



A conjugate gradient projection method for solving equations with convex constraints[☆]

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ABSTRACT

In this paper, we propose a derivative-free iterative method for a class of equations with convex constraints appearing in a variety of the practical problems such as compressing sensing, fluid mechanics, plasma physics, nonlinear optics and solid state. In the iteration, our search direction can be viewed as an extension of a modified three-term CG method. By an appropriate line search and the projection step, our method is convergent to the solution. Our method inherits the advantages of CG method and projection method, and thus is suitable for solving large-scale non-smooth problem. Under the assumption that F is Lipschitz continuous and satisfies a weaker condition of monotonicity, our method is globally convergent. Numerical results show the efficiency of the proposed method.

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1. Introduction

Let $F(x)$ be a continuous function with convex constraints, i.e.

$$F(x) = 0, \quad x \in \Omega, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is a nonempty closed convex set. This problem can be met in many areas, such as the ℓ_1 -norm regularized optimization arising from compressing sensing [1], the variational inequality problems [2], and nonlinear evolution problems in the quantum mechanics and nonlinear optics [3–6], fluid mechanics, plasma physics, and solid state [7–11]. Many iterative methods have been proposed to solve the problem (1.1) when $\Omega = \mathbb{R}^n$, such as Newton method, fixed-point method, quasi-Newton method, spectral gradient method and conjugate gradient method. Among these methods, the conjugate gradient (CG) method is one of the most suitable methods for solving large-scale problems because of the lower storage requirement and simple iterative structure. People may refer to [12–18] for details.

The constraint problem (1.1) is an important problem due to its extensive application, which attracts the attention of many researchers [19–22]. In 1998, Solodov and Svaiter [23] combined the Newton proximal point and projection method to solve the system of monotone equations when $\Omega \neq \mathbb{R}^n$. Under some appropriate assumptions, the super-linear rate of convergence is achieved. Due to the non-expansion property and the derivative-free feature, the projection methods are widely used to solve constraint or unconstrained optimization problems and equations. The most widely used methods for solving the problem (1.1) usually contain the following 3 steps: (i) finding a descent direction, which promises that the next iterative point is closer to the solution along the direction with some step-size; (ii) constructing

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the separation hyperplane, which separates the current iterative point from the solution point based on the separation theorem; (iii) projecting the current iterative point onto the separation hyperplane, which promises the convergence of the whole algorithm under certain assumptions. For example, based on the hyperplane projection technique of [23], Yu et al. [24] proposed a derivative-free method to solve (1.1) without solving any sub-problems. Their search direction, which is a negative value of function F at the next iterative point, is modified from the spectral gradient direction. Xiao and Zhu [25] modified the CG-descent method to solve convex constrained monotone problems, with application in compressive sensing problem. To establish a more effective method, Liu and Li [26] further studied the algorithm [25], by modifying the CGD method. In their method, the next search direction is composed by a weighted sum of a negative value of function F at the next iterative point and the current search direction. The numerical results indicate that their method is very promising. To improve the convergent behavior, many work have been done in recent years, such as [27–31].

It is noted that in [32], an efficient three-term CG method is proposed to solve unconstrained optimization problems, in which the search direction is sufficiently descent and independent of the line search. It is also noted that in [27], a projection method is successfully proposed for solving the equations with the convex constraint. In this paper, by employing the idea of the projection method introduced in [27], we effectively extend the three-term CG method in [32] for solving the equations with convex constraint. Our search direction is also in three-term with derivative-free feature and satisfies descent condition. Under appropriate assumptions that functions are Lipschitz continuous and monotone, the proposed method is convergent. Numerical experiments show the efficiency of our method. The paper is organized as follows. After introducing some properties of the projection operator, we introduce our method in Section 2. In Section 3, we analyze the global convergence of the algorithm. Numerical experiments are shown in Section 4. At last, we make a conclusion of our work in Section 5. Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm, $\langle \cdot, \cdot \rangle$ denotes inner product and F_k denotes $F(x_k)$ for convenience.

2. Algorithm

To illustrate our new method, we first give the definition of projection operator. The projection operator is defined as a mapping from R^n to a nonempty closed convex set Ω , i.e.,

$$P_{\Omega}(x) := \arg \min_{z \in \Omega} \|z - x\|, \quad x \in R^n.$$

This projection operator satisfies the properties:

$$\langle P_{\Omega}(x) - x, z - P_{\Omega}(x) \rangle \geq 0, \quad \forall x \in R^n, z \in \Omega, \quad (2.1)$$

$$\|P_{\Omega}(x) - P_{\Omega}(y)\| \leq \|x - y\|, \quad \forall x, y \in R^n. \quad (2.2)$$

In this paper, the next iterative point comes from projection, and the search direction d_k is defined as follows:

$$d_k = \begin{cases} -F_k, & k = 0, \\ -\sigma_1 F_k + \beta_k d_{k-1} + \sigma_2 c_k w_{k-1}, & k > 0, \end{cases} \quad (2.3)$$

where $\sigma_1, \sigma_2 > 0$,

$$c_k = \frac{F_k^T d_{k-1}}{w_{k-1}^T d_{k-1}},$$

$$\lambda_k = 1 + \max \left\{ 0, -\frac{y_{k-1}^T d_{k-1}}{\|d_{k-1}\|^2} \right\},$$

$$\beta_k = \frac{F_k^T w_{k-1} - 2c_k \|w_{k-1}\|^2}{d_{k-1}^T w_{k-1}}$$

and $y_{k-1} = F_k - F_{k-1}$, $w_{k-1} = y_{k-1} + \lambda_k d_{k-1}$. In the following, the flowchart of the method is given.

Algorithm 2.1

Step1. Given $x_0 \in R^n$, $\rho \in (0, 1)$, $\beta, \sigma, \varepsilon > 0$, and $\sigma_1, \sigma_2 > 0$. Set $k := 0$.

Step2. Compute F_k . If $\|F_k\| \leq \varepsilon$, stop.

Step3. Compute d_k by (2.3).

Step4. Set $z_k = x_k + \alpha_k d_k$, where $\alpha_k = \max\{\beta \rho^i, i = 0, 1, 2, \dots\}$ satisfies

$$-F(z_k)^T d_k \geq \sigma \alpha_k \|F(z_k)\| \cdot \|d_k\|^2.$$

Step5. The next iterative point: $x_{k+1} = P_{\Omega}(x_k - \tau_k F(z_k))$, where

$$\tau_k = \frac{\langle x_k - z_k, F(z_k) \rangle}{\|F(z_k)\|^2}.$$

Step6. Set $k := k + 1$ and turn to the Step2.

The following lemma indicates that the search direction d_k of Algorithm 2.1 satisfies the sufficiently descent property, which guarantees the existence of α_k in Step 4.

Lemma 2.1. *Let the sequences $\{d_k\}$ and $\{F_k\}$ be generated by Algorithm 2.1. If $0 < \sigma_1 \leq 1$ and $\sigma_1 - \frac{(1+\sigma_2)^2}{8\sigma_1} > 0$ hold, then there is a non-negative constant $\Gamma_1 > 0$ such that*

$$d_k^T F_k \leq -\Gamma_1 \|F_k\|^2, \quad \forall k \geq 0. \quad (2.4)$$

Proof. For $k = 0$, the conclusion (2.4) holds by choosing $\Gamma_1 = 1$. For $k \geq 1$, from the definition of d_k in (2.3), we have

$$\begin{aligned} & F_k^T d_k \\ &= -\sigma_1 \|F_k\|^2 + \beta_k d_{k-1}^T F_k + \sigma_2 c_k F_k^T w_{k-1} \\ &= -\sigma_1 \|F_k\|^2 + \frac{F_k^T w_{k-1} F_k^T d_{k-1}}{w_{k-1}^T d_{k-1}} - 2 \frac{(F_k^T d_{k-1})^2 \|w_{k-1}\|^2}{(w_{k-1}^T d_{k-1})^2} + \sigma_2 \frac{F_k^T d_{k-1} F_k^T w_{k-1}}{w_{k-1}^T d_{k-1}} \\ &\leq -\sigma_1 \|F_k\|^2 + \frac{F_k^T w_{k-1} F_k^T d_{k-1}}{w_{k-1}^T d_{k-1}} - 2\sigma_1 \frac{(F_k^T d_{k-1})^2 \|w_{k-1}\|^2}{(w_{k-1}^T d_{k-1})^2} + \sigma_2 \frac{F_k^T d_{k-1} F_k^T w_{k-1}}{w_{k-1}^T d_{k-1}} \\ &= -\sigma_1 \|F_k\|^2 + \frac{(1 + \sigma_2) F_k^T w_{k-1} F_k^T d_{k-1} w_{k-1}^T d_{k-1} - 2\sigma_1 (F_k^T d_{k-1})^2 \|w_{k-1}\|^2}{(w_{k-1}^T d_{k-1})^2} \\ &= -\sigma_1 \|F_k\|^2 + \frac{\frac{(1+\sigma_2)}{2\sqrt{2\sigma_1}} F_k^T (w_{k-1}^T d_{k-1}) 2\sqrt{2\sigma_1} w_{k-1} (F_k^T d_{k-1})}{(w_{k-1}^T d_{k-1})^2} - \frac{2\sigma_1 (F_k^T d_{k-1})^2 \|w_{k-1}\|^2}{(w_{k-1}^T d_{k-1})^2} \\ &\leq -\sigma_1 \|F_k\|^2 + \frac{\frac{(1+\sigma_2)^2}{8\sigma_1} \|F_k\|^2 (w_{k-1}^T d_{k-1})^2 + 2\sigma_1 \|w_{k-1}\|^2 (F_k^T d_{k-1})^2}{(w_{k-1}^T d_{k-1})^2} - \frac{2\sigma_1 (F_k^T d_{k-1})^2 \|w_{k-1}\|^2}{(w_{k-1}^T d_{k-1})^2} \\ &= -\left(\sigma_1 - \frac{(1 + \sigma_2)^2}{8\sigma_1}\right) \|F_k\|^2, \end{aligned}$$

where the first inequality holds by $0 < \sigma_1 \leq 1$, and the second inequality is satisfied as $2A^T B \leq \|A\|^2 + \|B\|^2$. By setting $\Gamma_1 = \sigma_1 - \frac{(1+\sigma_2)^2}{8\sigma_1}$, the proof can be done. The following lemma shows that the line search in Step 4 is executable, and can be proved in a similar way to that in [27].

Lemma 2.2. *Let the direction sequence $\{d_k\}$ be generated by Algorithm 2.1, then there always exists a step-size α_k satisfying the line search.*

3. Convergence analysis

In this section, we analyze the global convergence of Algorithm 2.1. For convenience, we assume $F_k \neq 0$, otherwise the solution can be obtained analytically. Additionally, we need some assumptions for the objective function F .

Assumption 3.1. The solution set of the problem (1.1) is denoted by S , and S is nonempty.

Assumption 3.2. The function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz continuous, i.e., there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \quad (3.1)$$

Assumption 3.3. For $\forall x \in S, y \in \mathbb{R}^n$, it holds that

$$\langle F(y), (y - x) \rangle \geq 0. \quad (3.2)$$

Obviously, Assumption 3.3 is a weaker condition than the monotonicity of $F(x)$.

Lemma 3.1. *Suppose that Assumptions 3.1–3.3 hold. Let the sequences $\{x_k\}$ and $\{z_k\}$ be generated by Algorithm 2.1, then the sequence $\{F(z_k)\}$ is bounded, and $\lim_{k \rightarrow \infty} \|z_k - x_k\| = 0$.*

Proof. For $\forall x^* \in S$, by Assumption 3.3 and the line search defined in Step 4, we have

$$\begin{aligned} \langle F(z_k), x_k - x^* \rangle &= \langle F(z_k), x_k - z_k \rangle + \langle F(z_k), z_k - x^* \rangle \\ &\geq \langle F(z_k), x_k - z_k \rangle \\ &\geq \sigma \|\alpha_k d_k\|^2 \|F(z_k)\| \geq 0. \end{aligned} \quad (3.3)$$

Table 1
Numerical performance on Problem 1.

Dim	Ini	Algorithm2.1	PDY	SP
		TI/NF/TIME/NROM	TI/NF/TIME/NROM	TI/NF/TIME/NROM
1000	x_1	11/25/0.019/1.82E−07	15/31/0.036/8.79E−07	39/41/0.011/9.66E−07
	x_2	10/21/0.004/1.62E−07	15/31/0.006/9.42E−07	40/42/0.004/7.57E−07
	x_3	11/26/0.003/1.85E−07	16/34/0.006/7.89E−07	42/45/0.004/8.73E−07
	x_4	12/39/0.003/1.85E−07	19/56/0.005/3.34E−07	47/57/0.004/8.87E−07
	x_5	12/35/0.005/1.86E−07	17/44/0.011/4.92E−07	46/53/0.005/9.17E−07
	x_6	9/20/0.002/9.56E−07	16/34/0.003/3.91E−07	43/46/0.003/8.35E−07
	x_7	9/19/0.003/4.44E−07	14/29/0.007/5.13E−07	37/39/0.002/6.84E−07
5000	x_1	11/25/0.023/4.07E−07	16/33/0.100/6.59E−07	41/43/0.032/9.60E−07
	x_2	10/21/0.008/3.61E−07	16/33/0.033/7.00E−07	42/44/0.018/7.53E−07
	x_3	11/26/0.008/4.14E−07	18/40/0.025/5.11E−07	44/47/0.017/8.67E−07
	x_4	12/39/0.010/4.14E−07	23/96/0.063/9.26E−07	49/59/0.022/8.81E−07
	x_5	12/35/0.010/4.16E−07	16/52/0.030/7.82E−07	48/55/0.028/9.11E−07
	x_6	10/22/0.004/3.42E−07	16/34/0.020/8.74E−07	45/48/0.037/8.30E−07
	x_7	9/19/0.004/9.94E−07	15/31/0.016/3.86E−07	39/41/0.017/6.80E−07
10 000	x_1	11/25/0.035/5.75E−07	16/33/0.045/9.32E−07	42/44/0.042/9.05E−07
	x_2	10/21/0.013/5.11E−07	16/33/0.023/9.90E−07	43/45/0.025/7.10E−07
	x_3	11/26/0.014/5.86E−07	18/40/0.019/7.22E−07	45/48/0.029/8.18E−07
	x_4	12/39/0.015/5.86E−07	28/134/0.063/5.11E−07	50/60/0.036/8.31E−07
	x_5	12/35/0.018/5.88E−07	24/84/0.042/4.92E−07	49/56/0.030/8.59E−07
	x_6	10/22/0.010/4.83E−07	17/36/0.018/4.15E−07	46/49/0.021/7.83E−07
	x_7	10/21/0.010/2.25E−07	15/31/0.018/5.46E−07	39/41/0.019/9.62E−07
30 000	x_1	11/25/0.066/9.96E−07	18/39/0.077/7.14E−07	44/46/0.092/6.97E−07
	x_2	10/21/0.036/8.85E−07	17/35/0.035/5.73E−07	44/46/0.078/8.20E−07
	x_3	12/28/0.043/1.62E−07	20/50/0.045/9.17E−07	46/49/0.062/9.44E−07
	x_4	11/40/0.042/9.97E−07	33/189/0.137/7.95E−07	51/61/0.076/9.59E−07
	x_5	13/37/0.053/1.63E−07	27/111/0.096/7.91E−07	50/57/0.072/9.92E−07
	x_6	10/22/0.025/8.37E−07	19/44/0.048/6.30E−07	47/50/0.071/9.04E−07
	x_7	10/21/0.024/3.90E−07	15/31/0.033/9.45E−07	41/43/0.055/7.40E−07
50 000	x_1	12/27/0.079/2.06E−07	18/39/0.103/9.22E−07	44/46/0.142/8.99E−07
	x_2	11/23/0.043/1.83E−07	17/35/0.054/7.39E−07	45/47/0.101/7.05E−07
	x_3	12/28/0.051/2.10E−07	21/56/0.082/9.65E−07	47/50/0.115/8.13E−07
	x_4	12/42/0.051/2.06E−07	40/264/0.311/8.52E−07	52/62/0.175/8.26E−07
	x_5	13/37/0.051/2.11E−07	32/151/0.200/3.72E−07	51/58/0.165/8.54E−07
	x_6	11/24/0.037/1.73E−07	19/44/0.064/8.13E−07	48/51/0.101/7.78E−07
	x_7	10/21/0.034/5.03E−07	16/33/0.057/4.04E−07	41/43/0.084/9.56E−07
100 000	x_1	12/27/0.146/2.91E−07	19/44/0.186/7.47E−07	45/47/0.319/8.48E−07
	x_2	11/23/0.087/2.59E−07	18/39/0.111/9.57E−07	45/47/0.211/9.98E−07
	x_3	12/28/0.084/2.96E−07	25/77/0.202/3.55E−07	48/51/0.204/7.66E−07
	x_4	14/56/0.147/2.97E−07	51/377/0.841/3.48E−07	53/63/0.250/7.78E−07
	x_5	13/40/0.135/2.98E−07	40/222/0.540/3.48E−07	52/59/0.235/8.05E−07
	x_6	11/24/0.098/2.45E−07	23/63/0.152/3.66E−07	49/52/0.209/7.33E−07
	x_7	10/21/0.084/7.11E−07	16/33/0.087/5.71E−07	42/44/0.181/9.01E−07

This also indicates that $\tau_k \geq 0$. From the property of projection operator (2.2), (3.3) and the definition of τ_k , we have

$$\begin{aligned}
 \|x_{k+1} - x^*\|^2 &= \|P_\Omega(x_k - \tau_k F(z_k)) - x^*\|^2 \\
 &\leq \|x_k - \tau_k F(z_k) - x^*\|^2 \\
 &= \|x_k - x^*\|^2 - 2\tau_k \langle F(z_k), x_k - x^* \rangle + \|\tau_k F(z_k)\|^2 \\
 &\leq \|x_k - x^*\|^2 - 2\tau_k \langle F(z_k), x_k - z_k \rangle + \|\tau_k F(z_k)\|^2 \\
 &= \|x_k - x^*\|^2 - \frac{\|\langle x_k - z_k, F(z_k) \rangle\|^2}{\|F(z_k)\|^2}.
 \end{aligned}$$

This inequality implies that $\|x_k - x^*\|^2$ is non-increasing. By using the above inequality iteratively, we have

$$\|x_{k+1} - x^*\|^2 \leq \|x_0 - x^*\|^2 - \sum_{i=0}^k \frac{\|\langle x_i - z_i, F(z_i) \rangle\|^2}{\|F(z_i)\|^2}, \quad (3.4)$$

which implies that the sequence $\{x_k - x^*\}$ is bounded. Furthermore, due to the boundedness and the non-increase of $\{\|x_k - x^*\|\}$, it is convergent. Thus $\{x_k\}$ is bounded. By the continuity of $F(x)$, the sequence $\{F_k\}$ is also bounded. By taking

Table 2
Numerical performance on Problem 2.

Dim	Ini	Algorithm2.1	PDY	SP
		TI/NF/TIME/NROM	TI/NF/TIME/NROM	TI/NF/TIME/NROM
1000	x_1	7/22/0.021/5.31E-07	17/51/0.031/3.68E-07	24/26/0.019/9.77E-07
	x_2	6/19/0.005/1.19E-07	4/9/0.003/4.50E-08	18/20/0.004/5.39E-07
	x_3	8/24/0.002/7.59E-08	18/54/0.008/9.15E-07	27/29/0.004/8.93E-07
	x_4	9/26/0.002/5.62E-07	20/59/0.007/5.37E-07	31/33/0.004/6.77E-07
	x_5	9/26/0.003/7.45E-08	20/59/0.011/8.70E-07	28/29/0.004/9.36E-07
	x_6	7/21/0.001/2.11E-07	18/54/0.009/4.84E-07	26/28/0.003/8.58E-07
	x_7	7/22/0.002/1.73E-07	5/11/0.003/7.33E-07	22/24/0.003/6.18E-07
5000	x_1	8/25/0.027/6.70E-08	17/51/0.051/8.22E-07	26/28/0.042/5.83E-07
	x_2	6/19/0.007/2.67E-07	4/9/0.007/1.01E-07	19/21/0.013/6.22E-07
	x_3	8/24/0.011/1.70E-07	19/57/0.020/7.42E-07	29/31/0.013/5.33E-07
	x_4	10/29/0.012/7.09E-08	25/80/0.037/7.55E-07	32/34/0.022/7.82E-07
	x_5	9/26/0.008/1.66E-07	23/70/0.038/4.66E-07	30/31/0.021/5.59E-07
	x_6	7/21/0.006/4.72E-07	19/57/0.022/3.92E-07	27/29/0.019/9.91E-07
	x_7	7/22/0.008/3.88E-07	6/13/0.007/6.18E-08	23/25/0.013/7.14E-07
10 000	x_1	8/25/0.043/9.47E-08	18/54/0.064/4.22E-07	26/28/0.059/8.25E-07
	x_2	6/19/0.014/3.77E-07	4/9/0.006/1.42E-07	19/21/0.021/8.80E-07
	x_3	8/24/0.016/2.40E-07	20/60/0.039/3.80E-07	29/31/0.038/7.53E-07
	x_4	10/29/0.019/1.00E-07	30/105/0.072/4.61E-07	33/35/0.046/5.71E-07
	x_5	9/26/0.024/2.35E-07	24/75/0.041/7.41E-07	30/31/0.028/7.90E-07
	x_6	7/21/0.018/6.67E-07	19/57/0.027/5.55E-07	28/30/0.029/7.24E-07
	x_7	7/22/0.016/5.48E-07	6/13/0.007/8.73E-08	24/26/0.023/5.22E-07
30 000	x_1	8/25/0.075/1.64E-07	18/54/0.117/7.31E-07	27/29/0.114/7.38E-07
	x_2	6/19/0.036/6.53E-07	4/9/0.020/2.47E-07	20/22/0.063/7.87E-07
	x_3	8/24/0.057/4.16E-07	21/65/0.085/8.31E-07	30/32/0.068/6.74E-07
	x_4	10/29/0.045/1.74E-07	35/138/0.210/9.34E-07	33/35/0.083/9.90E-07
	x_5	9/26/0.036/4.08E-07	28/95/0.134/7.49E-07	31/32/0.075/7.07E-07
	x_6	8/24/0.031/6.51E-08	19/58/0.070/9.09E-07	29/31/0.068/6.47E-07
	x_7	7/22/0.037/9.50E-07	6/13/0.020/1.51E-07	24/26/0.078/9.04E-07
50 000	x_1	8/25/0.126/2.12E-07	18/54/0.174/9.44E-07	27/29/0.167/9.52E-07
	x_2	6/19/0.072/8.43E-07	4/9/0.021/3.18E-07	21/23/0.114/5.25E-07
	x_3	8/24/0.083/5.37E-07	23/75/0.154/6.36E-07	30/32/0.173/8.70E-07
	x_4	10/29/0.088/1.86E-07	42/180/0.412/7.37E-07	34/36/0.157/6.60E-07
	x_5	9/26/0.060/5.26E-07	35/137/0.321/7.39E-07	31/32/0.120/9.12E-07
	x_6	8/24/0.050/8.41E-08	21/65/0.131/6.03E-07	29/31/0.112/8.36E-07
	x_7	8/25/0.052/6.91E-08	6/13/0.030/1.95E-07	25/27/0.097/6.03E-07
100 000	x_1	8/25/0.154/3.00E-07	19/58/0.293/9.21E-07	28/30/0.340/6.96E-07
	x_2	7/22/0.086/6.72E-08	4/9/0.043/4.50E-07	21/23/0.155/7.42E-07
	x_3	8/24/0.099/7.59E-07	23/75/0.306/9.00E-07	31/33/0.232/6.35E-07
	x_4	10/29/0.128/2.63E-07	51/242/1.062/5.21E-07	34/36/0.261/9.33E-07
	x_5	10/29/0.121/3.33E-07	38/153/0.653/5.37E-07	32/33/0.237/6.66E-07
	x_6	8/24/0.092/1.19E-07	22/71/0.279/6.93E-07	30/32/0.219/6.11E-07
	x_7	8/25/0.095/9.78E-08	19/58/0.237/8.01E-07	25/27/0.186/8.52E-07

$k \rightarrow +\infty$, the inequality (3.4) also indicates that

$$\sum_{k=0}^{\infty} \frac{\| \langle x_k - z_k, F(z_k) \rangle \|^2}{\| F(z_k) \|^2} < +\infty.$$

This implies that

$$\lim_{k \rightarrow \infty} \left\| \frac{(x_k - z_k)^T F(z_k)}{F(z_k)} \right\| = 0. \quad (3.5)$$

According to the line search defined in Step 4 of Algorithm 2.1 and the definition of z_k , we have

$$\begin{aligned} \| (x_k - z_k)^T F(z_k) \| &= \| F(x_k + \alpha_k d_k)^T \alpha_k d_k \| \\ &\geq \sigma \alpha_k^2 \| d_k \|^2 \| F(z_k) \| \\ &= \sigma \| x_k - z_k \|^2 \| F(z_k) \|. \end{aligned}$$

Thus, from (3.5) we have $\lim_{k \rightarrow \infty} \| x_k - z_k \| = 0$. Combining the boundedness of $\{x_k\}$ and the convergence of $\{\|x_k - z_k\|\}$, we have the sequence $\{z_k\}$ is bounded. By the boundedness of $\{z_k\}$ and the continuity of F , we have $\{F(z_k)\}$ is bounded.

Table 3

Numerical performance on Problem 3.

Dim	Ini	Algorithm2.1	PDY	SP
		TI/NF/TIME/NROM	TI/NF/TIME/NROM	TI/NF/TIME/NROM
1000	x_1	10/21/0.017/6.08E−07	16/33/0.023/7.24E−07	42/44/0.025/8.28E−07
	x_2	10/21/0.003/3.95E−07	15/31/0.006/9.24E−07	41/43/0.011/6.92E−07
	x_3	10/21/0.002/3.10E−07	17/35/0.005/3.47E−07	42/44/0.009/9.44E−07
	x_4	11/25/0.002/4.96E−07	20/49/0.008/6.28E−07	43/45/0.010/7.00E−07
	x_5	11/25/0.004/8.26E−07	18/39/0.007/6.32E−07	46/49/0.016/9.63E−07
	x_6	10/21/0.001/5.97E−07	17/35/0.005/3.60E−07	43/45/0.008/6.73E−07
	x_7	9/19/0.002/5.41E−07	14/29/0.003/4.96E−07	37/39/0.007/7.23E−07
5000	x_1	11/23/0.031/2.17E−07	17/35/0.066/5.41E−07	44/46/0.030/8.23E−07
	x_2	10/21/0.014/8.83E−07	16/33/0.011/6.84E−07	43/45/0.018/6.87E−07
	x_3	10/21/0.009/6.94E−07	18/39/0.012/8.38E−07	44/46/0.016/9.38E−07
	x_4	12/27/0.011/1.77E−07	23/66/0.020/9.05E−07	45/47/0.014/6.96E−07
	x_5	12/27/0.009/2.96E−07	21/56/0.019/8.86E−07	48/51/0.021/9.57E−07
	x_6	11/23/0.008/2.14E−07	17/35/0.011/8.05E−07	45/47/0.018/6.69E−07
	x_7	10/21/0.008/1.94E−07	15/31/0.010/3.74E−07	39/41/0.017/7.18E−07
10 000	x_1	11/23/0.030/3.08E−07	17/35/0.039/7.65E−07	45/47/0.042/7.76E−07
	x_2	11/23/0.013/2.00E−07	16/33/0.022/9.67E−07	43/45/0.025/9.72E−07
	x_3	10/21/0.011/9.81E−07	19/41/0.021/3.94E−07	45/47/0.024/8.84E−07
	x_4	12/27/0.011/2.51E−07	27/88/0.044/3.97E−07	45/47/0.027/9.84E−07
	x_5	12/27/0.011/4.18E−07	23/63/0.040/4.45E−07	49/52/0.033/9.02E−07
	x_6	11/23/0.010/3.02E−07	19/41/0.021/3.38E−07	45/47/0.022/9.46E−07
	x_7	10/21/0.009/2.74E−07	15/31/0.018/5.28E−07	40/42/0.017/6.77E−07
30 000	x_1	11/23/0.051/5.33E−07	19/41/0.071/4.27E−07	46/48/0.082/8.96E−07
	x_2	11/23/0.025/3.46E−07	17/35/0.042/5.61E−07	45/47/0.062/7.48E−07
	x_3	11/23/0.025/2.72E−07	21/51/0.060/5.55E−07	47/49/0.051/6.81E−07
	x_4	12/27/0.033/4.35E−07	35/146/0.121/5.77E−07	47/49/0.050/7.57E−07
	x_5	12/27/0.034/7.24E−07	28/95/0.086/5.64E−07	51/54/0.055/6.94E−07
	x_6	11/23/0.020/5.23E−07	19/41/0.040/5.86E−07	47/49/0.048/7.28E−07
	x_7	10/21/0.019/4.74E−07	15/31/0.030/9.15E−07	41/43/0.043/7.82E−07
50 000	x_1	11/23/0.040/6.88E−07	19/41/0.096/5.51E−07	47/49/0.180/7.71E−07
	x_2	11/23/0.029/4.47E−07	17/35/0.064/7.25E−07	45/47/0.120/9.66E−07
	x_3	11/23/0.034/3.51E−07	22/55/0.071/5.48E−07	47/49/0.086/8.79E−07
	x_4	13/32/0.051/9.53E−07	40/181/0.230/6.42E−07	47/49/0.085/9.78E−07
	x_5	12/27/0.037/9.35E−07	33/130/0.168/6.39E−07	51/54/0.093/8.97E−07
	x_6	11/23/0.032/6.76E−07	20/46/0.061/4.14E−07	47/49/0.085/9.40E−07
	x_7	10/21/0.027/6.12E−07	16/33/0.048/3.91E−07	42/44/0.070/6.73E−07
100 000	x_1	11/23/0.083/9.72E−07	20/45/0.302/4.91E−07	48/50/0.248/7.27E−07
	x_2	11/23/0.076/6.32E−07	18/37/0.107/3.40E−07	46/48/0.184/9.11E−07
	x_3	11/23/0.066/4.97E−07	24/67/0.179/7.07E−07	48/50/0.217/8.29E−07
	x_4	15/40/0.113/2.13E−07	53/285/0.953/6.68E−07	48/50/0.231/9.22E−07
	x_5	12/28/0.079/2.13E−07	39/171/0.445/6.71E−07	52/55/0.287/8.45E−07
	x_6	11/23/0.063/9.56E−07	22/55/0.147/6.22E−07	48/50/0.263/8.86E−07
	x_7	10/21/0.061/8.66E−07	16/33/0.086/5.53E−07	42/44/0.189/9.52E−07

Theorem 3.1. Suppose that [Assumptions 3.1–3.3](#) hold. Let the sequences $\{x_k\}$ and $\{z_k\}$ be generated by Algorithm 2.1, then

$$\lim_{k \rightarrow \infty} \inf \|F_k\| = 0. \quad (3.6)$$

Proof. By the second conclusion of [Lemma 3.1](#), we have

$$\lim_{k \rightarrow \infty} \alpha_k \|d_k\| = 0. \quad (3.7)$$

In the following we consider the two possible cases.

(i) Assume that $\lim_{k \rightarrow +\infty} \inf \|d_k\| = 0$. From [\(2.4\)](#) we can easily get

$$\|d_k\| \geq \Gamma_1 \|F_k\|, \quad \forall k \geq 0,$$

which implies that [\(3.6\)](#) holds.

(ii) Assume that $\lim_{k \rightarrow +\infty} \inf \|d_k\| \neq 0$. By the line search defined in Step 4 of Algorithm 2.1, we have

$$-F(x_k + \rho^{-1} \alpha_k d_k)^T d_k < \rho^{-1} \sigma \alpha_k \|d_k\|^2 \|F(x_k + \rho^{-1} \alpha_k d_k)\|. \quad (3.8)$$

Table 4
Numerical performance on Problem 4.

Dim	Ini	Algorithm2.1	PDY	SP
		TI/NF/TIME/NROM	TI/NF/TIME/NROM	TI/NF/TIME/NROM
1000	x_1	24/49/0.033/5.16E-07	16/33/0.029/9.43E-07	44/46/0.030/7.27E-07
	x_2	24/49/0.006/5.21E-07	17/35/0.009/4.08E-07	44/46/0.011/9.39E-07
	x_3	22/45/0.008/7.63E-07	16/33/0.009/3.94E-07	42/44/0.007/6.84E-07
	x_4	25/51/0.005/7.80E-07	17/35/0.009/9.72E-07	46/48/0.008/9.94E-07
	x_5	24/49/0.006/9.02E-07	17/35/0.009/5.12E-07	45/47/0.011/7.85E-07
	x_6	23/47/0.006/9.60E-07	16/33/0.007/6.69E-07	43/45/0.006/7.73E-07
	x_7	23/47/0.007/9.92E-07	17/35/0.005/4.82E-07	45/47/0.007/7.39E-07
5000	x_1	18/37/0.075/6.62E-07	17/35/0.056/7.08E-07	46/48/0.075/7.23E-07
	x_2	18/37/0.030/8.97E-07	17/35/0.025/9.14E-07	46/48/0.061/9.34E-07
	x_3	16/33/0.038/6.36E-07	16/33/0.026/8.83E-07	44/46/0.046/6.80E-07
	x_4	14/29/0.034/9.53E-07	20/45/0.036/5.70E-07	48/50/0.046/9.88E-07
	x_5	15/31/0.032/9.80E-07	18/37/0.025/3.81E-07	47/49/0.047/7.81E-07
	x_6	17/35/0.033/8.88E-07	17/35/0.022/5.02E-07	45/47/0.042/7.69E-07
	x_7	19/39/0.036/8.87E-07	18/37/0.023/3.58E-07	47/49/0.047/7.35E-07
10000	x_1	17/35/0.086/3.80E-07	18/37/0.083/3.32E-07	47/49/0.140/6.82E-07
	x_2	15/31/0.053/5.16E-07	19/41/0.065/5.83E-07	47/49/0.094/8.81E-07
	x_3	16/33/0.045/9.59E-07	17/35/0.045/4.19E-07	44/46/0.103/9.62E-07
	x_4	17/35/0.045/9.05E-07	23/59/0.069/6.38E-07	49/51/0.135/9.32E-07
	x_5	14/29/0.037/7.49E-07	20/45/0.058/4.25E-07	48/50/0.131/7.36E-07
	x_6	16/33/0.041/8.70E-07	17/35/0.044/7.10E-07	46/48/0.091/7.25E-07
	x_7	15/31/0.043/1.65E-07	20/45/0.053/4.00E-07	48/50/0.090/6.93E-07
30000	x_1	16/33/0.170/4.39E-07	20/45/0.191/4.54E-07	48/50/0.280/7.88E-07
	x_2	17/35/0.116/1.25E-07	21/48/0.142/3.65E-07	49/51/0.228/6.78E-07
	x_3	13/27/0.109/7.39E-07	17/35/0.109/7.25E-07	46/48/0.215/7.41E-07
	x_4	18/37/0.164/4.32E-07	29/91/0.242/5.80E-07	51/53/0.239/7.17E-07
	x_5	15/31/0.142/7.59E-07	22/55/0.167/7.05E-07	49/51/0.238/8.50E-07
	x_6	14/29/0.102/8.42E-07	19/41/0.134/5.54E-07	47/49/0.223/8.38E-07
	x_7	15/31/0.116/9.25E-07	22/55/0.165/6.64E-07	49/51/0.237/8.00E-07
50000	x_1	16/33/0.227/5.65E-07	21/48/0.276/3.65E-07	49/51/0.430/6.78E-07
	x_2	16/33/0.163/6.87E-07	22/55/0.248/7.26E-07	49/51/0.480/8.75E-07
	x_3	14/29/0.149/9.40E-07	17/35/0.173/9.36E-07	46/48/0.357/9.56E-07
	x_4	18/37/0.188/7.27E-07	32/110/0.431/6.56E-07	51/53/0.391/9.26E-07
	x_5	17/35/0.175/9.36E-07	23/59/0.255/7.52E-07	50/52/0.406/7.32E-07
	x_6	16/33/0.163/4.23E-07	20/45/0.205/4.16E-07	48/50/0.375/7.21E-07
	x_7	16/33/0.170/7.72E-07	23/59/0.259/7.08E-07	50/52/0.461/6.89E-07
100000	x_1	13/27/0.286/8.56E-07	23/59/0.577/6.57E-07	49/51/0.795/9.59E-07
	x_2	14/29/0.274/7.65E-07	25/69/0.701/6.39E-07	50/52/0.828/8.25E-07
	x_3	13/27/0.269/3.45E-07	19/41/0.399/5.97E-07	47/49/0.667/9.02E-07
	x_4	22/46/0.485/5.79E-07	39/154/1.382/6.47E-07	52/54/0.756/8.73E-07
	x_5	14/29/0.362/7.32E-07	28/86/0.751/6.30E-07	51/53/0.746/6.90E-07
	x_6	13/27/0.297/5.96E-07	21/48/0.450/3.66E-07	49/51/0.697/6.80E-07
	x_7	14/29/0.303/9.14E-07	28/86/0.693/5.93E-07	50/52/0.731/9.74E-07

From (2.4) and (3.1), we have

$$\begin{aligned}
 \Gamma_1 \|F_k\|^2 &\leq -F_k^T d_k \\
 &= (F(x_k + \rho^{-1} \alpha_k d_k) - F_k)^T d_k - F(x_k + \rho^{-1} \alpha_k d_k)^T d_k \\
 &\leq \rho^{-1} \alpha_k (L + \sigma \|F(x_k + \rho^{-1} \alpha_k d_k)\|) \|d_k\|^2.
 \end{aligned}$$

Then we obtain

$$\alpha_k \|d_k\| \geq \frac{\rho \Gamma_1 \|F_k\|^2}{(L + \sigma \|F(x_k + \rho^{-1} \alpha_k d_k)\|) \|d_k\|}. \quad (3.9)$$

From (3.7) we have $\lim_{k \rightarrow +\infty} \frac{\rho \Gamma_1 \|F_k\|^2}{(L + \sigma \|F(x_k + \rho^{-1} \alpha_k d_k)\|) \|d_k\|} = 0$, which implies that (3.6) holds.

4. Numerical experiments

In this section, we compare the performance of our method with the PDY method [27] and the SP method [24] for $F(x)$ with different dimensions. In our experiments, the parameters in PDY method and SP method are chosen as in [27] and [24] respectively, and we set $\rho = 0.6$, $\beta = 1$, $\sigma = 0.001$, $\sigma_1 = 0.7$, $\sigma_2 = 0.3$ in our method. All

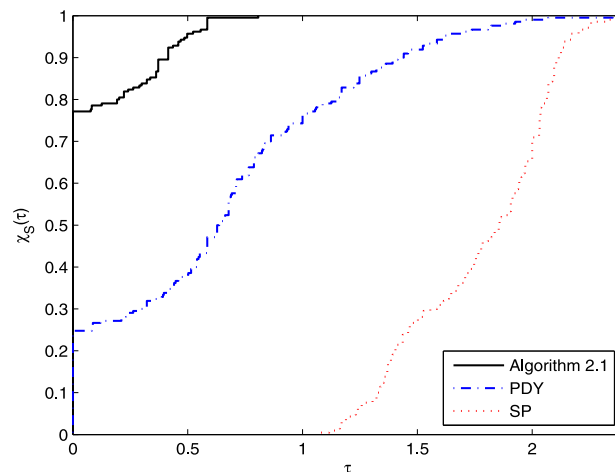


Fig. 1. Performance profiles with respect to number of iterations.

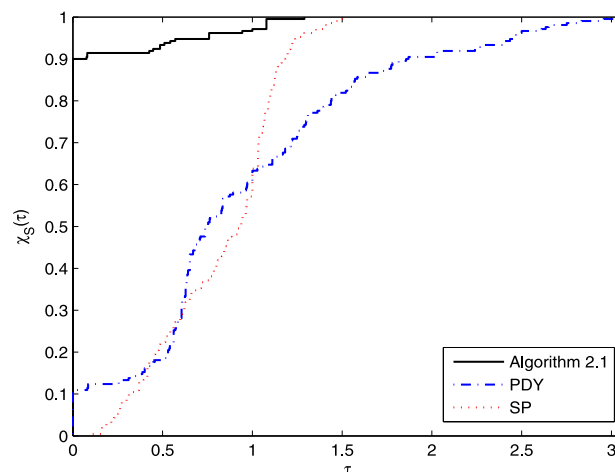


Fig. 2. Performance profiles with respect to number of function evaluation.

the tests were coded in MATLAB R2017a and run on a Thinkpad laptop with Intel(R) Core(TM) i7-7500U processor and CPU 2.70 GHz. To demonstrate the robustness of the algorithms, we test 7 different initial points. In Tables 1–5, we set $x_1 = (1, 1, \dots, 1)^T$, $x_2 = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})^T$, $x_3 = (2, 2, \dots, 2)^T$, $x_4 = (8, 8, \dots, 8)^T$, $x_5 = (\frac{11}{2}, \frac{11}{2}, \dots, \frac{11}{2})^T$, $x_6 = (\frac{3}{2}, \frac{3}{2}, \dots, \frac{3}{2})^T$, $x_7 = (\frac{1}{10}, \frac{1}{10}, \dots, \frac{1}{10})^T$ as the initial points. In the following tables, the dimension of the problem is denoted as *Dim*, the number of iteration is denoted as *TI*, the number of function evaluation is denoted as *NF*, the CPU time in seconds is denoted as *TIME*, the norm of the residual function $F(x)$ at the solution point is denoted as *NORM*. The iteration will be terminated when $\|F(x_k)\| \leq 10^{-6}$ or the number of iteration exceeds 10 000. In our experiment, it should be noted that the CPU time is counted once the initial point is given.

Problem 4.1. Strictly convex function taken from [33], i.e.,

$$F = e^x - 1,$$

where $\Omega = \mathbb{R}^+$.

Problem 4.2. Modified trigonometric function taken from [34], i.e.,

$$F_1 = x_1 + \sin x_1 - 1,$$

$$F_i = -x_{i-1} + 2x_i + \sin x_i - 1, \quad i = 2, 3, \dots, n-1,$$

$$F_n = x_n + \sin x_n - 1,$$

where $\Omega = [-3, +\infty]$.

Table 5
Numerical performance on Problem 5.

Dim	Ini	Algorithm2.1	PDY	SP
		TI/NF/TIME/NROM	TI/NF/TIME/NROM	TI/NF/TIME/NROM
1000	x_1	11/22/0.020/2.38E-07	16/32/0.019/7.80E-07	41/42/0.019/9.21E-07
	x_2	10/20/0.003/2.83E-07	15/30/0.004/6.98E-07	38/39/0.007/9.21E-07
	x_3	9/17/0.003/3.17E-07	17/34/0.005/8.59E-07	44/45/0.007/9.36E-07
	x_4	13/23/0.003/2.29E-07	20/39/0.004/4.82E-07	52/53/0.008/9.15E-07
	x_5	12/22/0.005/8.54E-07	18/35/0.005/6.78E-07	50/51/0.008/7.15E-07
	x_6	11/22/0.002/7.57E-07	17/34/0.003/5.13E-07	43/44/0.006/8.34E-07
	x_7	8/16/0.002/4.28E-07	12/24/0.003/7.68E-07	31/32/0.004/7.92E-07
5000	x_1	11/22/0.027/5.10E-07	17/34/0.035/5.71E-07	43/44/0.035/9.04E-07
	x_2	10/20/0.012/6.06E-07	16/32/0.013/5.12E-07	40/41/0.021/9.03E-07
	x_3	9/17/0.008/4.10E-07	18/36/0.013/6.27E-07	46/47/0.020/9.18E-07
	x_4	13/23/0.014/4.57E-07	25/57/0.018/6.53E-07	54/55/0.027/8.93E-07
	x_5	13/24/0.011/2.86E-07	22/46/0.017/3.63E-07	52/53/0.030/6.99E-07
	x_6	12/24/0.008/2.60E-07	18/36/0.013/3.75E-07	45/46/0.021/8.19E-07
	x_7	8/16/0.007/9.14E-07	13/26/0.011/5.44E-07	33/34/0.014/7.68E-07
10 000	x_1	11/22/0.040/7.17E-07	17/34/0.038/8.06E-07	44/45/0.054/8.51E-07
	x_2	10/20/0.015/8.53E-07	16/32/0.020/7.23E-07	41/42/0.034/8.50E-07
	x_3	9/17/0.010/5.28E-07	20/42/0.025/3.77E-07	47/48/0.043/8.64E-07
	x_4	13/23/0.015/6.36E-07	27/66/0.039/9.33E-07	55/56/0.046/8.39E-07
	x_5	13/24/0.018/4.00E-07	24/55/0.032/7.18E-07	52/53/0.041/9.87E-07
	x_6	12/24/0.016/3.65E-07	18/36/0.018/5.29E-07	46/47/0.033/7.70E-07
	x_7	9/18/0.012/2.06E-07	13/26/0.013/7.66E-07	34/35/0.024/7.22E-07
30 000	x_1	12/24/0.079/1.98E-07	18/37/0.083/4.83E-07	45/46/0.118/9.81E-07
	x_2	11/22/0.064/2.36E-07	17/34/0.056/4.16E-07	42/43/0.089/9.80E-07
	x_3	9/17/0.034/8.57E-07	20/42/0.055/6.52E-07	48/49/0.090/9.97E-07
	x_4	14/25/0.050/1.75E-07	35/102/0.118/4.73E-07	56/57/0.107/9.68E-07
	x_5	13/24/0.042/6.89E-07	30/82/0.098/4.86E-07	54/55/0.106/7.58E-07
	x_6	12/24/0.036/6.30E-07	20/42/0.050/3.86E-07	47/48/0.091/8.89E-07
	x_7	9/18/0.024/3.55E-07	14/28/0.034/4.48E-07	35/36/0.064/8.32E-07
50 000	x_1	12/24/0.091/2.55E-07	19/40/0.125/7.70E-07	46/47/0.185/8.44E-07
	x_2	11/22/0.078/3.04E-07	17/34/0.078/5.36E-07	43/44/0.137/8.43E-07
	x_3	10/19/0.052/1.75E-07	22/50/0.100/5.05E-07	49/50/0.157/8.58E-07
	x_4	14/25/0.081/2.25E-07	41/137/0.248/9.13E-07	57/58/0.191/8.33E-07
	x_5	13/24/0.079/8.88E-07	33/98/0.186/8.18E-07	54/55/0.204/9.79E-07
	x_6	12/24/0.068/8.13E-07	21/48/0.099/8.72E-07	48/49/0.194/7.65E-07
	x_7	9/18/0.051/4.58E-07	14/28/0.057/5.78E-07	36/37/0.109/7.16E-07
100 000	x_1	12/24/0.169/3.61E-07	21/48/0.256/6.15E-07	47/48/0.395/7.96E-07
	x_2	11/22/0.112/4.29E-07	17/35/0.151/9.65E-07	44/45/0.304/7.95E-07
	x_3	10/19/0.094/2.44E-07	25/65/0.267/7.44E-07	50/51/0.413/8.09E-07
	x_4	14/25/0.130/3.18E-07	51/194/0.719/8.64E-07	58/59/0.413/7.85E-07
	x_5	14/26/0.136/2.01E-07	41/144/0.542/8.89E-07	55/56/0.389/9.23E-07
	x_6	13/26/0.125/1.84E-07	22/51/0.214/6.23E-07	49/50/0.325/7.21E-07
	x_7	9/18/0.084/6.47E-07	14/28/0.121/8.17E-07	37/38/0.247/6.75E-07

Problem 4.3. Modified trigonometric function taken from [13], i.e.,

$$F_i = 2x_i - \sin x_i, \quad i = 1, 2, \dots, n,$$

where $\Omega = [-2, +\infty]$.**Problem 4.4.** Exponential problem taken from [35], i.e.,

$$F_1 = x_1 - e^{\cos \frac{x_1 + x_2}{n+1}},$$

$$F_i = -x_i - e^{\cos \frac{x_{i-1} + x_i + x_{i+1}}{n+1}}, \quad i = 2, 3, \dots, n-1,$$

$$F_n = x_n - e^{\cos \frac{x_{n-1} + x_n}{n+1}},$$

where $\Omega = \mathbb{R}^n$.**Problem 4.5.** Modified Logarithmic function taken from [18], i.e.,

$$F(x) = \ln(x+1) - \frac{x}{n},$$

where $\Omega = [-1, +\infty]$.

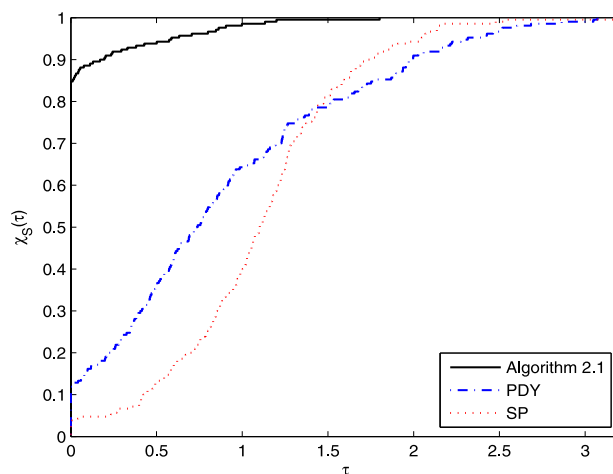


Fig. 3. Performance profiles with respect to the CPU time.

In Tables 1–5, we show the numerical results by using the method we propose, SP method and PDY method, respectively. Obviously, our method works very well in almost all the problems. All the performance of our method, including the number of iteration needed, the evaluation of $F(x)$ and the CPU time, is better than the other two methods. Furthermore, it can be observed that, with the increment of the dimension of the problem, the increment of CPU running time of our method is much slower than the other two methods, which indicates that it is more suitable for solving large-scale problems. To evaluate and compare the performance of different methods, Dolan and Moré proposed performance profiles in [36]. We plot the performance profile of the three methods in Figs. 1–3, in which the value of $\chi_S(\tau)$ is the probability that the solver will win over the rest of solvers. Figs. 1–3 show that our method won about 78%, 90%, 85% of the experiments in terms of the number of iterations TI , the number of function evaluation, and the CPU time, respectively.

5. Conclusion

In this paper, we proposed a new derivative-free iterative method for solving the equations with convex constraint, by combining a three-term CG method in [32] and a projection method [27]. The proposed method inherits the advantage of both CG method and projection method, i.e. the search direction is sufficiently descent, and the global convergence can be obtained under mild conditions. Numerical results show that our method is competitive for solving large-scale problems.

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