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Recursive computation of generalised Zernike polynomials

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ABSTRACT

An algorithmic approach for generating generalised Zernike polynomials by differential operators and connection matrices is proposed. This is done by introducing a new order of generalised Zernike polynomials such that it collects all the polynomials of the same total degree in a column vector. The connection matrices between these column vectors composed by the generalised Zernike polynomials and a family of polynomials generated by a Rodrigues formula are given explicitly. This yields a Rodrigues type formula for the generalised Zernike polynomials themselves with properly defined differential operators. Another consequence of our approach is the fact that the generalised Zernike polynomials obey a rather simple partial differential equation. We recall also how to define Hermite–Zernike polynomials.

1. Introduction

In this paper we establish a recursive method for computing the generalised Zernike polynomials which are known to be orthogonal on the unit disc of $\mathbb{R}^2$ with respect to the weight function

$$\rho(x, y; \lambda) = (1 - x^2 - y^2)^{\lambda}, \quad \lambda > -1. \tag{1}$$

The use of Zernike polynomials [1] for describing the classical aberrations of an optical system is well known [2–4]. There have been many other applications, such as to describe the statistical strength of aberrations produced by atmospheric turbulence, atmospheric thermal blooming effects, optical testing, opthalmic optics, corneal topography, interferometer measurements, ocular aberrometry, just to mention a few of them.

The main difficulties when dealing with Zernike polynomials come from the different ordering used in different sources in the literature. The most common is the Noll ordering [5], but Wyant and Creath [6] and Malacara [7] suggest different ones. In general, these orderings preserve the radial power increasing order (that is, the n index), but differ in the order of

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the m term (that is, the angular term). Other orderings motivated by the desire to match certain boundary conditions can be found in [8,9].

In this paper we propose an algorithmic approach for generating recursively the real generalised Zernike polynomials defined as a product of angular functions and radial polynomials

$$Z_{m,j}(\varrho,\theta;\lambda)=\begin{cases} R_m^0(\varrho;\lambda), & m=[j/2],\\ R_m^{m-[j/2]}(\varrho;\lambda)\cos\left(\theta\,(m-[j/2])\right), & m-[j/2]>0,\ j+m^2 \text{ odd},\\ R_m^{m-[j/2]}(\varrho;\lambda)\sin\left(\theta\,(m-[j/2])\right), & m-[j/2]>0,\ j+m^2 \text{ even},\end{cases} \tag{2}$$

for $0\le j\le 2m$, where $0\le\varrho<1$, $0\le\theta<2\pi$, $[x]$ denotes the integer part of x, and ordered in accordance with their total degree. The radial part of $Z_{m,j}(\varrho,\theta;\lambda)$ is

$$R_n^m(\varrho;\lambda)=\sum_{s=0}^{n-m}\frac{(-1)^s(\lambda-m+n+1)_n\varrho^{-m+2n-2s}(\lambda+n+1)_{-m+n-s}}{s!(n-s)!\binom{2n-m}{n-m}(-m+n-s)!}, \tag{3}$$

where $0\le m\le n$, $\lambda>-1$ and $(A)_s=A(A+1)(A+s-1)$, $(A)_0=1$, denotes the Pochhammer symbol. For $\lambda=0$ they coincide with Zernike polynomials [1]. Moreover, these generalised Zernike polynomials are the real and imaginary parts of those in complex variables introduced in [10] and applied in quantum optics in [11]. Operational formulas and generating functions for these complex generalised Zernike polynomials have been obtained in [12].

In order to build the desired algorithm, consider first the column vector of all generalised Zernike polynomials of a fixed total degree s ordered as follows. For odd degree $s=2p+1$, the corresponding vectors of $2p+2$ polynomials are

$$\mathbb{Z}_1^\lambda=\begin{pmatrix}Z_{1,0}(\varrho,\theta;\lambda)\\ Z_{1,1}(\varrho,\theta;\lambda)\end{pmatrix},\qquad \mathbb{Z}_3^\lambda=\begin{pmatrix}Z_{2,2}(\varrho,\theta;\lambda)\\ Z_{2,3}(\varrho,\theta;\lambda)\\ Z_{3,0}(\varrho,\theta;\lambda)\\ Z_{3,1}(\varrho,\theta;\lambda)\end{pmatrix},$$

$$\mathbb{Z}_{2p+1}^\lambda=\begin{pmatrix}Z_{p+1,2p}(\varrho,\theta;\lambda)\\ Z_{p+1,2p+1}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-2}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-1}(\varrho,\theta;\lambda)\\ \cdots\\ Z_{2p+1,0}(\varrho,\theta;\lambda)\\ Z_{2p+1,1}(\varrho,\theta;\lambda)\end{pmatrix},\quad p\ge 2,$$

and for even degree $s=2p$, the vectors of $2p+1$ generalised Zernike polynomials are

$$\mathbb{Z}_2^\lambda=\begin{pmatrix}Z_{1,2}(\varrho,\theta,\lambda)\\ Z_{2,1}(\varrho,\theta,\lambda)\\ Z_{2,0}(\varrho,\theta,\lambda)\end{pmatrix},\qquad \mathbb{Z}_{2p}^\lambda=\begin{pmatrix}Z_{p,2p}(\varrho,\theta;\lambda)\\ Z_{p+1,2p-2}(\varrho,\theta;\lambda)\\ Z_{p+1,2p-1}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-4}(\varrho,\theta;\lambda)\\ Z_{p+2,2p-3}(\varrho,\theta;\lambda)\\ \cdots\\ Z_{2p,0}(\varrho,\theta;\lambda)\\ Z_{2p,1}(\varrho,\theta;\lambda)\end{pmatrix},\quad p\ge 2.$$

We denote by $(\mathbb{Z}_s^\lambda)_i$ the ith polynomial in the column vector $\mathbb{Z}_s^\lambda$. Notice that $\mathbb{Z}_s^\lambda$ contains the $s+1$ generalised Zernike polynomials of total degree s. Thus, we have for $1\le k\le s+1$ and odd s,

$$\left(\mathbb{Z}_s^\lambda\right)_k=\begin{cases}Z_{(s+k)/2,s-k}(\varrho,\theta;\lambda), & k=1,3,5,\dots,s-2,s,\\ Z_{(s+k-1)/2,s-k+2}(\varrho,\theta;\lambda), & k=2,4,6,\dots,s-1,s+1.\end{cases} \tag{4}$$

Moreover, for even $s\ge 4$

$$\left(\mathbb{Z}_s^\lambda\right)_k=\begin{cases}Z_{s/2,s-k+1}(\varrho,\theta;\lambda), & k=1,\\ Z_{s/2+[k/2],s-k+2}(\varrho,\theta;\lambda), & k=3,5,\dots,s-1,s+1,\\ Z_{s/2+[k/2],s-k}(\varrho,\theta;\lambda), & k=2,4,6,\dots,s-2,s.\end{cases} \tag{5}$$

Observe that generalised Zernike polynomials (2) can be written for odd s as

$$\mathbb{Z}_1^\lambda=(\lambda+1)\begin{pmatrix}\mathrm{Re}(P_{1,0}^\lambda(z,z^*))\\ \mathrm{Im}(P_{1,0}^\lambda(z,z^*))\end{pmatrix},\qquad \mathbb{Z}_3^\lambda=\frac{(\lambda+1)_3}{3!}\begin{pmatrix}\mathrm{Im}(P_{2,1}^\lambda(z,z^*))\\ \mathrm{Re}(P_{2,1}^\lambda(z,z^*))\\ \mathrm{Re}(P_{3,0}^\lambda(z,z^*))\\ \mathrm{Im}(P_{3,0}^\lambda(z,z^*))\end{pmatrix}, \tag{6}$$

and for $s > 3$,

$$\left(\mathbb{Z}_s^\lambda\right)_k = \frac{(\lambda+1)_s}{s!}\left(\sigma_{[k/2]}\mathrm{Re}\left(P^\lambda_{(s+1)/2+[(k-1)/2],(s-1)/2-[(k-1)/2]}(z,z^*)\right)\right.$$
$$\left.+\,\sigma_{[k/2]+1}\,\mathrm{Im}\left(P^\lambda_{(s+1)/2+[(k-1)/2],(s-1)/2-[(k-1)/2]}(z,z^*)\right)\right), \tag{7}$$

where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary part of z,

$$\sigma_n = \frac{1+(-1)^n}{2}.$$

If follows from [10, Eq. (2.5)] that

$$P^\lambda_{m,n}(z,z^*) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k m!n!\Gamma(\lambda+1)}{k!(m-k)!(n-k)!\Gamma(k+\lambda+1)}(1-zz^*)^k z^{m-k}(z^*)^{n-k} \tag{8}$$

with $z = x+iy$, $z^* = x-iy$. Moreover, for even s the generalised Zernike polynomials (2) can be written as

$$\left(\mathbb{Z}_s^\lambda\right)_1 = \frac{(\lambda+1)_s}{s!}\mathrm{Re}\left(P^\lambda_{s/2,s/2}(z,z^*)\right), \tag{9}$$

and for $k > 1$,

$$\left(\mathbb{Z}_s^\lambda\right)_k = \frac{(\lambda+1)_s}{s!}\left(\sigma_{[(k+1)/2]}\mathrm{Re}\left(P^\lambda_{s/2+[k/2],s/2-[k/2]}(z,z^*)\right) + \sigma_{[(k+3)/2]}\mathrm{Im}\left(P^\lambda_{s/2+[k/2],s/2-[k/2]}(z,z^*)\right)\right). \tag{10}$$

The generalised Zernike polynomials form a complete orthogonal set on the unit disc $\mathbb{D} = \{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 \le 1\}$ with respect to the inner product (see [10]),

$$\iint_{\mathbb{D}} (\mathbb{Z}_n^\lambda)(\mathbb{Z}_m^\lambda)^T(1-x^2-y^2)^\lambda dxdy = \Lambda_{n,m}\,\delta_{n,m}, \quad \lambda > -1,$$

where A^T denotes the transpose of A, $\Lambda_{n,m}$ is a $(n+1)\times(m+1)$ matrix which is zero for $n \ne m$ and a diagonal invertible matrix for $n = m$, and $\delta_{n,m}$ denotes the Kronecker delta.

Our recursive approach is based on a relation between $\mathbb{Z}_s^\lambda$ and the bivariate polynomials defined by the Rodrigues-type formula [13]

$$P_{n,m}(x,y;\lambda) = \frac{1}{(1-x^2-y^2)^\lambda}\frac{\partial^{n+m}}{\partial x^n \partial y^m}\left[(1-x^2-y^2)^{n+m+\lambda}\right]. \tag{11}$$

We shall use the following notation for the column vector of $(n+1)$ polynomials of total degree exactly n:

$$\mathbb{P}_n^\lambda = \left(P_{n,0}(x,y;\lambda), P_{n-1,1}(x,y;\lambda), \ldots, P_{0,n}(x,y;\lambda)\right)^T, \tag{12}$$

where the elements are arranged according to the lexicographical order [14,15].

The column vector $\mathbb{P}_n^\lambda$ contains one of the polynomial solutions of the following second order partial differential equation of hypergeometric type [16]

$$(1-x^2)\frac{\partial^2}{\partial x^2}u(x,y) + (1-y^2)\frac{\partial^2}{\partial y^2}u(x,y) - 2xy\frac{\partial^2}{\partial x \partial y}u(x,y)$$
$$-(2\lambda+3)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u(x,y) + n(n+2\lambda+2)u(x,y) = 0. \tag{13}$$

As it was shown by Hermite [13], the polynomials $\mathbb{P}_n^\lambda$, defined in (12), are orthogonal within subspaces (orthogonal to all polynomials of no higher than $n-1$ [17]) on the unit disc $\mathbb{D}$ with respect to the same weight function $\rho(x,y;\lambda)$ defined in (1), i.e.,

$$\iint_{\mathbb{D}} (\mathbb{P}_n^\lambda)\,(\mathbb{P}_m^\lambda)^T\,(1-x^2-y^2)^\lambda dxdy = H_{n,m}\delta_{n,m}, \quad \lambda > -1,$$

where $H_{n,m}$ is a $(n+1)\times(m+1)$ matrix which is zero for $n \ne m$ and a non-diagonal matrix for $n = m$.

2. The algorithm and the main results

We shall present our algorithm and its outcome. Let I be the identity operator and define the operators

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \mathcal{E} = \frac{\partial}{\partial x}, \qquad \mathcal{F} = \frac{\partial}{\partial y}, \tag{14}$$

and do the following initialisation $\mathcal{Y}_{0,1} = I$, $\mathcal{Y}_{1,1} = \mathcal{E}$, and $\mathcal{Y}_{1,2} = \mathcal{F}$.

For each n, compute recursively the following $n+1$ differential operators $\mathcal{Y}_{n,k}$, $1 \le k \le n+1$, by the following procedure that we call the

Algorithm 1

for $m \ge 1$ **do**
  if $n = 2m$ **then** $\mathcal{Y}_{n,1} = \Delta \circ \mathcal{Y}_{n-2,1}$,
    for $j = 1, 2, \ldots, m-1$ **do**
      $\mathcal{Y}_{n,2j} = \Delta \circ \mathcal{Y}_{n-2,2j+1}, \qquad \mathcal{Y}_{n,2j+1} = \Delta \circ \mathcal{Y}_{n-2,2j}$,
    end
    $\mathcal{Y}_{n,n} = 2\,\mathcal{Y}_{m,m} \circ \mathcal{Y}_{m,m+1}$,
    $\mathcal{Y}_{n,n+1} = (-1)^{m+1}(\mathcal{Y}_{m,m} \circ \mathcal{Y}_{m,m} - \mathcal{Y}_{m,m+1} \circ \mathcal{Y}_{m,m+1})$.
  else
    for $j = 0, \ldots, m-1$ **do**
      $\mathcal{Y}_{n,2j+1} = \Delta \circ \mathcal{Y}_{n-2,2j+2}, \qquad \mathcal{Y}_{n,2j+2} = \Delta \circ \mathcal{Y}_{n-2,2j+1}$,
    end
    $\mathcal{Y}_{n,n} = \Delta^m \circ \mathcal{E} - 2\,\mathcal{Y}_{2[(m+2)/2]-1,2[(m+2)/2]} \circ \mathcal{Y}_{2[(m+1)/2],2[(m+1)/2]}$,
    $\mathcal{Y}_{n,n+1} = (-1)^m \Delta^m \circ \mathcal{F} + 2\mathcal{Y}_{2[(m+2)/2]-1,2[(m+2)/2]-1} \circ \mathcal{Y}_{2[(m+1)/2],2[(m+1)/2]}$.
  end
end

Then our main result states that

Theorem 2.1. *For each n and for* $1 \le k \le n+1$,

$$\mathcal{Y}_{n,k}\left[(1-x^2-y^2)^{n+\lambda}\right] = (-1)^n\, 2^n\, n!\, (1-x^2-y^2)^{\lambda}\, (\mathbb{Z}_n^{\lambda})_k, \tag{15}$$

where the differential operators $\mathcal{Y}_{n,k}$ *are computed by the proposed algorithm and* $\mathbb{Z}_n^{\lambda}$ *denotes the column vector of generalised Zernike polynomials defined in* (6), (7), (9), *and* (10).

Proof. The result can be proved by induction.

Corollary 2.2. *For each n, the generalised Zernike polynomials defined in* (6), (7), (9), *and* (10) *are generated by the Rodrigues-type formula*

$$(\mathbb{Z}_n^{\lambda})_k = \frac{(-1)^n}{2^n\, n!}\, (1-x^2-y^2)^{-\lambda}\, \mathcal{Y}_{n,k}\left[(1-x^2-y^2)^{n+\lambda}\right], \quad 1 \le k \le n+1, \tag{16}$$

where the differential operators $\mathcal{Y}_{n,k}$ *are computed by Algorithm* 1.

It is worth emphasising that Algorithm 1 implies that $\mathcal{Y}_{n,k}$ is a homogeneous linear differential operator of order n with constant coefficients, so that it can be written as

$$\mathcal{Y}_{n,k} = \sum_{r=1}^{n+1} a_{k,r}^{(n)}\, \frac{\partial^n}{\partial x^{n+1-r}\, \partial y^{r-1}}, \quad 1 \le k \le n+1. \tag{17}$$

Let $\mathbb{A}_n$ be the $(n+1)\times(n+1)$ matrix whose entries are the coefficients which appear in (17):

$$\mathbb{A}_n = \left(a_{k,r}^{(n)}\right)_{1\le k,r\le n+1}. \tag{18}$$

Algorithm 1 implies that

Corollary 2.3. *For each n, the connection matrices $\mathbb{A}_n$ linking the column vector polynomials $\mathbb{P}_n^\lambda$ defined in (12) and the column vector of generalised Zernike polynomials $\mathbb{Z}_n^\lambda$ defined in (6), (7), (9), and (10), that is, the matrices $\mathbb{A}_n$ that obey*

$$\mathbb{Z}_n^\lambda = \frac{(-1)^n}{2^n\, n!}\mathbb{A}_n\mathbb{P}_n^\lambda, \tag{19}$$

can be computed in a recursive way by Algorithm 1.

2.1. Results and discussion

From Algorithm 1, the first matrices $\mathbb{A}_n$ defined in (17) and (18) are given by

$$\mathbb{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{A}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \mathbb{A}_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix},$$

$$\mathbb{A}_4 = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 1 & 0 & -6 & 0 & 1 \end{pmatrix}, \qquad \mathbb{A}_5 = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 3 & 0 & 2 & 0 & -1 \\ 1 & 0 & -2 & 0 & -3 & 0 \\ 1 & 0 & -10 & 0 & 5 & 0 \\ 0 & 5 & 0 & -10 & 0 & 1 \end{pmatrix},$$

where, as we have already noticed, the coefficients $a_{k,r}^{(n)}$ do not depend on λ.

From the Rodrigues formulae (11) and (19) we obtain the following column vectors for the first generalised Zernike polynomials in Cartesian coordinates following the order presented in Section 1:

$$\mathbb{Z}_1^\lambda = \begin{pmatrix} (\mathbb{Z}_1^\lambda)_1 \\ (\mathbb{Z}_1^\lambda)_2 \end{pmatrix} = \begin{pmatrix} (\lambda+1)x \\ (\lambda+1)y \end{pmatrix},$$

$$\mathbb{Z}_2^\lambda = \begin{pmatrix} (\mathbb{Z}_2^\lambda)_1 \\ (\mathbb{Z}_2^\lambda)_2 \\ (\mathbb{Z}_2^\lambda)_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\lambda+2}{2}\left((\lambda+2)(x^2+y^2)-1\right) \\ (\lambda+1)(\lambda+2)xy \\ \dfrac{(\lambda+1)(\lambda+2)}{2}(x^2-y^2) \end{pmatrix},$$

$$\mathbb{Z}_3^\lambda = \begin{pmatrix} (\mathbb{Z}_3^\lambda)_1 \\ (\mathbb{Z}_3^\lambda)_2 \\ (\mathbb{Z}_3^\lambda)_3 \\ (\mathbb{Z}_3^\lambda)_4 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{6}(\lambda+2)(\lambda+3)y\left((\lambda+3)(x^2+y^2)-2\right) \\ \dfrac{1}{6}(\lambda+2)(\lambda+3)x\left((\lambda+3)(x^2+y^2)-2\right) \\ \dfrac{1}{6}(\lambda+1)(\lambda+2)(\lambda+3)x\left(x^2-3y^2\right) \\ -\dfrac{1}{6}(\lambda+1)(\lambda+2)(\lambda+3)y\left(y^2-3x^2\right) \end{pmatrix},$$

$$\mathbb{Z}_4^\lambda = \begin{pmatrix} (\mathbb{Z}_4^\lambda)_1 \\ (\mathbb{Z}_4^\lambda)_2 \\ (\mathbb{Z}_4^\lambda)_3 \\ (\mathbb{Z}_4^\lambda)_4 \\ (\mathbb{Z}_4^\lambda)_5 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{24}(\lambda+3)_2\left((\lambda+3)\left(x^2+y^2\right)\left((\lambda+4)\left(x^2+y^2\right)-4\right)+2\right) \\ \dfrac{1}{24}(\lambda+2)_3\left((\lambda+4)\left(x^4-y^4\right)+3\left(y^2-x^2\right)\right) \\ \dfrac{1}{12}(\lambda+2)_3xy\left((\lambda+4)\left(x^2+y^2\right)-3\right) \\ \dfrac{1}{6}(\lambda+1)_4xy(x-y)(x+y) \\ \dfrac{1}{24}(\lambda+1)_4\left(x^4-6x^2y^2+y^4\right) \end{pmatrix},$$

$$\mathbb{Z}_5^\lambda = \begin{pmatrix} (\mathbb{Z}_5^\lambda)_1 \\ (\mathbb{Z}_5^\lambda)_2 \\ (\mathbb{Z}_5^\lambda)_3 \\ (\mathbb{Z}_5^\lambda)_4 \\ (\mathbb{Z}_5^\lambda)_5 \\ (\mathbb{Z}_5^\lambda)_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{120}(\lambda+3)_3 x\left((\lambda+4)\left(x^2+y^2\right)\left((\lambda+5)\left(x^2+y^2\right)-6\right)+6\right) \\ \frac{1}{120}(\lambda+3)_3 y\left((\lambda+4)\left(x^2+y^2\right)\left((\lambda+5)\left(x^2+y^2\right)-6\right)+6\right) \\ -\frac{1}{120}(\lambda+2)_4 y\left(y^2-3x^2\right)\left((\lambda+5)\left(x^2+y^2\right)-4\right) \\ \frac{1}{120}(\lambda+2)_4 x\left(x^2-3y^2\right)\left((\lambda+5)\left(x^2+y^2\right)-4\right) \\ \frac{1}{120}(\lambda+1)_5 x\left(x^4-10x^2y^2+5y^4\right) \\ \frac{1}{120}(\lambda+1)_5 y\left(5x^4-10x^2y^2+y^4\right) \end{pmatrix}.$$

2.2. Hermite–Zernike polynomials

The idea to reorder the generalised Zernike polynomials as in Section 1 is rather natural from mathematical point of view. It is clear from (2) that $Z_{m,j}(\varrho,\theta;\lambda)$ are represented as radial polynomials R_m^k multiplied by sines and cosines. This suggests extensions of the Zernike polynomials to any radially symmetric weight function. Here we recall a basis of bivariate polynomials orthogonal with respect to the radial normal distribution. Indeed, let

$$w(x,y) = \exp(-x^2-y^2).$$

We have introduced in [18] the polynomials

$$H^*_{n,m,c}(\varrho,\theta) = \varrho^m L_n^{(m)}(\varrho^2)\cos(m\theta), \qquad H^*_{n,m,s}(\varrho,\theta) = \varrho^m L_n^{(m)}(\varrho^2)\sin(m\theta), \tag{20}$$

where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials [19, p. 98], orthogonal with respect to the weight $x^\alpha\exp(-x)$ on $[0,\infty)$. Since the polynomials defined in (20) are natural analogues of the generalised Zernike polynomials $Z_{m,j}(\varrho,\theta;\lambda)$, we call $H^*_{n,m,c}$ and $H^*_{n,m,s}$ the Hermite–Zernike polynomials. The fact that they form an orthogonal basis with respect to the weight function $w(x,y)$ can be easily established by straightforward integration and a polar change of variables.

It is known that the Laguerre polynomials $L_n^{(\alpha)}(x)$ are generated by the Rodrigues type formula [19, (4.6.19)]

$$e^{-x}x^\alpha L_n^{(\alpha)}(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left[e^{-x}x^{n+\alpha}\right].$$

Therefore a slight modification of the latter implies that the radial parts of the Hermite–Zernike polynomials are generated by a modified formula of this type.

3. Conclusions

We have obtained the Rodrigues type formula (16), where the operators $\mathcal{Y}_{n,k}$ are generated recursively and can be constructed explicitly by means of Algorithm 1. It is clear from their definition that they are homogeneous partial differential operators with constant coefficients that do not depend on λ. Therefore, while constructing the operators by the recursive procedure, the coefficients $a_{k,r}^{(n)}$ in (17), and then the matrix $\mathbb{A}_n$ can be determined explicitly. This yields the equivalent but alternative way of building the generalised Zernike polynomials $\mathbb{Z}_n^\lambda$ via (19). It is worth emphasising that the connection matrices $\mathbb{A}_n$, defined in (18), link the family of polynomials $\mathbb{P}_n^\lambda$ orthogonal only in subspaces to the complete orthogonal system $\mathbb{Z}_n^\lambda$ (see [10]). As we have just mentioned, a remarkable feature of Algorithm 1 proposed in the present paper is that the matrices $\mathbb{A}_n$ connecting the vector column of generalised Zernike polynomials $\mathbb{Z}_n^\lambda$ and the bivariate orthogonal polynomials $\mathbb{P}_n^\lambda$ do not depend on the parameter λ. This immediately implies another interesting result: the generalised Zernike polynomials $\mathbb{Z}_n^\lambda$ are solution of the same second-order linear partial differential equation (13) as $\mathbb{P}_n^\lambda$.

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