \rho_2'(\sigma_2' - 1) + \frac{\rho_2'}{2}$ and $\gamma_2' = \rho_2'\sigma_2'$. After simplifications, Eqs. (44)–(47) can be written as follows:

$$b_{1,1} = \frac{1}{2^{m+1} \times 2^{n+1}} \sum_{p=1}^{2^{m+1}}\sum_{q=1}^{2^{n+1}} F_{p,q}. \tag{48}$$

$$b_{i,1} = \frac{1}{2\rho_1' \times 2^{n+1}} \left(\sum_{p=2\alpha_1'-1}^{2\beta_1'}\sum_{q=1}^{2^{n+1}} F_{p,q} - \sum_{p=2\beta_1'+1}^{2\gamma_1'}\sum_{q=1}^{2^{n+1}} F_{p,q}\right), \quad i = 2, 3, \ldots, 2^m, \tag{49}$$

$$b_{1,j} = \frac{1}{2^{m+1} \times \rho_2} \left(\sum_{p=1}^{2^{m+1}} \sum_{q=\alpha_2}^{\beta_2} F_{p,q} - \sum_{p=1}^{2^{m+1}} \sum_{q=\beta_2+1}^{\gamma_2} F_{p,q} \right), \quad j = 2, 3, \ldots, 2^{n+1}, \tag{50}$$

$$\begin{aligned} b_{i,j} \; = \; & \frac{1}{2\rho_1' \times \rho_2} \left(\sum_{p=2\alpha_1'-1}^{2\beta_1'} \sum_{q=\alpha_2}^{\beta_2} F_{p,q} - \sum_{p=2\alpha_1'-1}^{2\beta_1'} \sum_{q=\beta_2+1}^{\gamma_2} F_{p,q} - \sum_{p=2\beta_1'+1}^{2\gamma_1'} \sum_{q=\alpha_2}^{\beta_2} F_{p,q} \right. \\ & \left. + \sum_{p=2\beta_1'+1}^{2\gamma_1'} \sum_{q=\beta_2+1}^{\gamma_2} F_{p,q} \right), \quad i = 2, 3, \ldots, 2^m, \; j = 2, 3, \ldots, 2^{n+1}. \end{aligned} \tag{51}$$

Now let us suppose that

$$\begin{aligned} &\tau_1 = \tau_1' = 2^{\lfloor \log_2 (i-1) \rfloor}, \\ &\sigma_1 = \sigma_1' = i - \tau_1, \\ &\rho_1 = 2\rho_1' = \frac{2^{n+1}}{\tau_1}, \\ &\alpha_1 = 2\alpha_1' - 1 = \rho_1(\sigma_1 - 1) + 1, \\ &\beta_1 = 2\beta_1' = \rho_1(\sigma_1 - 1) + \frac{\rho_1}{2}, \\ &\gamma_1 = 2\gamma_1' = \rho_1\sigma_1; \end{aligned}$$

then Eqs. (48)–(51) yield (24)–(27) for $i = 1, 2, \ldots, 2^m$ and $j = 1, 2, \ldots, 2^{n+1}$. Next, we add equations corresponding to $p = k$, $q = 2l - 1$ and $p = k$, $q = 2l$ for $k = 1, 2, \ldots, 2^{m+1}$ and $l = 1, 2, \ldots, 2^n$, and in a similar way as above we obtain (24)–(27) for $i = 1, 2, \ldots, 2^{m+1}$ and $j = 1, 2, \ldots, 2^n$. It only remains to obtain Eq. (27) for $i = 2^m + 1, 2^m + 2, \ldots, 2^{m+1}$ and $j = 2^n + 1, 2^n + 2, \ldots, 2^{n+1}$. For this, we first add equations corresponding to $p = 2k - 1$, $q = 2l - 1$ and $p = 2k$, $q = 2l$, and then from the resultant equation obtained we subtract equations corresponding to $p = 2k$, $q = 2l - 1$ and $p = 2k - 1$, $q = 2l$ for $k = 1, 2, \ldots, 2^m$ and $l = 1, 2, \ldots, 2^n$. Thus we obtain the following values:

$$b_{2^m+k, 2^n+l} = \frac{1}{4} \left(F_{2k-1,2l-1} + F_{2k,2l} - F_{2k-1,2l} - F_{2k,2l-1} \right). \tag{52}$$

These values agree with the values obtained from Eq. (27) for $k = 2^m + 1, 2^m + 2, \ldots, 2^{m+1}$ and $l = 2^n + 1, 2^n + 2, \ldots, 2^{n+1}$. Hence the theorem is proved. □

3.3. *Fredholm integral equations*

Consider the Fredholm integral equation (1). The kernel function $K(x, t, u(t))$ is approximated using the two-dimensional Haar wavelets given in Eq. (22). Substituting this approximation of the kernel function in the Fredholm integral equation (1), we obtain the following:

$$u(x) = f(x) + \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_{i,j} \mathrm{haar}_i(x) \int_0^1 \mathrm{haar}_j(t) \, \mathrm{d}t. \tag{53}$$

Using the following property of Haar wavelets:

$$\int_0^1 \mathrm{haar}_i(t) \, \mathrm{d}t = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{for } i = 2, 3, \ldots, \end{cases} \tag{54}$$

Eq. (53) reduces to the following equation:

$$u(x) = f(x) + \sum_{i=1}^{2M} b_{i,1} \mathrm{haar}_i(x). \tag{55}$$

Substituting the collocation points defined in Eq. (10), we obtain the following:

$$u(x_r) = f(x_r) + \sum_{i=1}^{2M} b_{i,1} \mathrm{haar}_i(x_r), \quad r = 1, 2, \ldots, 2M. \tag{56}$$

Now the coefficients $b_{i,1}$ can be replaced with their expressions given in (24)–(25), and we obtain the following system of nonlinear equations:

$$u(x_r) = f(x_r) + \frac{1}{2M \times 2M} \sum_{p=1}^{2M} \sum_{q=1}^{2M} K(x_p, t_q, u(t_q))\mathrm{haar}_1(x_r) + \sum_{i=2}^{2M} \frac{1}{\rho_1 \times 2M} \left(\sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2M} K(x_p, t_q, u(t_q)) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2M} K(x_p, t_q, u(t_q)) \right) \mathrm{haar}_i(x_r), \quad r = 1, 2, \ldots, 2M. \tag{57}$$

Let us introduce the following notation:

$$u(x_r) = u(t_r) = u_r, \quad r = 1, 2, \ldots, 2M. \tag{58}$$

With this notation, the above system can be written as

$$u_r = f(x_r) + \frac{1}{2M \times 2M} \sum_{p=1}^{2M} \sum_{q=1}^{2M} K(x_p, t_q, u_q)\mathrm{haar}_1(x_r) + \sum_{i=1}^{2M} \frac{1}{\rho_1 \times 2M} \times \left(\sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2M} K(x_p, t_q, u_q) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2M} K(x_p, t_q, u_q) \right) \mathrm{haar}_i(x_r), \quad r = 1, 2, \ldots, 2M. \tag{59}$$

Now Eq. (59) represents a $2M \times 2M$ nonlinear system with unknown u_r, $r = 1, 2, \ldots, 2M$. We may use Newton's or Broyden's method with initial guess $\mathbf{u}^{(0)} = \mathbf{0}$ to solve this system. The algorithm is terminated when the required tolerance has been achieved; that is, when

$$\|\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}\| < \epsilon,$$

where ϵ is the required tolerance. This system directly gives us the solution of the Fredholm integral equation (1) at the collocation points. Once the solution at collocation points is known, we can find solution at any point with the help of Theorem 1.

3.4. Volterra integral equations

Consider the Volterra integral equation (2). Substituting the approximation of the kernel function given in Eq. (22) in the Volterra integral equation (2), we obtain the following equation:

$$u(x) = f(x) + \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_{i,j}\mathrm{haar}_i(x) \int_0^x \mathrm{haar}_j(t)\, dt. \tag{60}$$

Using the property of Haar functions, this equation reduces to the following:

$$u(x) = f(x) + \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_{i,j}\mathrm{haar}_i(x)\mathrm{ihaar}_j(x). \tag{61}$$

Substituting the collocation points, we obtain the following system:

$$u(x_r) = f(x_r) + \sum_{i=1}^{2M} \sum_{j=1}^{2M} b_{i,j}\mathrm{haar}_i(x_r)\mathrm{ihaar}_j(x_r) \quad r = 1, 2, \ldots, 2M. \tag{62}$$

Now the coefficients $b_{i,j}$ can be replaced with their expressions given in Eqs. (24)–(27):

$$u_r = f(x_r) + \frac{1}{2M \times 2M} \sum_{p=1}^{2M} \sum_{q=1}^{2M} K(x_p, t_q, u_q)\mathrm{haar}_1(x_r)\mathrm{ihaar}_1(x_r) + \sum_{i=2}^{2M} \frac{1}{\rho_1 \times 2M} \left(\sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2M} K(x_p, t_q, u_q) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2M} K(x_p, t_q, u(t_q)) \right) \mathrm{haar}_i(x_r)\mathrm{ihaar}_1(x_r) + \sum_{j=2}^{2M} \frac{1}{2M \times \rho_2} \left(\sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} K(x_p, t_q, u_q) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2} K(x_p, t_q, u_q) \right) \mathrm{haar}_1(x_r)\mathrm{ihaar}_j(x_r)$$

Table 1
Comparison of errors with the Haar wavelet method [46] for Example 1.

$2M$	Haar wavelet method [46]	Present method
4	3.3E−3	3.7E−3
8	2.7E−3	1.0E−3
16	1.1E−3	2.6E−4
32	3.7E−4	6.6E−5
64	1.1E−4	1.7E−5
128	3.1E−5	4.2E−6

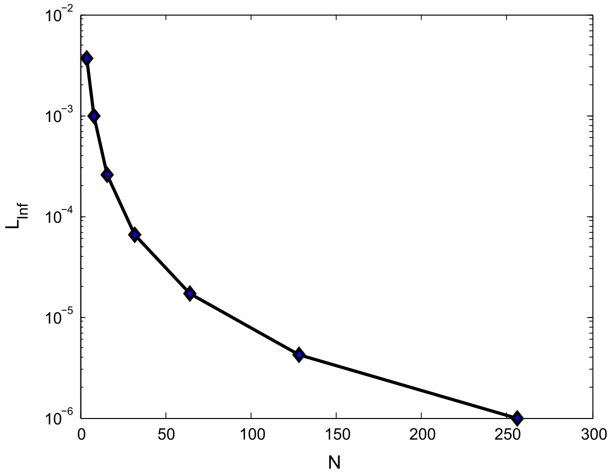

Fig. 1. Comparison of exact and approximate solutions for Example 1.

$$+\sum_{i=2}^{2M}\sum_{j=2}^{2M}\frac{1}{\rho 1\rho 2}\left(\sum_{p=\alpha_1}^{\beta_1}\sum_{q=\alpha_2}^{\beta_2}K(x_p,t_q,u_q)-\sum_{p=\alpha_1}^{\beta_1}\sum_{q=\beta_2+1}^{\gamma_2}K(x_p,t_q,u_q)\right.$$
$$\left.-\sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=\alpha_2}^{\beta_2}K(x_p,t_q,u_q)+\sum_{p=\beta_1+1}^{\gamma_1}\sum_{q=\beta_2+1}^{\gamma_2}K(x_p,t_q,u_q)\right)\mathrm{haar}_i(x_r)\mathrm{ihaar}_j(x_r),\quad r=1,2,\ldots,2M. \tag{63}$$

Proceeding as in the previous case, we can obtain the solution of the Volterra integral equation at the collocation points.

4. Numerical experiments

Example 1. Consider the following nonlinear Fredholm integral equation [46]:

$$u(x)=-x^2-\frac{x}{3}(2\sqrt{2}-1)+2+\int_0^1 xt\sqrt{u(t)}\mathrm{d}t. \tag{64}$$

The exact solution of this problem is $u(x)=2-x^2$. In Table 1 we have compared the maximum absolute errors of the proposed method with those from the Haar wavelet method [46]. The table shows that the performance of our method is better than that of the Haar wavelet method [46]. In Fig. 1 we have shown the absolute errors for this problem. It is evident from the figure that the maximum absolute error decreases with the increase in number of collocation points. The added advantage of the new method in comparison to the Haar wavelet method [46] is that it does not involve numerical integration. In our case we approximate the integrand with the Haar wavelet basis and perform exact integration of Haar functions.

Example 2. Consider the following nonlinear Fredholm integral equation [13]:

$$u(x)=\sin(\pi x)+\frac{1}{5}\int_0^1\cos(\pi x)\sin(\pi t)(u(t))^3\mathrm{d}t. \tag{65}$$

The exact solution of this problem is

$$u(x)=\sin(\pi x)+\frac{20-\sqrt{391}}{3}\cos(\pi x). \tag{66}$$

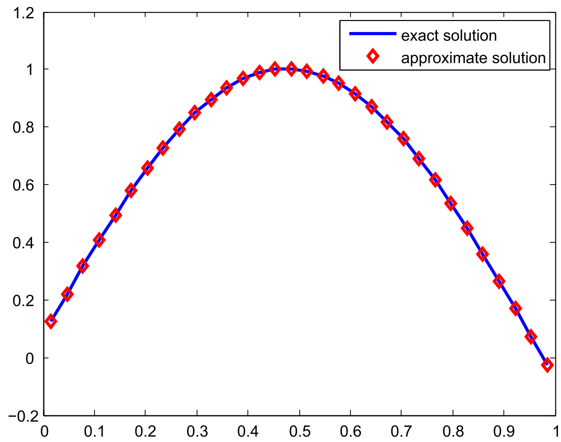

Fig. 2. Comparison of exact and approximate solutions for Example 2.

Table 2
Comparison of errors with the triangular function method [13] for Example 2.

$2M$	Triangular factorization method [13]	Present method
4	3.9E−2	2.8E−16
8	9.9E−3	2.9E−16
16	2.5E−3	2.8E−16
32	7.6E−4	3.3E−16

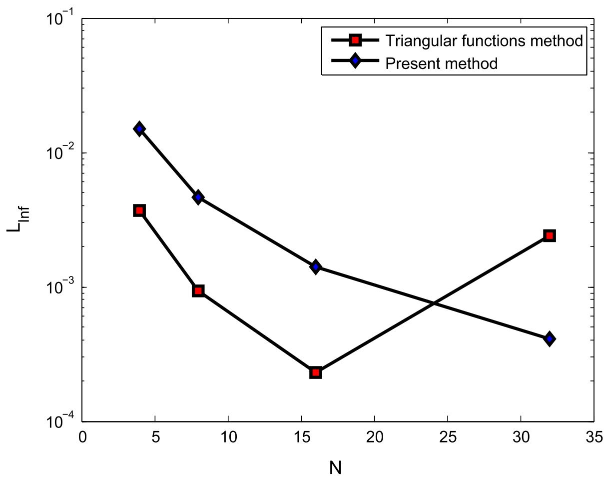

Fig. 3. Comparison of exact and approximate solutions for Example 3.

In Fig. 2, we have shown a comparison of the approximate solution with the exact solution. Table 2 shows the comparison of errors of the present method with those of the triangular function method [13]. It is evident from the table that our method performed far better than the triangular function method [13] for this problem.

Example 3. Consider the following nonlinear Volterra integral equation [13]:

$$u(x) = \frac{3}{2} - \frac{1}{2}\exp(-2x) - \int_0^x (u(t)^2 + u(t))\mathrm{d}t. \tag{67}$$

The exact solution of this problem is $\exp(-x)$. In Fig. 3, we have compared the maximum absolute errors of the present method with those of the triangular function method [13]. The figure shows that better accuracy is obtained with the present method with the increase of number of collocation points. This shows the stability of our method while the errors in the triangular function method [13] show an oscillatory behavior.

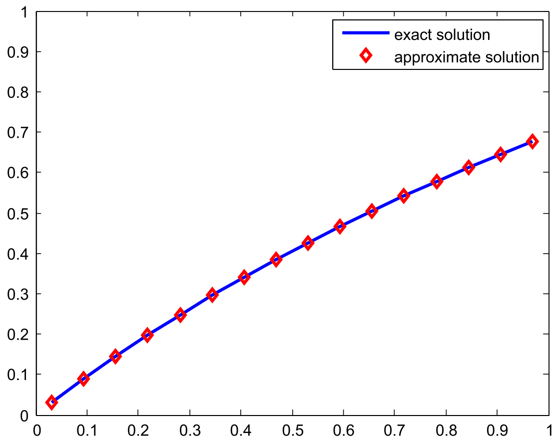

Fig. 4. Comparison of exact and approximate solutions for Example 4.

Example 4. Consider the following nonlinear Volterra integral equation [49]:

$$u(x) = f(x) + \int_0^x xt^2(u(t))^2 \mathrm{d}t, \tag{68}$$

where

$$f(x) = \left(1 + \frac{11}{9}x + \frac{2}{3}x^2 - \frac{1}{3}x^3 + \frac{2}{9}x^4\right)\ln(x+1) - \frac{1}{3}(x + x^4)(\ln(x+1))^2 - \frac{11}{9}x^2 + \frac{5}{18}x^3 - \frac{2}{27}x^4. \tag{69}$$

The exact solution of this problem is $u(x) = \ln(x+1)$. In Fig. 4, we have shown the comparison of the approximate solution with the exact solution.

5. Conclusion

Two new generic algorithm are proposed for the numerical solution of nonlinear Fredholm integral equations of the second kind and nonlinear Volterra integral equations of the second kind. A two-dimensional Haar wavelet basis is used for this purpose. The algorithms are established theoretically alongside numerical validations. The new algorithms do not need any linear system solution for evaluation of the wavelet coefficients and are more efficient than conventional Haar wavelet based methods. Different types of integral equation can be solved numerically by the same method more accurately than previously.

References

[1] Y. Liu, Application of Chebyshev polynomial in solving Fredholm integral equations, Math. Comput. Modelling 50 (2009) 465–469.
[2] M. Javidi, A. Golbabai, Modified homotopy perturbation method for solving non-linear Fredholm integral equations, Chaos Solitons Fractals 40 (2009) 1408–1412.
[3] A. Golbabai, B. Keramati, Solution of non-linear Fredholm integral equations of the first kind using modified homotopy perturbation, Chaos Solitons Fractals 39 (2009) 2316–2321.
[4] M.H. Reihani, Z. Abadi, Rationalized Haar functions method for solving Fredholm and Volterra integral equations, J. Comput. Appl. Math. 200 (2007) 12–20.
[5] Y. Ordokhani, M. Razzaghi, Solution of nonlinear Volterra–Fredholm–Hammerstein integral equations via a collocation method and rationalized Haar functions, Appl. Math. Lett. 21 (2008) 4–9.
[6] S. Yousefi, M. Razzaghi, Legendre wavelets method for the nonlinear Volterra–Fredholm integral equations, Math. Comput. Simul. 70 (2005) 1–8.
[7] E. Babolian, A. Shahsavaran, Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, J. Comput. Appl. Math. 225 (2009) 87–95.
[8] A. Alipanah, S. Esmaeili, Numerical solution of the two-dimensional Fredholm integral equations using Gaussian radial basis function, J. Comput. Appl. Math. 235 (2011) 5342–5347.
[9] A. Alipanah, M. Dehghan, Numerical solution of the nonlinear Fredholm integral equations by positive definite functions, Appl. Math. Comput. 190 (2007) 1754–1761.
[10] K. Maleknejad, E. Hashemizadeh, R. Ezzati, A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation, Commun. Nonlinear Sci. 16 (2011) 647–655.
[11] M.A. Abdou, M.M. El-Borai, M.M. El-Kojok, Toeplitz matrix method and nonlinear integral equation of Hammerstein type, J. Comput. Appl. Math. 223 (2009) 765–776.
[12] J.-P. Kauthen, A survey of singularly perturbed Volterra equations, Appl. Numer. Math. 24 (1997) 95–114.
[13] K. Maleknejad, H. Almasieh, M. Roodaki, Triangular functions (TF) method for the solution of nonlinear Volterra–Fredholm integral equations, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3293–3298.
[14] Siraj-ul-Islam, I. Aziz, B. Šarler, The numerical solution of second-order boundary-value problems by collocation method with the Haar wavelets, Math. Comput. Modelling 50 (2010) 1577–1590.
[15] G.-W. Jang, Y.Y. Kim, K.K. Choi, Remesh-free shape optimization using the wavelet-Galerkin method, Internat. J. Solids Struct. 41 (2004) 6465–6483.

[16] L.A. Diaz, M.T. Martin, V. Vampa, Daubechies wavelet beam and plate finite elements, Finite Elem. Anal. Des. 45 (2009) 200–209.
[17] Y. Liu, Y. Liu, Z. Cen, Daubechies wavelet meshless method for 2-D elastic problems, Tsinghua Sci. Technol. 13 (2008) 605–608.
[18] W. Dahmen, A. Kurdila, P. Oswald, Multiscale Wavelet Methods for Partial Differential Equations, Academic Press, 1997.
[19] H. Hashish, S.H. Behiry, N.A. El-Shamy, Numerical integration using wavelets, Appl. Math. Comput. 211 (2009) 480–487.
[20] Siraj-ul-Islam, I. Aziz, F. Haq, A comparative study of numerical integration based on Haar wavelets and hybrid functions, Comput. Math. Appl. 59 (6) (2010) 2026–2036.
[21] I. Aziz, Siraj-ul-Islam, W. Khan, Quadrature rules for numerical integration based on Haar wavelets and hybrid functions, Comput. Math. Appl. 61 (9) (2011) 2770–2781.
[22] K. Maleknejad, T. Lotfi, K. Mahdiani, Numerical solution of first kind Fredholm integral equations with wavelets-Galerkin method and wavelets precondition, Appl. Math. Comput. 186 (2007) 794–800.
[23] H. Saeedi, M.M. Moghadam, N. Mollahasani, G.N. Chuev, A CAS wavelet method for solving nonlinear Fredholm integro-differential equations of fractional order, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1154–1163.
[24] M. Dehghan, M. Lakestani, Numerical solution of nonlinear system of second-order boundary value problems using cubic *B*-spline scaling functions, Int. J. Comput. Math. 85 (2008) 1455–1461.
[25] Siraj-ul-Islam, B. Šarler, I. Aziz, F. Haq, Haar wavelet collocation method for the numerical solution of boundary layer fluid flow problems, Int. J. Therm. Sci. 50 (2011) 686–697.
[26] M. Mehra, N.K.-R. Kevlahan, An adaptive wavelet collocation method for the solution of partial differential equations on the sphere, J. Comput. Phys. 227 (2008) 5610–5632.
[27] V. Comincioli, G. Naldi, T. Scapolla, A wavelet-based method for numerical solution of nonlinear evolution equations, Appl. Numer. Math. 33 (2000) 291–297.
[28] H. Jafari, S.A. Yousefi, M.A. Firoozjaee, S. Momani, C.M. Khalique, Application of Legendre wavelets for solving fractional differential equations, Comput. Math. Appl. 62 (2011) 1038–1045.
[29] J.L. Wu, A wavelet operational method for solving fractional partial differential equations numerically, Appl. Math. Comput. 214 (2009) 31–40.
[30] X. Zhu, G. Lei, G. Pan, On application of fast and adaptive Battle–Lemarie wavelets to modelling of multiple lossy transmission lines, J. Comput. Phys. 132 (1997) 299–311.
[31] J. Biazar, H. Ebrahimi, Chebyshev wavelets approach for nonlinear systems of Volterra integral equations, Comput. Math. Appl. 63 (2012) 608–616.
[32] E. Babolian, F. Fattahzdeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, Appl. Math. Comput. 188 (2007) 417–426.
[33] K. Maleknejad, T. Lotfi, Y. Rostami, Numerical computational method in solving Fredholm integral equations of the second kind by using Coifman wavelet, Appl. Math. Comput. 186 (2007) 212–218.
[34] E. Banifatemi, M. Razzaghi, S. Yousefi, Two-dimensional Legendre wavelets method for the mixed Volterra–Fredholm integral equations, J. Vib. Control 13 (2007) 1667–1675.
[35] X. Shang, D. Han, Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets, Appl. Math. Comput. 191 (2007) 440–444.
[36] Ü. Lepik, Numerical solution of evolution equations by the Haar wavelet method, Appl. Math. Comput. 185 (2007) 695–704.
[37] C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed-parameter systems, IEE Proc., Control Theory Appl. 144 (1997) 87–94.
[38] C.H. Hsiao, Haar wavelet approach to linear stiff systems, Math. Comput. Simul. 64 (2004) 561–567.
[39] C.H. Hsiao, W.J. Wang, Haar wavelet approach to nonlinear stiff systems, Math. Comput. Simul. 57 (2001) 347–353.
[40] Ü. Lepik, Haar wavelet method for nonlinear integro-differential equations, Appl. Math. Comput. 176 (2006) 324–333.
[41] K. Maleknejad, F. Mirzaee, Using rationalized Haar wavelet for solving linear integral equations, Appl. Math. Comput. 160 (2005) 579–587.
[42] K. Maleknejad, R. Mollapourasl, M. Alizadeh, Numerical solution of Volterra type integral equation of the first kind with wavelet basis, Appl. Math. Comput. 194 (2007) 400–405.
[43] Y. Ordokhani, Solution of nonlinear Volterra–Fredholm–Hammerstein integral equations via rationalized Haar functions, Appl. Math. Comput. 180 (2006) 436–443.
[44] Ü. Lepik, Application of the Haar wavelet transform to solving integral and differential equations, Proc. Estonian Acad. Sci. Phys. Math. 56 (1) (2007) 28–46.
[45] E. Babolian, S. Bazm, P. Lima, Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1164–1175.
[46] Ü. Lepik, E. Tamme, Solution of nonlinear Fredholm integral equations via the Haar wavelet method, Proc. Estonian Acad. Sci. Phys. Math. 56 (2007) 17–27.
[47] J. Gao, Y.-L. Jiang, Trigonometric Hermite wavelet approximation for the integral equations of second kind with weakly singular kernel, J. Comput. Appl. Math. 215 (2008) 242–259.
[48] S. Yousefi, A. Banifatemi, Numerical solution of Fredholm integral equations by using CAS wavelets, Appl. Math. Comput. 183 (2006) 458–463.
[49] Y. Mahmoudi, Wavelet Galerkin method for numerical solution of nonlinear integral equation, Appl. Math. Comput. 167 (2005) 1119–1129.