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A three-level linear implicit conservative scheme for the Rosenau-KdV-RLW equation

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Abstract

A three-level linear conservative implicit finite difference scheme for solving the Rosenau-KdV-RLW equation is proposed. The existence, uniqueness, and conservation for energy of the difference solution are proved by the discrete energy method. The new scheme is 2nd-order convergent and unconditionally stable. Several numerical experiments verify that the theoretical results are accurate and the scheme is efficient and reliable.

Key words: Rosenau-KdV-RLW equation, conservative scheme, discrete energy method, unconditionally stable

1. Introduction

The general Rosenau-KdV equation coupling with the Rosenau-RLW equation is of the form [1, 2]

$$u_t + au_x + b(u^p)_x - cu_{xxt} + du_{xxx} + eu_{xxxxt} = 0.$$

Some particular cases of the above equation are:

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- 1) Well-known KdV equation [3, 4, 5, 6, 7] ($a=0, b=0.5, c=0, d=1, e=0, p=2$)

$$u_t + uu_x + u_{xxx} = 0,$$

which has been used in very wide applications, such as magnetic fluid wave and longitudinal astigmatic wave [1].

- 2) Rosenau equation [8, 9, 10] ($a=1, b=0.5, c=0, d=0, e=1, p=2$)

$$u_t + u_x + uu_x + u_{xxxxt} = 0,$$

where the existence and uniqueness of the solution for the Rosenau equation were proved by Park [8]. This equation has been generalized to a more general equation, that is [9]

$$u_t + u_x + u^p u_x + u_{xxxxt} = 0.$$

- 3) Regularized Long Wave (RLW) equation [11] ($a=1, b=0.5, c=1, d=0, e=0, p=2$)

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

which is usually used to simulate the long wave in nonlinear emanative media.

- 4) Rosenau-KdV equation [12, 13] ($a=1, b=0.5, c=0, d=1, e=1, p=2$)

$$u_t + u_x + uu_x + u_{xxx} + u_{xxxxt} = 0.$$

Zuo [12] discussed the solitary wave solutions and periodic solutions for the Rosenau-KdV equation. An average linear 2nd-order convergent finite difference scheme for the numerical solution of the initial-boundary value problem of the Rosenau-KdV equation was suggested in [13].

5) Generalized Rosenau-RLW equation [14] ($a=1, b=1, c=1, d=0, e=1$)

$$u_t + u_x + (u^p)_x - u_{xxt} + u_{xxxxt} = 0,$$

in which when $p = 2$, it is called the usual Rosenau-RLW equation, and when $p \geq 3$, it is called the modified Rosenau-RLW equation.

In this article, we consider the following initial-boundary value problem of the general Rosenau-KdV-RLW equation as follows:

$$u_t + au_x + b(u^p)_x - cu_{xxt} + du_{xxx} + u_{xxxxt} = 0, \quad x \in \Omega = [\alpha, \beta], \quad t \in [0, T], \quad (1)$$

with an initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega = [\alpha, \beta], \quad (2)$$

and boundary conditions

$$u(\alpha, t) = u(\beta, t) = 0, \quad u_x(\alpha, t) = u_x(\beta, t) = 0, \quad t \in [0, T], \quad (3)$$

where a, b, c and d are any non-negative real numbers, $p \geq 2$ is an integer, $u_0(x)$ is a smooth function. Since $u_0(x)$ tends to zero when $\alpha \ll 0$ and $\beta \gg 0$ in most cases, it is reasonable to assume that $u(\alpha, t) = u(\beta, t) = 0$ and $u_x(\alpha, t) = u_x(\beta, t) = 0$, so that the initial condition (2) and the boundary conditions (3) are consistent each other [14].

Solitary waves, shock waves and singular solitons along with conservation laws of the Rosenau-KdV-RLW equation with power law nonlinearity that models the dynamics of shallow water waves were obtained [15]. Ahmed *et. al* [16] computed the conservation laws of the Rosenau-KdV-RLW equation with power law nonlinearity by

the aid of multiplier approach in Lie symmetry analysis. Sanchez *et. al* [17] obtained the solutions of the perturbed Rosenau-KdV-RLW equation. Wongsaijai *et. al* [1] proposed a three-level weighted average implicit finite difference scheme to solve the Rosenau-KdV-RLW equation. However, in their proof of Theorem 3, the authors should consider the parameter $\gamma \geq 0$, otherwise there may exist some difficulty to obtain $\|u\|_{L_2} \leq C$ and $\|u_{\tilde{x}\tilde{x}}\|_{L_2} \leq C$ when $\gamma < 0$. In this study, we will give a modified proof of this theorem. In addition, Pan *et. al* [2] developed a new Crank-Nicolson (C-N) conservative numerical scheme for the Rosenau-KdV-RLW equation, and their scheme is two-level pseudo-compact, unconditionally stable, and of second-order accuracy. Compared by the C-N nonlinear scheme, the new three-level scheme in this article is linear and hence simplifies the computation. In the next sections, we will prove the existence, uniqueness, convergence, and stability of the new three-level finite difference scheme. Theoretical and experimental results also verify the scheme to be 2nd-order convergent, unconditionally stable, and conservative for energy.

2. Solution properties

We define the following Sobolev spaces [18]:

$$H^k(\Omega) = \{u(x) : \int_{\Omega} (u^{(j)})^2 dx < +\infty, j = 0, 1, 2, \dots, k-1, k\},$$

$$H_0^k(\Omega) = \{u(x) \in H^k(\Omega) : \partial^i u / \partial x^i = 0 \text{ on } \partial\Omega, i = 0, 1, 2, \dots, k-1\},$$

where $u^{(j)}$ is the j^{th} order derivative. The initial-boundary value problem (1)-(3) possesses the following conservative properties:

Lemma 2.1 (See [19]) Suppose $u_0(x) \in H_0^2[\alpha, \beta]$, then the solution of Eqs. (1)-(3) satisfies

$$Q(t) = \int_{\alpha}^{\beta} u(x, t) dx = \int_{\alpha}^{\beta} u(x, 0) dx = \int_{\alpha}^{\beta} u_0(x) dx = Q(0).$$

Theorem 2.2 Suppose $u_0(x) \in H_0^2[\alpha, \beta]$, then the solution of Eqs. (1)-(3) satisfies

$$E(t) = \int_{\alpha}^{\beta} (u^2 + cu_x^2 + u_{xx}^2) dx = \|u\|_{L_2}^2 + c\|u_x\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0), \quad c \geq 0.$$

Proof. Considering $u_t - cu_{xxt} + u_{xxxxt} = -au_x - b(u^p)_x - du_{xxx}$, we have

$$\begin{aligned} \frac{dE(t)}{dt} &= 2 \int_{\alpha}^{\beta} (uu_t + cu_x u_{xt} + u_{xx} u_{xxt}) dx \\ &= 2 \int_{\alpha}^{\beta} (uu_t - cuu_{xxt} - u_x u_{xxx}) dx + 2(cuu_{xt} + u_x u_{xxt}) \Big|_{\alpha}^{\beta} \\ &= 2 \int_{\alpha}^{\beta} u(u_t - cu_{xxt} + u_{xxxx}) dx - 2(uu_{xxx}) \Big|_{\alpha}^{\beta} \\ &= -2 \int_{\alpha}^{\beta} u \left[au_x + b(u^p)_x + du_{xxx} \right] dx \\ &= - \left[au^2 + \frac{2bp}{p+1} u^{p+1} + 2duu_{xx} - d(u_x)^2 \right] \Big|_{\alpha}^{\beta} = 0. \end{aligned}$$

Therefore, $E(t)$ is a constant function only depending on the initial data.

Theorem 2.3 Suppose $u_0(x) \in H_0^2[\alpha, \beta]$, then the solution of Eqs. (1)-(3) satisfies

$$\|u\|_{L_2} \leq C, \quad \|u_x\|_{L_2} \leq C, \quad \|u_{xx}\|_{L_2} \leq C, \quad \text{and hence } \|u\|_{L_{\infty}} \leq C, \quad \|u_x\|_{L_{\infty}} \leq C.$$

Proof. From Theorem 2.2, we know that $\|u\|_{L_2} \leq C, \|u_{xx}\|_{L_2} \leq C$. Using Hölder inequality and the Schwartz inequality [13], we get

$$\|u_x\|_{L_2}^2 = \int_{\alpha}^{\beta} u_x u_x dx = - \int_{\alpha}^{\beta} u u_{xx} dx \leq \|u\|_{L_2} \|u_{xx}\|_{L_2} \leq \frac{1}{2} \left[\|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 \right] \leq C.$$

By the Sobolev inequality [22], we obtain

$$\|u\|_{L_\infty} \leq C_1\|u\|_{L_2} + C_2\|u_x\|_{L_2} \leq C,$$

$$\|u_x\|_{L_\infty} \leq C_1\|u_x\|_{L_2} + C_2\|u_{xx}\|_{L_2} \leq C.$$

Theorem 2.4 Suppose $u_0(x) \in H_0^2[\alpha, \beta]$, then the problem in Eqs. (1)-(3) is well-posed.

Proof. Assume that u_1 and u_2 are two solutions of Eqs. (1)-(3) satisfying the initial conditions $u_0^{(1)}$ and $u_0^{(2)}$, respectively. Let $\delta = u_1 - u_2$, then δ satisfies

$$\delta_t - c\delta_{xxt} + \delta_{xxxxt} = -a\delta_x - b[(u_1)^p]_x + b[(u_2)^p]_x - d\delta_{xxx},$$

and the initial-boundary conditions:

$$\delta(x, 0) = u_0^{(1)} - u_0^{(2)}, \quad \delta(\alpha, t) = \delta(\beta, t) = 0, \quad \delta_x(\alpha, t) = \delta_x(\beta, t) = 0,$$

where $t \in [0, T]$, $x \in \Omega = [\alpha, \beta]$. Letting

$$E(t) = \int_{\alpha}^{\beta} (\delta^2 + c\delta_x^2 + \delta_{xx}^2) dx, \quad c \geq 0,$$

we use a similar derivation as that in the proof of Theorem 2.2 and obtain

$$\begin{aligned} \frac{dE(t)}{dt} &= 2 \int_{\alpha}^{\beta} (\delta\delta_t + c\delta_x\delta_{xt} + \delta_{xx}\delta_{xxt}) dx \\ &= -2 \int_{\alpha}^{\beta} \delta \left[a\delta_x + b[(u_1)^p]_x - b[(u_2)^p]_x + d\delta_{xxx} \right] dx \\ &= 2b \int_{\alpha}^{\beta} \delta \left[[(u_2)^p]_x - [(u_1)^p]_x \right] dx - \left[a\delta^2 + 2d\delta\delta_{xx} - d(\delta_x)^2 \right] \Big|_{\alpha}^{\beta} \end{aligned}$$

$$\begin{aligned}
 &= 2bp \int_{\alpha}^{\beta} \delta \left[(u_2)^{p-1} (u_2)_x - (u_1)^{p-1} (u_1)_x \right] dx \\
 &= 2bp \int_{\alpha}^{\beta} \delta \delta_x (u_2)^{p-1} dx + 2bp \int_{\alpha}^{\beta} \delta \sum_{k=0}^{p-2} (u_2)^{p-2-k} (u_1)^k (u_1)_x \delta dx
 \end{aligned} \tag{4}$$

By Theorem 2.3, we obtain

$$\begin{aligned}
 \left| \int_{\alpha}^{\beta} \delta \delta_x (u_2)^{p-1} dx \right| &\leq C \int_{\alpha}^{\beta} |\delta| \cdot |\delta_x| dx \leq C \left(\int_{\alpha}^{\beta} \delta^2 dx + \int_{\alpha}^{\beta} (\delta_x)^2 dx \right), \\
 \left| \int_{\alpha}^{\beta} \delta \sum_{k=0}^{p-2} (u_2)^{p-2-k} (u_1)^k (u_1)_x \delta dx \right| &\leq C \int_{\alpha}^{\beta} \delta^2 dx,
 \end{aligned}$$

where C is a constant. Substituting the above two inequalities into Eq. (4), we obtain

$$\frac{dE(t)}{dt} \leq CE(t), \quad t \in [0, T]$$

This leads to $E(t) \leq e^{CT} E(0)$, $0 \leq t \leq T$. Thus, if $u_0^{(1)} = u_0^{(2)}$, we have $\delta(x, 0) = 0$ and hence $E(0) = 0$, implying that $E(t) = 0$. By the Sobolev inequality [22], we obtain $\|\delta\|_{L_{\infty}} = 0$ and $u_1 = u_2$. Furthermore, if $\delta(x, 0) < \varepsilon$, $\delta_x(x, 0) < \varepsilon$, $\delta_{xx}(x, 0) < \varepsilon$, we obtain $E(0) < \varepsilon$ and hence $E(t) \leq e^{CT} E(0) \leq \varepsilon e^{CT}$, $0 \leq t \leq T$, implying that the solution is continuously dependent on the initial condition. We conclude that Eqs. (1)-(3) is well-posed.

3. Difference scheme

In this section, we propose a three-level linear implicit finite difference scheme for the initial-boundary value problem of Eqs. (1)-(3). As usual, let $h = (\beta - \alpha)/J$ and τ be the uniform step sizes in the spatial and temporal directions, respectively. Denote $x_j = \alpha + jh$, $t_n = n\tau$, $0 \leq j \leq J$, $0 \leq n \leq N$, $\left\lceil \frac{T}{\tau} \right\rceil = N$, $u_j^n \approx u(x_j, t_n)$ and

$Z_h^0 = \{u = (u_j) | u_{-1} = u_0 = u_J = u_{J+1} = 0, -1 \leq j \leq J+1\}$. Throughout this article, we denote C as a general positive constant, which may have different values independent of h and τ , that varies in different occurrences in the context.

We then define the difference operators, inner product, and norms as follows:

$$\begin{aligned} (u_j^n)_{\bar{x}} &= \frac{u_{j+1}^n - u_j^n}{h}, \quad (u_j^n)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h}, \quad (u_j^n)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \\ (u_j^n)_{\bar{t}} &= \frac{u_j^{n+1} - u_j^n}{2\tau}, \quad \bar{u}_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2}, \quad (u_j^n)_{\hat{t}} = \frac{u_j^{n+1} - u_j^{n-1}}{\tau}, \\ \langle u^n, v^n \rangle &= h \sum_{j=1}^{J-1} u_j^n v_j^n, \quad \|u^n\|^2 = \langle u^n, u^n \rangle, \quad \|u^n\|_{\infty} = \max_{1 \leq j \leq J-1} |u_j^n|. \end{aligned}$$

Since $(u^p)_x = \frac{p}{p+1}[u^{p-1}u_x + (u^p)_x]$, we consider the following finite difference scheme for the problem (1)-(3):

$$\begin{aligned} (u_j^n)_{\bar{t}} + a(\bar{u}_j^n)_{\hat{x}} + \frac{bp}{p+1} \left\{ (u_j^n)^{p-1}(\bar{u}_j^n)_{\hat{x}} + \left[(u_j^n)^{p-1} \bar{u}_j^n \right]_{\hat{x}} \right\} - c(u_j^n)_{\bar{x}\bar{x}\hat{t}} \\ + d(\bar{u}_j^n)_{\bar{x}\bar{x}\hat{x}} + (u_j^n)_{\bar{x}\bar{x}\bar{x}\hat{t}} = 0, \quad 2 \leq j \leq J-2, \quad 2 \leq n \leq N, \end{aligned} \quad (5)$$

$$u_j^0 = u_0(x_j), \quad 0 \leq j \leq J, \quad (6)$$

$$u_0^n = u_J^n = 0, \quad u_{-1}^n = u_1^n = 0, \quad u_{J-1}^n = u_{J+1}^n = 0, \quad 1 \leq n \leq N. \quad (7)$$

It should be pointed out that since $u_0^n = 0$ and $(u_x)_0^n = 0$ based on Eq. (3), we may assume $u_{-1}^n = u_1^n = 0$ for simplicity. Similarly, we assume $u_{J-1}^n = u_J^n = u_{J+1}^n = 0$, where $j = -1$ and $J+1$ are ghost points, $1 \leq n \leq N$.

The following lemmas and theorems are some properties of the above finite difference scheme (5)-(7). They are essential for existence, uniqueness, convergence, and stability of the numerical solution.

Lemma 3.1 (See [14, 20]) For any two mesh functions, $u, v \in Z_h^0$, we have

$$\langle u_{\tilde{x}}, v \rangle = -\langle u, v_{\tilde{x}} \rangle, \quad \langle u_{\hat{x}}, v \rangle = -\langle u, v_{\hat{x}} \rangle, \quad \langle u_{\tilde{x}\tilde{x}}, v \rangle = -\langle u_{\tilde{x}}, v_{\tilde{x}} \rangle.$$

Then, we have $\langle u_{\tilde{x}\tilde{x}}, u \rangle = -\langle u_{\tilde{x}}, u_{\tilde{x}} \rangle = -\|u_{\tilde{x}}\|^2$. Furthermore, if $(u_0^n)_{\tilde{x}\tilde{x}} = (u_J^n)_{\tilde{x}\tilde{x}} = 0$, then $\langle u, u_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} \rangle = \|u_{\tilde{x}\tilde{x}}\|^2$.

Lemma 3.2 (See [1, 2]) For any mesh function $u \in Z_h^0$, we have

$$\langle u_{\hat{x}}, u \rangle = 0, \quad \langle u_{\tilde{x}\tilde{x}\hat{x}}, u \rangle = 0. \quad (8)$$

Lemma 3.3 (Discrete Sobolev's inequality [21]) For any discrete function $\{u_j^n | j = 0, 1, 2, \dots, J\}$ on the finite interval $[\alpha, \beta]$, there exist two constants C_1 and C_2 such that

$$\|u^n\|_\infty \leq C_1 \|u^n\| + C_2 \|u_{\tilde{x}}^n\|.$$

Theorem 3.4 For any mesh function $u \in Z_h^0$, we have

$$\langle (u^n)^{p-1}(\bar{u}^n)_{\hat{x}} + [(u^n)^{p-1}\bar{u}^n]_{\hat{x}}, \bar{u}^n \rangle = 0. \quad (9)$$

Proof. For any mesh function $u \in Z_h^0$, according to the discrete boundary conditions in Eq. (7) and the definition of Z_h^0 , we obtain

$$\begin{aligned} & \langle (u^n)^{p-1}(\bar{u}^n)_{\hat{x}} + [(u^n)^{p-1}\bar{u}^n]_{\hat{x}}, \bar{u}^n \rangle \\ &= h \sum_{j=1}^{J-1} \{ (u_j^n)^{p-1}(\bar{u}_j^n)_{\hat{x}} + [(u_j^n)^{p-1}\bar{u}_j^n]_{\hat{x}} \} \bar{u}_j^n \\ &= \frac{1}{2} \sum_{j=1}^{J-1} (u_j^n)^{p-1}(\bar{u}_{j+1}^n - \bar{u}_{j-1}^n) \bar{u}_j^n + [(u_{j+1}^n)^{p-1}\bar{u}_{j+1}^n - (u_{j-1}^n)^{p-1}\bar{u}_{j-1}^n] \bar{u}_j^n \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{j=1}^{J-1} \{ [(u_j^n)^{p-1} \bar{u}_{j+1}^n \bar{u}_j^n - (u_{j+1}^n)^{p-1} \bar{u}_{j+1}^n \bar{u}_j^n] - [(u_{j-1}^n)^{p-1} \bar{u}_j^n \bar{u}_{j-1}^n - (u_j^n)^{p-1} \bar{u}_j^n \bar{u}_{j-1}^n] \} \\
 &= 0
 \end{aligned}$$

Theorem 3.5 Suppose $u_0(x) \in H_0^2[\alpha, \beta]$, then the scheme in Eqs. (5)-(7) is conservative for discrete energy; that is, $E^n = E^{n-1} = \dots = E^0$, where

$$E^n = \|u^{n+1}\|^2 + \|u^n\|^2 + c(\|u_{\tilde{x}}^{n+1}\|^2 + \|u_{\tilde{x}}^n\|^2) + \|u_{\tilde{x}\tilde{x}}^{n+1}\|^2 + \|u_{\tilde{x}\tilde{x}}^n\|^2, \quad c \geq 0. \quad (10)$$

Proof. Computing the inner product of Eq. (5) with $2\bar{u}^n$, we obtain

$$\begin{aligned}
 &\langle (u^n)_{\hat{t}}, 2\bar{u}^n \rangle + a \langle (\bar{u}^n)_{\hat{x}}, 2\bar{u}^n \rangle + b \langle P, 2\bar{u}^n \rangle - c \langle (u^n)_{\tilde{x}\tilde{x}\hat{t}}, 2\bar{u}^n \rangle \\
 &\quad + d \langle (\bar{u}^n)_{\tilde{x}\tilde{x}\hat{x}}, 2\bar{u}^n \rangle + \langle (u^n)_{\tilde{x}\tilde{x}\tilde{x}\hat{t}}, 2\bar{u}^n \rangle = 0,
 \end{aligned} \quad (11)$$

where $P_j = \frac{p}{p+1} [(u_j^n)^{p-1} (\bar{u}_j^n)_{\hat{x}} + [(u_j^n)^{p-1} \bar{u}_j^n]_{\hat{x}}]$. By the definition of $(u_j^n)_{\hat{t}}$, it follows from the first term of Eq. (11) that

$$\langle (u^n)_{\hat{t}}, 2\bar{u}^n \rangle = \frac{1}{2\tau} (\|u^{n+1}\|^2 - \|u^{n-1}\|^2). \quad (12)$$

According to Lemma 3.2 and Theorem 3.4, we have

$$\langle (\bar{u}^n)_{\hat{x}}, 2\bar{u}^n \rangle = 0, \quad \langle (\bar{u}^n)_{\tilde{x}\tilde{x}\hat{x}}, 2\bar{u}^n \rangle = 0, \quad (13)$$

$$\langle P, 2\bar{u}^n \rangle = \frac{2p}{p+1} \langle [(u^n)^{p-1} (\bar{u}^n)_{\hat{x}} + [(u^n)^{p-1} \bar{u}^n]_{\hat{x}}], \bar{u}^n \rangle = 0. \quad (14)$$

By the definition of $(u_j^n)_{\hat{t}}$ and Lemma 3.1, it follows from the fourth term of Eq. (11)

that

$$\langle (u^n)_{\tilde{x}\tilde{x}\hat{t}}, 2\bar{u}^n \rangle = -\frac{1}{2\tau} (\|u_{\tilde{x}}^{n+1}\|^2 - \|u_{\tilde{x}}^{n-1}\|^2). \quad (15)$$

According to the discrete boundary conditions in Eq. (7) and Lemma 3.1, it follows from the sixth term of Eq. (11) that

$$\langle (u^n)_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{t}}, 2\bar{u}^n \rangle = \frac{1}{2\tau} (\|u_{\bar{x}\bar{x}}^{n+1}\|^2 - \|u_{\bar{x}\bar{x}}^{n-1}\|^2). \quad (16)$$

Substituting Eqs. (12)-(16) into Eq. (11), we have

$$\|u^{n+1}\|^2 + c\|u_{\bar{x}}^{n+1}\|^2 + \|u_{\bar{x}\bar{x}}^{n+1}\|^2 = \|u^{n-1}\|^2 + c\|u_{\bar{x}}^{n-1}\|^2 + \|u_{\bar{x}\bar{x}}^{n-1}\|^2, \quad c \geq 0. \quad (17)$$

By the definition of E^n in Eq. (10), Theorem 3.5 holds.

Theorem 3.6 Suppose $u_0(x) \in H_0^2[\alpha, \beta]$, then the solution u^n of Eqs. (5)-(7) satisfies $\|u^n\| \leq C$, $\|u_{\bar{x}}^n\| \leq C$, $\|u_{\bar{x}\bar{x}}^n\| \leq C$, which yield $\|u^n\|_\infty \leq C$, $\|u_{\bar{x}}^n\|_\infty \leq C$, $\|u_{\bar{x}\bar{x}}^n\|_\infty \leq C$, $\|u_{\bar{x}}^n\|_\infty \leq C$, $n = 1, 2, 3, \dots, N$.

Proof. It follows from Theorem 3.5 that $\|u^n\| \leq C$, $\|u_{\bar{x}\bar{x}}^n\| \leq C$. By Lemma 3.1 and the Cauchy-Schwartz inequality [13, 14], we get

$$\|u_{\bar{x}}^n\| \leq \|u_{\bar{x}\bar{x}}^n\|, \quad \|u_{\bar{x}}^n\|^2 \leq \|u^n\| \|u_{\bar{x}\bar{x}}^n\| \leq \frac{1}{2} (\|u^n\|^2 + \|u_{\bar{x}\bar{x}}^n\|^2) \leq C.$$

According to Lemma 3.3, we have $\|u^n\|_\infty \leq C$, $\|u_{\bar{x}}^n\|_\infty \leq C$, $\|u_{\bar{x}\bar{x}}^n\|_\infty \leq C$ and hence $\|u_{\bar{x}}^n\|_\infty \leq C$, $n = 1, 2, 3, \dots, N$.

4. Existence and uniqueness

In this section, we prove the solvability of the difference scheme in Eqs. (5)-(7). This guarantees the existence and uniqueness of the numerical solution.

Theorem 4.1 The difference scheme in Eqs. (5)-(7) is uniquely solvable.

Proof. Let $\bar{u}^n = (u_0^n, u_1^n, \dots, u_{J-1}^n, u_J^n)^T$, $0 \leq n \leq N$, and it is obvious that $\bar{u}^0$ is uniquely determined by the initial condition in Eq. (6). We can choose a second-order

method to compute $\vec{u}^1$ (such as the C-N scheme). Suppose that $\vec{u}^0, \vec{u}^1, \vec{u}^2, \dots, \vec{u}^n$ are solved uniquely, we consider the homogeneous Eq. (5) for $\vec{u}^{n+1}$ as

$$\begin{aligned} \frac{1}{2\tau}u_j^{n+1} + \frac{a}{2}(u_j^{n+1})_{\hat{x}} + \frac{bp}{2(p+1)} \left[(u_j^n)^{p-1}(u_j^{n+1})_{\hat{x}} + [(u_j^n)^{p-1}u_j^{n+1}]_{\hat{x}} \right] \\ - \frac{c}{2\tau}(u_j^{n+1})_{\hat{x}\hat{x}} + \frac{d}{2}(u_j^{n+1})_{\hat{x}\hat{x}\hat{x}} + \frac{1}{2\tau}(u_j^{n+1})_{\hat{x}\hat{x}\hat{x}\hat{x}} = 0. \end{aligned} \quad (18)$$

Taking the inner product of Eq. (18) with u_j^{n+1} , we can obtain from the boundary conditions in Eq. (7) and Lemma 3.1 that

$$\frac{1}{2\tau}\|u^{n+1}\|^2 + \frac{c}{2\tau}\|u_{\hat{x}}^{n+1}\|^2 + \frac{1}{2\tau}\|u_{\hat{x}\hat{x}}^{n+1}\|^2 = 0, \quad c \geq 0. \quad (19)$$

That is, Eq. (19) only admits a zero solution, implying there exists unique $\vec{u}^{n+1}$ that satisfies the scheme in Eqs. (5)-(7). This completes the proof of the existence and uniqueness of the new scheme.

5. Convergence and stability

In this section, we prove the convergence and stability of the scheme in Eqs. (5)-(7). The following lemma is essential for the proofs of convergence and stability of the new scheme.

Lemma 5.1 (Discrete Gronwall's inequality [10, 14]) Suppose that $w(k)$ and $\rho(k)$ are nonnegative functions and $\rho(k)$ is nondecreasing. If

$$w(k) \leq \rho(k) + C\tau \sum_{l=0}^{k-1} w(l), \quad C > 0,$$

then $w(k) \leq \rho(k)e^{C\tau k}$.

Theorem 5.2 Supposing $u_0(x) \in H_0^2[\alpha, \beta]$, then the solution $\bar{u}^n$ of the scheme in Eqs. (5)-(7) converges to the solution of the problem (1)-(3) and the rate of convergence is $O(\tau^2 + h^2)$ by the $\|\cdot\|_\infty$ norm.

Proof. Let $e_j^n = v_j^n - u_j^n$, where v_j^n and u_j^n are the solutions of the problem in Eqs. (1)-(3) and the scheme in Eqs. (5)-(7), respectively. Then, we obtain the following error equation:

$$R_j^n = (e_j^n)_t + a(\bar{e}_j^n)_{\hat{x}} + b(\Lambda_j^n + \Pi_j^n) - c(e_j^n)_{\hat{x}\hat{x}t} + d(\bar{e}_j^n)_{\hat{x}\hat{x}\hat{x}} + (e_j^n)_{\hat{x}\hat{x}\hat{x}t}, \quad (20)$$

and $e_{-1}^n = e_0^n = e_1^n = 0$, $e_{J-1}^n = e_J^n = e_{J+1}^n = 0$, where $R_j^n = O(\tau^2 + h^2)$ and

$$\Lambda_j^n = \frac{p}{p+1} [(v_j^n)^{p-1} (\bar{v}_j^n)_{\hat{x}} - (u_j^n)^{p-1} (\bar{u}_j^n)_{\hat{x}}],$$

$$\Pi_j^n = \frac{p}{p+1} [(v_j^n)^{p-1} \bar{v}_j^n - (u_j^n)^{p-1} \bar{u}_j^n]_{\hat{x}}.$$

Taking the inner product of Eq. (20) with $2\bar{e}_j^n$, we get

$$\begin{aligned} & \|e^{n+1}\|^2 - \|e^{n-1}\|^2 + c(\|e_{\hat{x}}^{n+1}\|^2 - \|e_{\hat{x}}^{n-1}\|^2) + \|e_{\hat{x}\hat{x}}^{n+1}\|^2 - \|e_{\hat{x}\hat{x}}^{n-1}\|^2 \\ &= 2\tau \langle R^n, 2\bar{e}^n \rangle - 2\tau b \langle \Lambda^n, 2\bar{e}^n \rangle - 2\tau b \langle \Pi^n, 2\bar{e}^n \rangle. \end{aligned} \quad (21)$$

From Theorem 2.3 and Theorem 3.6, we have

$$\begin{aligned} -\langle \Lambda^n, 2\bar{e}^n \rangle &= -\frac{2ph}{p+1} \sum_{j=1}^{J-1} \{ (v_j^n)^{p-1} (\bar{v}_j^n)_{\hat{x}} - (u_j^n)^{p-1} (\bar{u}_j^n)_{\hat{x}} \} \bar{e}_j^n \\ &= -\frac{2ph}{p+1} \sum_{j=1}^{J-1} (v_j^n)^{p-1} (\bar{e}_j^n)_{\hat{x}} \bar{e}_j^n - \frac{2ph}{p+1} \sum_{j=1}^{J-1} [(v_j^n)^{p-1} - (u_j^n)^{p-1}] (\bar{u}_j^n)_{\hat{x}} \bar{e}_j^n \\ &= -\frac{2ph}{p+1} \sum_{j=1}^{J-1} (v_j^n)^{p-1} (\bar{e}_j^n)_{\hat{x}} \bar{e}_j^n - \frac{2ph}{p+1} \sum_{j=1}^{J-1} \left[e_j^n \sum_{k=0}^{p-2} (v_j^n)^{p-2-k} (u_j^n)^k \right] (\bar{u}_j^n)_{\hat{x}} \bar{e}_j^n \end{aligned}$$

$$\begin{aligned}
 &\leq Ch \sum_{j=1}^{J-1} \left| (\bar{e}_j^n)_{\hat{x}} \right| \cdot \left| \bar{e}_j^n \right| + Ch \sum_{j=1}^{J-1} \left| e_j^n \right| \cdot \left| \bar{e}_j^n \right| \\
 &\leq C(\|\bar{e}_{\hat{x}}^n\|^2 + \|e^n\|^2 + \|\bar{e}^n\|^2) \\
 &\leq C(\|e_{\hat{x}}^{n+1}\|^2 + \|e_{\hat{x}}^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2). \tag{22}
 \end{aligned}$$

From Theorem 2.3, Theorem 3.1 and Theorem 3.6, we have

$$\begin{aligned}
 -\langle \Pi^n, 2\bar{e}^n \rangle &= -\frac{p}{p+1} \langle [(v^n)^{p-1} \bar{v}^n - (u^n)^{p-1} \bar{u}^n]_{\hat{x}}, 2\bar{e}^n \rangle \\
 &= \frac{p}{p+1} \langle [(v^n)^{p-1} \bar{v}^n - (u^n)^{p-1} \bar{u}^n], 2\bar{e}_{\hat{x}}^n \rangle \\
 &= \frac{2ph}{p+1} \sum_{j=1}^{J-1} \{ (v_j^n)^{p-1} \bar{e}_j^n + [(v_j^n)^{p-1} - (u_j^n)^{p-1}] \bar{u}_j^n \} (\bar{e}_j^n)_{\hat{x}} \\
 &= \frac{2ph}{p+1} \sum_{j=1}^{J-1} (v_j^n)^{p-1} \bar{e}_j^n (\bar{e}_j^n)_{\hat{x}} + \frac{2ph}{p+1} \sum_{j=1}^{J-1} \left\{ \bar{u}_j^n \sum_{k=0}^{p-2} \left[(v_j^n)^{p-2-k} (u_j^n)^k \right] e_j^n (\bar{e}_j^n)_{\hat{x}} \right\} \\
 &\leq Ch \sum_{j=1}^{J-1} \left| (\bar{e}_j^n)_{\hat{x}} \right| \cdot \left| \bar{e}_j^n \right| + Ch \sum_{j=1}^{J-1} \left| e_j^n \right| \cdot \left| (\bar{e}_j^n)_{\hat{x}} \right| \\
 &\leq C(\|\bar{e}_{\hat{x}}^n\|^2 + \|e^n\|^2 + \|\bar{e}^n\|^2) \\
 &\leq C(\|e_{\hat{x}}^{n+1}\|^2 + \|e_{\hat{x}}^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2). \tag{23}
 \end{aligned}$$

Furthermore, we have

$$\langle R^n, 2\bar{e}^n \rangle = \langle R^n, e^{n+1} + e^{n-1} \rangle \leq \|R^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2). \tag{24}$$

Substituting Eqs. (22)-(24) into Eq. (21), we have

$$\begin{aligned} & \|e^{n+1}\|^2 - \|e^{n-1}\|^2 + c(\|e_{\tilde{x}}^{n+1}\|^2 - \|e_{\tilde{x}}^{n-1}\|^2) + \|e_{\tilde{x}\tilde{x}}^{n+1}\|^2 - \|e_{\tilde{x}\tilde{x}}^{n-1}\|^2 \\ & \leq 2\tau\|R^n\|^2 + C\tau(\|e_{\tilde{x}}^{n+1}\|^2 + \|e_{\tilde{x}}^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2). \end{aligned} \quad (25)$$

Setting $B^n = \|e^{n+1}\|^2 + \|e^n\|^2 + c(\|e_{\tilde{x}}^{n+1}\|^2 + \|e_{\tilde{x}}^n\|^2) + \|e_{\tilde{x}\tilde{x}}^{n+1}\|^2 + \|e_{\tilde{x}\tilde{x}}^n\|^2$, then Eq. (25) can be rewritten as

$$B^n - B^{n-1} \leq 2\tau\|R^n\|^2 + C\tau(B^n + B^{n-1}).$$

Hence, we obtain

$$(1 - C\tau)(B^n - B^{n-1}) \leq 2\tau\|R^n\|^2 + 2C\tau B^{n-1}.$$

If τ is sufficiently small such that $1 - C\tau > 0$, then

$$B^n - B^{n-1} \leq C\tau\|R^n\|^2 + C\tau B^{n-1}. \quad (26)$$

Summarizing Eq. (26) from 1 to n , we get

$$B^n \leq B^0 + C\tau \sum_{l=1}^n \|R^l\|^2 + C\tau \sum_{l=1}^n B^{l-1}.$$

Thus, if we use a second-order method to compute $\vec{u}^1$ such that

$$B^0 \leq O(\tau^2 + h^2)^2, \quad \tau \sum_{l=1}^n \|R^l\|^2 \leq n\tau \max_{1 \leq l \leq n} \|R^l\|^2 \leq T \cdot O(\tau^2 + h^2)^2,$$

then we obtain from Lemma 5.1 that $B^n \leq O(\tau^2 + h^2)^2$. It implies

$$\|e^n\| \leq O(\tau^2 + h^2), \quad \|e_{\tilde{x}\tilde{x}}^n\| \leq O(\tau^2 + h^2).$$

By Lemma 3.1 and the Cauchy-Schwartz inequality [13], we get

$$\|e_{\tilde{x}}^n\| \leq \|e^n\|^{\frac{1}{2}} \|e_{\tilde{x}\tilde{x}}^n\|^{\frac{1}{2}} \leq \frac{1}{2}(\|e^n\| + \|e_{\tilde{x}\tilde{x}}^n\|) \leq O(\tau^2 + h^2).$$

According to Lemma 3.3, we obtain $\|e^n\|_{\infty} \leq O(\tau^2 + h^2)$. This completes the proof.

Using a similar proof for Theorem 5.2, we can obtain Theorem 5.3.

Theorem 5.3. Supposing $u_0(x) \in H_0^2[\alpha, \beta]$, then the solution $\vec{u}^n$ of the scheme in Eqs. (5)-(7) is stable in the sense of norm $\|\cdot\|_{\infty}$.

6. Computation procedure

The computation procedure of the scheme in Eqs. (5)-(7) can be described as follows:

Step 1 Solve $\vec{u}^0$ based on Eq. (6);

Step 2 Solve $\vec{u}^1$ based on the following two-level finite difference scheme as:

$$\begin{aligned} (u_j^0)_{\tilde{t}} + a(u_j^{0.5})_{\tilde{x}} + \frac{bp}{p+1} \left[(u_j^0)^{p-1} (u_j^{0.5})_{\tilde{x}} + [(u_j^0)^{p-1} (u_j^{0.5})]_{\tilde{x}} \right] - c(u_j^0)_{\tilde{x}\tilde{x}\tilde{t}} \\ + d(u_j^{0.5})_{\tilde{x}\tilde{x}\tilde{x}} + (u_j^0)_{\tilde{x}\tilde{x}\tilde{x}\tilde{t}} = 0, \quad 0 \leq j \leq J, \end{aligned} \quad (27)$$

with $u_j^0 = u_0(x_j)$, $u_0^1 = u_J^1 = 0$, $u_{-1}^1 = u_1^1 = 0$, $u_{J-1}^1 = u_{J+1}^1 = 0$, where $u_j^{0.5} = \frac{1}{2}(u_j^0 + u_j^1)$.

Step 3 Solve $\vec{u}^{n+1}$ based on Eqs. (5)-(7).

7. Numerical experiments

In this section, three numerical examples will be demonstrated to examine the robustness and accuracy of the three-level scheme in Eqs. (5)-(7) and Eq. (27). The

accuracy of the scheme is measured by calculating the L_∞ and L_2 error norms of the solution.

Example 1 Consider the parameters ($a=1$, $b=0.5$, $c=0$, $d=1$, $p=2$) in Eq. (1), which is the Rosenau-KdV equation [1]

$$u_t + u_x + 0.5(u^2)_x + u_{xxx} + u_{xxxxt} = 0, \quad \alpha \leq x \leq \beta, \quad t \in [0, T]. \quad (28)$$

The initial condition was chosen to be $u_0(x) = k_{11} \operatorname{sech}^4(k_{12}x)$ such that the exact solitary wave solution is $u(x, t) = k_{11} \operatorname{sech}^4[k_{12}(x - k_{13}t)]$, where [23]

$$k_{11} = -\frac{35}{24} + \frac{35}{312}\sqrt{313}, \quad k_{12} = \frac{1}{24}\sqrt{-26 + 2\sqrt{313}}, \quad k_{13} = \frac{1}{2} + \frac{\sqrt{313}}{26}.$$

Errors and rate of convergence results in term of L_∞ and L_2 norms at the time $T=20$ by using $\alpha = -70$ and $\beta=100$ are reported in Tables 1-2. It can be seen that the computational efficiency of the present scheme in Eqs. (5)-(7) is slightly better than that of the method in [23], and the second-order convergence of the numerical solutions is also verified. We continue with the examination of the soliton profile in Figure 1. From Figure 1, one can find that the patterns of the numerical solutions are in excellent agreement with the exact solutions.

Example 2 Consider the parameters ($a=1$, $b=1$, $c=0$, $d=1$, $p=5$) in Eq. (1), which is the Rosenau-KdV equation [13]

$$u_t + u_x + (u^5)_x + u_{xxx} + u_{xxxxt} = 0, \quad \alpha \leq x \leq \beta, \quad t \in [0, T]. \quad (29)$$

The initial condition was chosen to be $u_0(x) = k_{21} \operatorname{sech}(k_{22}x)$ such that the exact solitary wave solution is $u(x, t) = k_{21} \operatorname{sech}[k_{22}(x - k_{23}t)]$, where

$$k_{21} = \sqrt[4]{\frac{4}{15}(-5 + \sqrt{34})}, \quad k_{22} = \frac{1}{3}\sqrt{-5 + \sqrt{34}}, \quad k_{23} = \frac{1}{10}(5 + \sqrt{34}).$$

We took $\alpha = -60$, $\beta = 90$, $T = 40$ and conducted numerical simulations in different time steps and space steps, respectively. The comparison on wave graph from numerical solution at various times is given in Figure 2. The figure shows that the height of the wave graph at different times is almost identical. It implies that the energy is conservative. We also listed the comparison on errors using L_∞ at $T = 40$ in Table 3. From Table 3, one can find that the computational efficiency of the new scheme is slightly better than those of the methods in [13]. Finally, we listed the conservative invariant E^n at different times in Table 4. The obtained results testify that the new scheme is conservative for energy.

Example 3 Consider the parameters ($a=1$, $b=0.5$, $c=0.1$, $d=1$, $p=2$) in Eq. (1), which is the Rosenau-KdV-RLW equation [2]

$$u_t + u_x + 0.5(u^2)_x - 0.1u_{xxt} + u_{xxx} + u_{xxxxt} = 0, \quad \alpha \leq x \leq \beta, \quad t \in [0, T]. \quad (30)$$

The initial condition was chosen to be $u_0(x) = k_{31} \text{sech}^4(k_{32}x)$ such that the exact solitary wave solution is $u(x, t) = k_{31} \text{sech}^4[k_{32}(x - k_{33}t)]$, where [1]

$$k_{31} = -\frac{5}{456}(25 - 13\sqrt{457}), k_{32} = \left(\frac{-13 + \sqrt{457}}{288}\right)^{1/2}, \quad k_{33} = \frac{241 + 13\sqrt{457}}{266}.$$

We took $\alpha = -40$, $\beta = 100$, and listed the comparison of errors using L_∞ and L_2 at $T = 30$ in Table 5. From Table 5, one can find that the computational efficiency of the new scheme is slightly better than that of the method in [1]. Then, we listed the errors using L_2 - and L_∞ -norm and rates of convergence for $h = \tau$ at $T = 30$ in Table 6. As shown in Table 6, the estimated rates of convergence are close to the theoretically predicted second-order rate of convergence. The convergence order of $\log(L)$ - $\log(h)$ is given in Figure 3(a) under various mesh steps h and τ . From

Figure 3(a), it is obvious that scheme in Eqs. (5)-(7) is convergent in the L_2 - and L_∞ -norm, and the convergence order is $O(\tau^2 + h^2)$. Figure 3(b) is presented to show the conservative laws of discrete energy E^n computed by scheme in Eqs. (5)-(7) when $h = \tau = 0.25$. It is seen from Figure 3(b) that scheme in Eqs. (5)-(7) preserves the discrete energy very well, implying that it can be used for computing for a long period. Exact solutions of $u(x, t)$ at $T = 0$ and numerical solutions at $T = 10, 30$ with $h = \tau = 0.5$ and 0.25 are given in Figure 4. The figure shows that the patterns of the numerical solutions are in excellent agreement with the exact solutions. The wave graph comparison of numerical solution between different time steps and space steps ($\tau = h = 0.125, 0.0625$) at various times is given in Figure 5. The figure shows that the height of the wave graph at different times ($T = 0, 20, 40$) is almost identical. It implies that the energy is also conservative.

8. Conclusion

A three-level linear conservative finite difference scheme for the general Rosenau-KdV-RLW equation has been introduced and analyzed, which gives the well resolution for the Rosenau-KdV-RLW equation. The present method gives an implicit linear system, which can be easily implemented by solving a penta-diagonal system. The solvability, energy conservation, stability, and convergence with order of $O(\tau^2 + h^2)$ have been analyzed in details. Numerical experiments show that the present method supports the theoretical analysis of convergence rate.

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Table 1. Comparison of errors using L_∞ -norm at $T = 20$.

Scheme	$\tau = h = 0.2$	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$
Scheme [23]	—	1.1314E-03	2.8359E-04	7.0979E-05
Scheme (5)-(7)	7.8920E-04	1.8771E-04	4.6987E-05	1.1751E-05

Table 2. Errors and rates of convergence when $h = \tau$ at $T = 20$.

h	L_2	Rate	L_∞	Rate
0.2	9.00216E-04	—	7.8920E-04	—
0.1	1.13342E-04	1.98959	1.8771E-04	2.071876
0.05	1.41962E-05	1.99710	4.6987E-05	1.998163
0.025	1.77567E-06	1.99908	1.1751E-05	1.999484

Table 3. Comparison of errors using L_∞ when $h = \tau$ at $T = 40$.

h	L_∞ [13]	L_∞ Scheme (5)-(7)
0.25	1.79985E-02	9.23107E-03
0.125	4.56804E-03	2.33207E-03
0.0625	1.14689E-03	5.84749E-04
0.03125	2.87085E-04	1.46333E-04

Table 4. Energy E^n in different time steps and space steps when $\tau = h$.

h	$T = 10$	$T = 20$	$T = 30$	$T = 40$
0.25	6.221349804819	6.221349804820	6.221349804820	6.221349804820
0.125	6.221405877565	6.221405877551	6.221405877549	6.221405877541
0.0625	6.221419928242	6.221419928522	6.221419928339	6.221419928294
0.03125	6.221423440815	6.221423442055	6.221423439509	6.221423438484

Table 5. Comparison of errors using L_∞ and L_2 at $T = 30$ when $h = \tau$.

h	L_∞ [1]	L_∞ Scheme (5)-(7)	L_2 [1]	L_2 Scheme (5)-(7)
0.25	9.86753E-01	6.99597E-01	2.94337E+00	1.86617E-00
0.125	2.14488E-01	1.97127E-01	8.05629E-01	5.18662E-01
0.0625	5.19201E-02	5.06954E-02	2.05276E-01	1.33174E-01
0.03125	1.28858E-02	1.27669E-02	5.15696E-02	3.35296E-02

Table 6. Errors and rates of convergence when $h = \tau$ at $T = 30$.

h	L_2	Rate	L_∞	Rate
0.5	1.86617E-00	—	6.99597E-01	—
0.25	5.18662E-01	1.84721	1.97127E-01	1.82740
0.125	1.33174E-01	1.96149	5.06954E-02	1.95920
0.0625	3.35296E-02	1.98980	1.27669E-02	1.98945

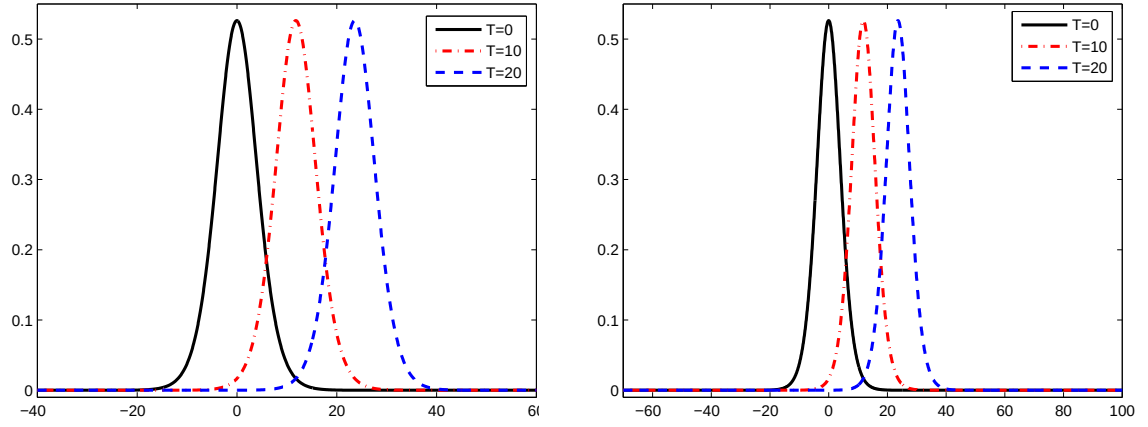


Figure 1: Exact solutions of $u(x, t)$ at $T = 0$ and numerical solutions of the Rosenau-KdV equation with $\tau = h = 0.05$ at $T = 10, 20$, $\alpha = -40$, $\beta = 60$ (left) and $\alpha = -70$, $\beta = 100$ (right).

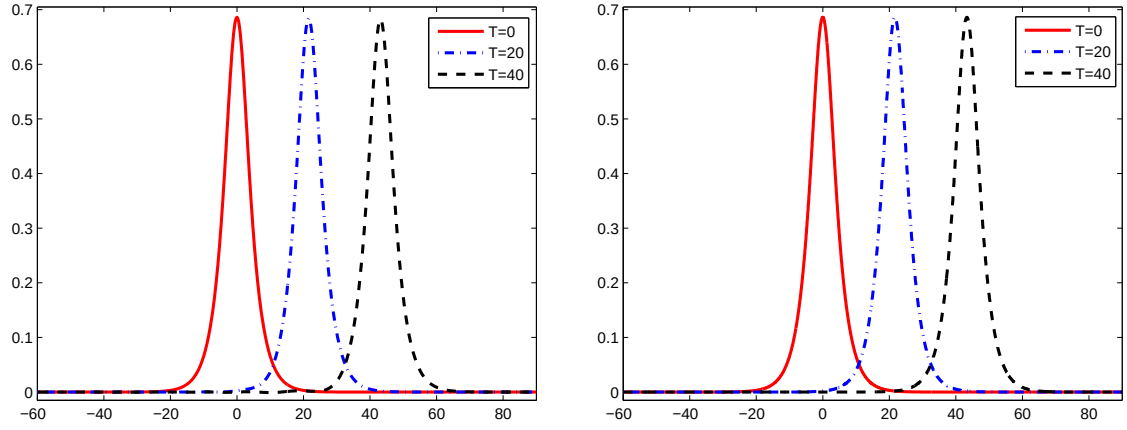


Figure 2: Wave graph of $u(x, t)$ at various times when $\tau = h = 0.25$ (left) and $\tau = h = 0.03125$ (right).

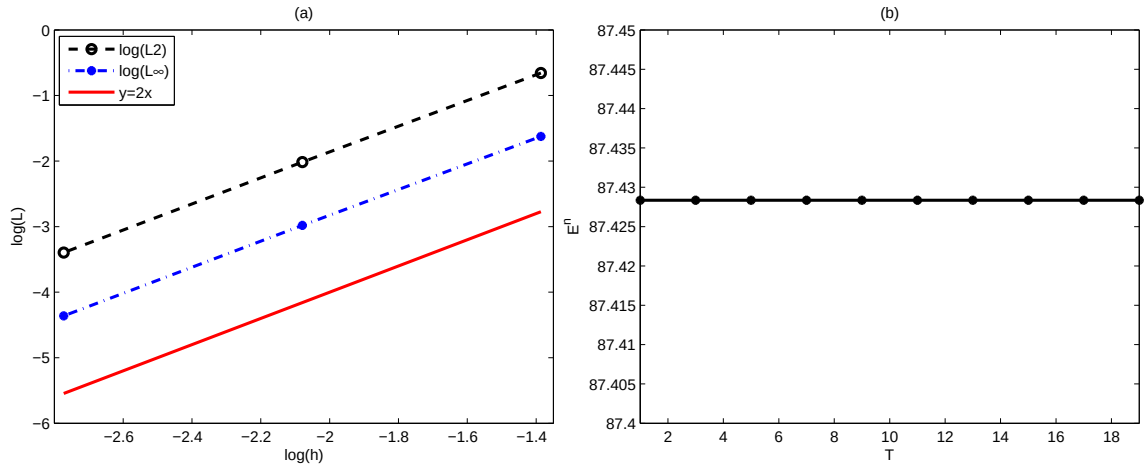


Figure 3: The convergence order (left) and discrete conservative energy (right).

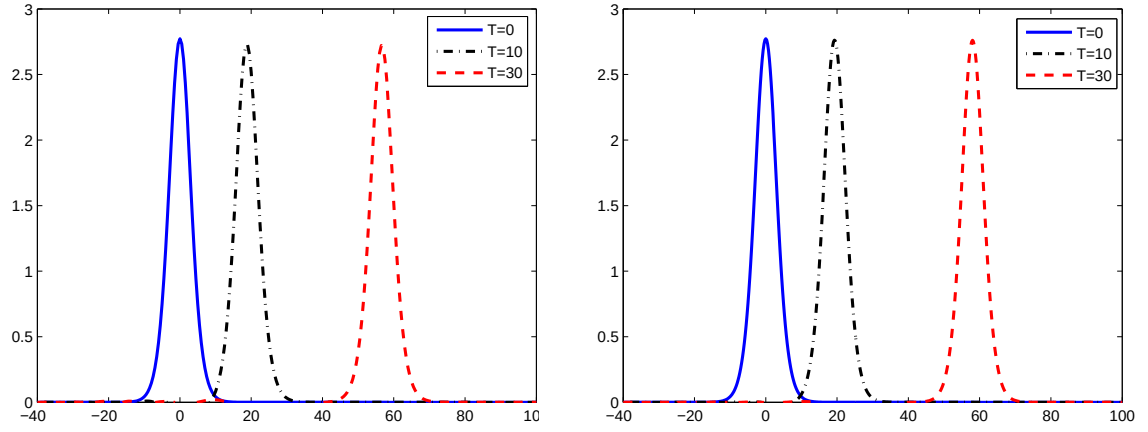


Figure 4: Exact solutions of $u(x, t)$ at $T = 0$ and numerical solutions when $\alpha = -40$, $\beta = 100$ at $T = 10, 30$ with $h = \tau = 0.5$ (left) and $h = \tau = 0.25$ (right).

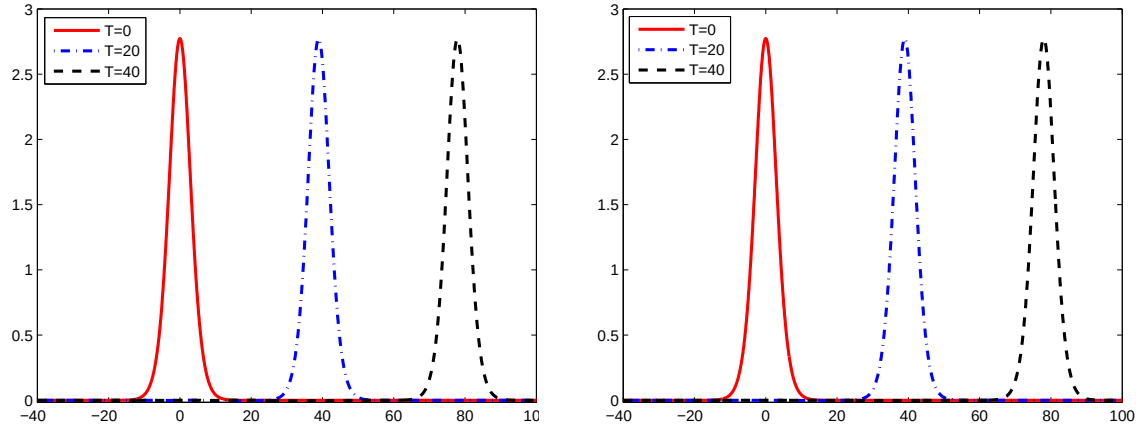


Figure 5: Wave graph of $u(x, t)$ at various times when $\tau = h = 0.125$ (left) and $\tau = h = 0.0625$ (right).