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Cubic spline fractal solutions of two-point boundary value problems with a non-homogeneous nowhere differentiable term

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ABSTRACT

Fractal interpolation functions (FIFs) supplement and subsume all classical interpolants. The major advantage by the use of fractal functions is that they can capture either the irregularity or the smoothness associated with a function. This work proposes the use of cubic spline FIFs through moments for the solutions of a two-point boundary value problem (BVP) involving a complicated non-smooth function in the non-homogeneous second order differential equation. In particular, we have taken a second order linear BVP: $y''(x) + Q(x)y'(x) + P(x)y(x) = R(x)$ with the Dirichlet's boundary conditions, where $P(x)$ and $Q(x)$ are smooth, but $R(x)$ may be a continuous nowhere differentiable function. Using the discretized version of the differential equation, the moments are computed through a tridiagonal system obtained from the continuity conditions at the internal grids and endpoint conditions by the derivative function. These moments are then used to construct the cubic fractal spline solution of the BVP, where the non-smooth nature of y'' can be captured by fractal methodology. When the scaling factors associated with the fractal spline are taken as zero, the fractal solution reduces to the classical cubic spline solution of the BVP. We prove that the proposed method is convergent based on its truncation error analysis at grid points. Numerical examples are given to support the advantage of the fractal methodology.

1. Introduction

Differential equations have been used to describe or model real life phenomena. They occur frequently in many areas of scientific and engineering disciplines like physics, chemistry, architecture, ecology, chemical kinetics, mechanical engineering, quantum mechanics, electrical engineering, civil engineering, meteorology, and a relatively new science called chaos. The ultimate goal of solving a differential equation is finding an explicit or an analytical solution which can be used to describe the underlying physical phenomena or scientific experiment.

On the other hand, many of the real world and experimental signals are irregular and rarely render a sensation of smoothness in their structures. Thereby traditional interpolants with simple geometric structures may not describe these fine microscopic patterns effectively. Thus, there is a need for the search of interpolation techniques which can produce interpolants that fail to be differentiable in a dense subset of the interpolation interval. To address this cause, Barnsley [1] introduced the notion of fractal interpolation using the theory of iterated function system. Graph of the fractal

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interpolant possesses a non-integer dimension which can be used as a quantifier for the complexity of the underlying phenomenon. Barnsley and Harrington [2] initiated the construction of smooth FIFs, and unfolded the notable relationship between fractal functions and splines. Smooth FIFs can be applied to generalize the classical interpolation techniques (see, for instance, [3–9]). Therefore, fractal interpolation offers the flexibility of choosing either a smooth or non-smooth interpolant depending on the modeling problem at hand.

A boundary value problem (BVP) is a differential equation together with a set of additional constraints called the boundary conditions. A solution to a boundary value problem satisfies the differential equation and the prescribed boundary conditions. There are many methods developed in the literature for solving higher order BVPs. Bickley [10] used general cubic splines and Albasiny [11] used cubic splines through moments for the solution of linear (regular) two-point BVPs. Fyfe [12] discussed the application of deferred corrections to the method proposed by Bickley. Jain and Aziz [13] proposed a parametric spline function consisting of trigonometric and polynomial maps of order 1 for the solution of the system of ordinary and partial differential equations. Al-Said [14] presented a method to approximate the solution of linear second order BVP and also to determine first, second and third order derivatives at every point of the range of integration. Ramos and others [15–17] used different types of block methods to the direct approximation of the solution of fourth order Boundary Value Problems. Recently, differential or integral equation is discretized by using a novel fractional step in [18].

Sometimes variables representing the derivatives may be irregular in various physical phenomena. For examples: (i) a sphere falling in a wormlike micellar solution undergoes continual oscillations as it falls [19] (ii) the motion of a pendulum on a cart has varying irregularity in the study of acceleration (2nd derivative) in nonlinear control systems [20]. Therefore, if we can model the motion of these phenomena by using 2nd order ODE with certain boundary conditions, then use of fractal splines may be advantageous to capture the irregularity/fractality associated with the derivative. Thus, we expect fractal spline solutions of BVP for these nonlinear phenomena.

In this paper, we consider a two point BVP whose solution is a curve obtained by a fractal methodology through discretization of the domain, and this solution is an extension of the cubic spline defined in Albasiny [11]. The classical cubic spline does not give appropriate solution if the non-homogeneous differential equation involves a continuous function but not differentiable in a dense subset of the interpolation interval. Since the classical cubic spline is a particular case of fractal spline, the proposed method can be used to consider any type of continuous function in a BVP. We define the prerequisite material on (smooth) fractal functions and the general two-point second order boundary value problem in Section 2. We develop the description of our methods to solve the two-point BVPs using the cubic spline FIFs through moments in Section 3. Truncation error analysis is studied in Section 4. Finally, the proposed methods are used to solve BVPs in Section 5.

2. Preliminaries

To equip ourselves with the requisite general material, we will discuss the construction of fractal functions in Section 2.1, and the basics of a 2nd order BVP in Section 2.2.

2.1. Fractal interpolation function

This section targets to equip a novice reader with the basics of fractal interpolation. These materials are collected from the well-known treatises [1,2,21].

For $r \in \mathbb{N}$, denote $\mathbb{N}_r$ as the set of first r natural numbers. For $N > 1$, let $x_0 < x_1 < \cdots < x_N$ be real numbers and $I = [x_0, x_N]$. Let the prescribed set of interpolation data be $\{(x_j, y_j) \in I \times \mathbb{R} : j \in \mathbb{N}_N \cup 0\}$. We want to construct a finite number of contraction maps in $\mathbb{R}^2$ for an iterated function system (IFS) so that its fixed point is the required fractal function. The abscissa values are defined through L_i and the ordinate values are defined through F_i (see [1] for details). For $i \in \mathbb{N}_N$, set $I_i = [x_{i-1}, x_i]$ and let $L_i : I \to I_i$ be contractive homeomorphisms such that

$$\left.\begin{aligned} L_i(x_0) = x_{i-1},\ L_i(x_N) = x_i \ \forall\, i \in \mathbb{N}_N, \\ |L_i(x) - L_i(x^*)| \le l_i|x - x^*| \ \forall\, x, x^* \in I \text{ for some } 0 < l_i < 1. \end{aligned}\right\} \tag{2.1}$$

Let $F_i : I \times \mathbb{R} \to \mathbb{R}$ be continuous maps satisfying

$$\left.\begin{aligned} &F_i(x_0, y_0) = y_{i-1},\ F_i(x_N, y_N) = y_i,\ i \in \mathbb{N}_N, \\ &|F_i(x, y) - F_i(x, y')| \le k_i|y - y'|,\ x \in I; y, y' \in \mathbb{R};\ \text{for some } 0 \le k_i < 1. \end{aligned}\right\} \tag{2.2}$$

For $i \in \mathbb{N}_N$, define functions $\omega_i(x, y) = \big(L_i(x), F_i(x, y)\big)$. It is easy to check that each ω_i is a contraction map in $\mathbb{R}$. We need the required IFS $\{I \times \mathbb{R}; \omega_i : i \in \mathbb{N}_N\}$ for construction of a fractal function. The following is the most fundamental result in the field of fractal interpolation.

Theorem 2.1 (*[21]*)**.** *The following hold:*

(i) The IFS $\{I \times \mathbb{R}; \omega_i : i \in \mathbb{N}_N\}$ has a unique attractor G such that G is the graph of a continuous function $f : I \to \mathbb{R}$.
(ii) The function f interpolates the data set, i.e., $f(x_j) = y_j$ for all j.

(iii) Let $\mathcal{G} := \{g \in \mathcal{C}(I) : g(x_0) = y_0, g(x_N) = y_N\}$ be endowed with the uniform metric $d(g, \tilde{g}) := \max\{|g(x) - \tilde{g}(x)| : x \in I\}$. If $T : \mathcal{G} \to \mathcal{G}$ is defined by $Tg(x) = F_i\big(L_i^{-1}(x), g \circ L_i^{-1}(x)\big)$, $x \in I_i$, $i \in \mathbb{N}_N$, and for all $g \in \mathcal{G}$, then T has a unique fixed point f, and $f = \lim_{n\to\infty} T^n(g)$ for any $g \in \mathcal{G}$. Further, the fixed point f is the function satisfying conditions given in (i)–(ii).

The function which has made its debut in the foregoing theorem is termed a fractal interpolation function (FIF), and it satisfies the functional equation:

$$f(x) = F_i\big(L_i^{-1}(x), f \circ L_i^{-1}(x)\big), \quad x \in I_i,\ i \in \mathbb{N}_N. \tag{2.3}$$

The adjective *fractal* is used to emphasize that the graph of f may have non-integer Hausdorff and Minkowski dimension. Also, the graph $G(f)$ is a union of transformed copies of itself, specifically $G(f) = \bigcup_{i=1}^{N} \omega_i\big(G(f)\big)$. The most extensively studied FIF in theory and applications is formed by

$$L_i(x) = a_i x + e_i, \qquad F_i(x, y) = \alpha_i y + q_i(x), \quad i \in \mathbb{N}_N, \tag{2.4}$$

where α_i are constants satisfying $0 \le |\alpha_i| < 1$ and q_i are continuous functions so that the "join-up conditions" in (2.2) imposed on the bivariate functions F_i are satisfied. The multiplier α_i is called a vertical scaling factor for the transformation ω_i and the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in (-1, 1)^N$ is called a scale vector of the FIF. The prescriptions in (2.1) uniquely determine the constants a_i and b_i appearing in the affine map L_i as

$$a_i = \frac{x_i - x_{i-1}}{x_N - x_0}, \qquad e_i = \frac{x_{i-1}x_N - x_i x_0}{x_N - x_0}.$$

Let us recall that the function f determined by the IFS in (2.4), which takes the form

$$f(x) = \alpha_i f\big(L_i^{-1}(x)\big) + q_i\big(L_i^{-1}(x)\big), \quad x \in I_i,\ i \in \mathbb{N}_N, \tag{2.5}$$

is, in general, non-smooth in nature. The following theorem provides the conditions on α_i and functions q_i so that the FIF is $\mathcal{C}^p$-continuous.

Theorem 2.2 ([2]). *Let $\{(x_j, y_j) : j \in \mathbb{N}_N \cup \{0\}\}$ be a given data set with strictly increasing abscissae. Let $L_i(x) = a_i x + b_i$ satisfy (2.1) and $F_i(x, y) = \alpha_i y + q_i(x)$ obeys (2.2) for $i \in \mathbb{N}_N$. Suppose that for some integer $p \ge 0$, $|\alpha_i| < a_i^p$ and $q_i \in \mathcal{C}^p(I)$, $i \in \mathbb{N}_N$. Let*

$$F_{i,k}(x, y) = \frac{\alpha_i y + q_i^{(k)}(x)}{a_i^k}, \qquad y_{0,k} = \frac{q_1^{(k)}(x_0)}{a_1^k - \alpha_1}, \qquad y_{N,k} = \frac{q_N^{(k)}(x_N)}{a_N^k - \alpha_N}, \quad k = 1, 2, \dots, p.$$

If $F_{i-1,k}(x_N, y_{N,k}) = F_{i,k}(x_0, y_{0,k})$ for $i \in \mathbb{N}_N$ and $k \in \mathbb{N}_p$, then the IFS $\big\{I \times \mathbb{R};\ \big(L_i(x), F_i(x, y)\big) : i \in \mathbb{N}_N\big\}$ determines a FIF $f \in \mathcal{C}^p(I)$, and $f^{(k)}$ is the FIF defined by $\big\{I \times \mathbb{R};\ \big(L_i(x), F_{i,k}(x, y)\big) : i \in \mathbb{N}_N\big\}$ for $k \in \mathbb{N}_p$.

Based on this theorem, smooth polynomial FIFs are constructed in [3,6,9].

2.2. A two-point boundary value problem

Definition 2.1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a given function. The problem

$$y'' = f(x, y, y'), \quad x \in [x_0, x_N], \tag{2.6}$$

along with the boundary conditions

$$\begin{cases} \beta_1 y(x_0) + \beta_2 y'(x_0) = \beta_3, \\ \gamma_1 y(x_N) + \gamma_2 y'(x_N) = \gamma_3, \end{cases} \tag{2.7}$$

where $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2$ and γ_3 are real numbers, is called a two-point second order boundary value problem (BVP). The existence and uniqueness of the solution of two point BVP have been discussed by Keller [22]. Bickley [10] considered this BVP when (2.6) is taken as

$$P(x)u'' + Q(x)u' + R(x)u = S(x).$$

A BVP (2.6)–(2.7) should be well posed in order to be useful in the applications, which means that there exists a unique solution, which continuously depends on the input imposed to the problem. There are several types of schemes proposed to obtain numerical solutions like shooting method, finite difference schemes, finite element and finite volume methods at discretized points of the domain. When the number of discretized points is very large, the numerical solution is close to the original solution if the concerned method is convergent. Similarly, various types of splines have been used to get the solution of BVP in a closed manner. But this closed forms of the classical solutions does not match with the exact solution, when $R(x)$ is not differentiable in a dense subset of the domain. Thus, we propose a deterministic fractal cubic spline solution for this second order BVP with Dirichlet's boundary conditions, which is more general than the classical cubic spline solution.

3. Solution of BVP by fractal splines

In this section, we solve the two-point BVP by using a cubic spline FIF through moments, which was proposed in [3]. The main idea is to use the condition of continuity of the derivative of the fractal spline at the grid points in the discretization of (2.6) for the computation of moments so that we can write the fractal cubic spline solution explicitly.

3.1. Cubic spline FIF through moments formulation

In the first part, we describe the general construction of cubic spline FIFs $f \in \mathcal{C}^2[x_0, x_N]$ through the moments, $M_j = f''(x_j)$, $j \in \mathbb{N}_N \cup \{0\}$, which passes through the interpolation data set $\{(x_j, y_j) : j \in \mathbb{N}_N \cup \{0\}\}$ (see [3] for details). Let $\triangle$ represent the partition $x_0 < x_1 < \cdots < x_N$ of the domain of the interval $I := [x_0, x_N]$ of the BVP. The graph of the cubic spline FIF f is the attractor of the IFS $\mathcal{I} := \{I \times \mathbb{R}; \omega_i(x, y) = (L_i(x), F_i(x, y)), i \in \mathbb{N}_N\}$, where $L_i(x)$ satisfies (2.1) and $F_i(x, y) = a_i^2(\alpha_i y + q_i(x))$, $|\alpha_i| < 1$, and $q_i(x)$ is a suitable cubic polynomial.

Using the moments $M_j, j \in \mathbb{N}_N \cup \{0\}$, $f'(x_0)$ and $f'(x_N)$ in the structure of a $\mathcal{C}^2$-fractal cubic spline, we will get $N + 1$ equations. In order to compute these $N + 3$ unknowns, we have to put two additional boundary conditions similar to the construction of the classical $\mathcal{C}^2$-cubic splines. This construction helps us solve a system of order at most $N + 3$ equations instead of a system of $4N$ equations for the computations of the cubic polynomials $q_i(x)$, $i \in \mathbb{N}_N$. Thus, we will assume the structure of the cubic spline FIF through moments in the following:

$$f(L_i(x)) = a_i^2 \left. \begin{cases} \alpha_i f(x) + \frac{(M_i - \alpha_i M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{i-1} - \alpha_i M_0)(x_N - x)^3}{6(x_N - x_0)} \\ - \frac{(M_{i-1} - \alpha_i M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_i - \alpha_i M_N)(x_N - x_0)(x - x_0)}{6} \\ + \left(\frac{y_{i-1}}{a_i^2} - \alpha_i y_0\right)\frac{x_N - x}{x_N - x_0} + \left(\frac{y_i}{a_i^2} - \alpha_i y_N\right)\frac{x - x_0}{x_N - x_0} \end{cases} \right\}, \quad i \in \mathbb{N}_N, \tag{3.1}$$

where

$$L_i(x) = a_i x + e_i, \qquad a_i = \frac{x_i - x_{i-1}}{x_N - x_0}, \qquad \text{and} \qquad e_i = \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}.$$

Now differentiating (3.1) with respect to x, we obtain

$$\begin{aligned} f'(L_i(x)) = a_i \Bigg\{ & \alpha_i f'(x) + \frac{(M_i - \alpha_i M_N)(x - x_0)^2}{2(x_N - x_0)} - \frac{(M_{i-1} - \alpha_i M_0)(x_N - x)^2}{2(x_N - x_0)} \\ & + \frac{(M_{i-1} - \alpha_i M_0)(x_N - x_0)}{6} - \frac{(M_i - \alpha_i M_N)(x_N - x_0)}{6} \\ & - \left(\frac{y_{i-1}}{a_i^2} - \alpha_i y_0\right)\frac{1}{x_N - x_0} + \left(\frac{y_i}{a_i^2} - \alpha_i y_N\right)\frac{1}{x_N - x_0} \Bigg\}, \end{aligned}$$

Let $\ell = x_N - x_0$. Then, the above equation can be written as

$$\begin{aligned} f'(L_i(x)) = a_i \Bigg\{ & \alpha_i f'(x) + \frac{(M_i - \alpha_i M_N)(x - x_0)^2}{2\ell} - \frac{(M_{i-1} - \alpha_i M_0)(x_N - x)^2}{2\ell} \\ & + \frac{\ell}{6}\Big\{M_{i-1} - M_i + \alpha_i(M_N - M_0)\Big\} + \frac{1}{\ell}\Big(\frac{y_i - y_{i-1}}{a_i^2} - \alpha_i(y_N - y_0)\Big) \Bigg\}. \end{aligned} \tag{3.2}$$

Since $f'(x)$ is continuous at the knots x_i, $i = 1 \ldots, N - 1$, we have $\lim_{x \to x_i^-} f'(x) = \lim_{x \to x_i^+} f'(x)$ for $i \in \mathbb{N}_{N-1}$. As per assumption (2.1), $L_i(x_N) = L_{i+1}(x_0) = x_i$. For $d_N := f'(x_N)$, substituting $x = x_N$ in (3.2), we deduce

$$\begin{aligned} f'(x_i^-) = a_i \Bigg\{ & \alpha_i d_N + \frac{\ell}{2}(M_i - \alpha_i M_N) + \frac{\ell}{6}\Big[M_{i-1} - M_i + \alpha_i(M_N - M_0)\Big] \\ & + \frac{1}{\ell}\Big(\frac{y_i - y_{i-1}}{a_i^2} - \alpha_i(y_N - y_0)\Big) \Bigg\}, \quad i \in \mathbb{N}_N. \end{aligned} \tag{3.3}$$

Now, for $d_0 := f'(x_0)$, putting $x = x_0$ in $f'(L_{i+1}(x))$, we get

$$\begin{aligned} f'(x_i^+) = a_{i+1}\Bigg\{ & \alpha_{i+1} d_0 - \frac{\ell}{2}(M_i - \alpha_{i+1} M_0) + \frac{\ell}{6}\Big[M_i - M_{i+1} + \alpha_{i+1}(M_N - M_0)\Big] \\ & + \frac{1}{\ell}\Big(\frac{y_{i+1} - y_i}{a_{i+1}^2} - \alpha_{i+1}(y_N - y_0)\Big)\Bigg\}, \quad i \in \mathbb{N}_{N-1} \cup \{0\}. \end{aligned} \tag{3.4}$$

Denote $h_i = x_i - x_{i-1}$, $i \in \mathbb{N}_N$. Using (3.3)–(3.4) and continuity at internal grid points, we get the following equations:

$$\begin{aligned}&\frac{h_i}{6}M_{i-1} + \frac{h_i + h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \\ &\frac{1}{h_i}y_{i-1} - \Big(\frac{1}{h_i} + \frac{1}{h_{i+1}}\Big)y_i + \frac{1}{h_{i+1}}y_{i+1} - \Big\{a_i\alpha_i d_N - a_{i+1}\alpha_{i+1}d_0 - \frac{1}{6}(2h_i\alpha_i \\ &+ h_{i+1}\alpha_{i+1})M_N - \frac{1}{6}(h_i\alpha_i + 2h_{i+1}\alpha_{i+1})M_0 + \frac{1}{\ell}(a_{i+1}\alpha_{i+1} - a_i\alpha_i)(y_N - y_0)\Big\} \\ &\text{for } i = 1, 2, \dots, N-1. \end{aligned} \tag{3.5}$$

Substituting $i = 1$ and $x = x_0$ in (3.2), the functional relation for d_0 is

$$\begin{aligned} d_0 &= a_1\alpha_1 d_0 - \frac{h_1}{2}(1-\alpha_1)M_0 + \frac{h_1}{6}((1-\alpha_1)M_0 - M_1 + \alpha_1 M_N) \\ &\quad + \frac{a_1}{\ell}\Big(\frac{y_1 - y_0}{a_1^2} - \alpha_1(y_N - y_0)\Big), \\ \Rightarrow (1 - a_1\alpha_1)d_0 &= \frac{y_1 - y_0}{h_1} - \frac{a_1\alpha_1}{\ell}(y_N - y_0) - \frac{h_1}{3}(1-\alpha_1)M_0 \\ &\quad - \frac{h_1}{6}(M_1 - \alpha_1 M_N). \end{aligned} \tag{3.6}$$

Similarly, substituting $i = N$ and $x = x_N$ in (3.2), the functional relation for d_N is

$$\begin{aligned} d_N &= a_N\alpha_N d_N + \frac{h_N}{2}((1-\alpha_N)M_N) + \frac{h_N}{6}\Big[M_{N-1} - M_N + \alpha_N(M_N \\ &\quad - M_0)\Big] + \frac{a_N}{\ell}\Big(\frac{y_N - y_{N-1}}{a_N^2} - \alpha_N(y_N - y_0)\Big), \\ \Rightarrow (1 - a_N\alpha_N)d_N &= \frac{y_N - y_{N-1}}{h_N} - \frac{a_N\alpha_N}{\ell}(y_N - y_0) + \frac{h_N}{3}(1-\alpha_N)M_N \\ &\quad + \frac{h_N}{6}(M_{N-1} - \alpha_N M_0). \end{aligned} \tag{3.7}$$

Thus, we have $N+1$ equations in (3.5)–(3.7) for the computation of IFS parameters $M_j, j \in \mathbb{N}_N \cup \{0\}$, $d_0 = f'(x_0)$ and $d_N = f'(x_N)$. Our aim is to replace these M_j with functional values at grid points with suitable discretization of the given differential equation.

3.2. Discretization of BVP by fractal splines

Now we wish to solve a linear two-point second order BVP of the form

$$y''(x) + P(x)y'(x) + Q(x)y(x) = R(x) \tag{3.8}$$

with Dirichlet boundary conditions

$$y(x_0) = \beta^* \quad \text{and} \quad y(x_N) = \gamma^*, \tag{3.9}$$

where $R(x)$ may be continuous and nowhere differentiable function in $I := [x_0, x_N]$. If we use any classical spline solution ψ of the BVP (3.8)–(3.9), then $\psi''(x) + P(x)\psi'(x) + Q(x)\psi(x)$ is a piecewise differentiable function which does not match with $R(x)$. Therefore, we need a $\mathcal{C}^2$-function whose 2nd derivative may be similar to a continuous and nowhere differentiable function. This can be easily achieved by a fractal cubic spline. First, we will discretize the domain $[x_0, x_N]$ as $x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N$. We have to find the approximate values of the solution as y_j at the grid x_j for $j = 1, 2, \dots, N-1$ through the fractal cubic spline solution (3.1). We compute the approximate solution of the BVP using (3.3)–(3.7) in our methodology.

At $x = x_i$, (3.8) can be written in the discretization form as

$$M_i + P_i d_i + Q_i y_i = R_i \quad \text{for } i = 0, 1, 2, \dots, N, \tag{3.10}$$

where $M_i = y''(x_i)$, $P_i = P(x_i)$, $d_i = y'(x_i)$, $Q_i = Q(x_i)$, $y_i = y(x_i)$ and $R_i = R(x_i)$. Using (3.10) in (3.4), we obtain the following equations for $i = 0, 1, \dots, N-1$:

$$\Big(1 - \frac{h_{i+1}}{3}P_i\Big)M_i - \frac{h_{i+1}P_i}{6}M_{i+1} = R_i + \Big(\frac{P_i}{h_{i+1}} - Q_i\Big)y_i - \frac{P_i}{h_{i+1}}y_{i+1} + K_{0,N}^1 a_{i+1}\alpha_{i+1}P_i, \tag{3.11}$$

where $K_{0,N}^1 = -\Big[d_0 + \frac{\ell}{6}(M_N - M_0) + \frac{\ell}{2}M_0 - \frac{1}{\ell}(y_N - y_0)\Big]$.

Similarly, using (3.10) in (3.3), we obtain the following equations for $i = 1, \ldots, N$:

$$\frac{h_iP_i}{6}M_{i-1} + \left(1 + \frac{h_i}{3}P_i\right)M_i = R_i - \left(\frac{P_i}{h_i} + Q_i\right)y_i + \frac{P_i}{h_i}y_{i-1} + K^2_{0,N}a_i\alpha_iP_i, \tag{3.12}$$

where $K^2_{0,N} = -\Big[d_N + \frac{\ell}{6}(M_N - M_0) - \frac{\ell}{2}M_N - \frac{1}{\ell}(y_N - y_0)\Big]$.
It is clear that (3.11) and (3.12) constitute a system of $2N$ equations with the $2N + 4$ unknowns, $M_0, M_1, \ldots, M_N$ and $y_0, y_1, \ldots, y_N$ and d_0, d_N. Elimination of M_j's leads directly to $N+1$ unknowns $y_1, y_2, \ldots, y_{N-1}$ and d_0, d_N, The continuity conditions at the internal grid points along with the two boundary conditions are sufficient for their determination.
Addition of (3.11) and (3.12) yields the following relations for $i = 1, 2, \ldots, N-1$:

$$\begin{aligned}\frac{h_i}{6}P_iM_{i-1} + \left[2 - \frac{h_{i+1} - h_i}{3}\right]M_i - \frac{h_{i+1}}{6}P_iM_{i+1} = 2R_i + \frac{P_i}{h_i}y_{i-1} - \frac{P_i}{h_{i+1}}y_{i+1}\\ - \left[\frac{h_{i+1} - h_i}{h_ih_{i+1}} + 2Q_i\right]y_i + (K^1_{0,N}a_{i+1}\alpha_{i+1} + K^2_{0,N}a_i\alpha_i)P_i.\end{aligned} \tag{3.13}$$

Elimination of M_i from (3.13) and (3.5), leads us to the following equations:

$$A_{1,i}M_{i-1} + A_{2,i}M_{i+1} = A_{3,i}y_{i-1} + A_{4,i}y_i + A_{5,i}y_{i+1} - A_{6,n}, \tag{3.14}$$

where

$$\begin{aligned}
A_{1,i} &= \frac{h_i}{3}\left(1 - \frac{h_{i+1}P_i}{3}\right), \qquad A_{2,i} = \frac{h_{i+1}}{3}\left(1 + \frac{h_iP_i}{3}\right),\\
A_{3,i} &= \frac{2}{h_i}\left(1 - \frac{h_{i+1}P_i}{3}\right), \qquad A_{5,i} = \frac{2}{h_{i+1}}\left(1 + \frac{h_iP_i}{3}\right),\\
A_{4,i} &= -2\left\{(h_i + h_{i+1})\left[\frac{1}{h_ih_{i+1}} - \frac{Q_i}{3}\right] + \frac{h_i^2 - h_{i+1}^2}{3h_ih_{i+1}}P_i\right\},\\
A_{6,i} &= \frac{h_i + h_{i+1}}{3}\left[\left(K^1_{0,N}a_{i+1}\alpha_{i+1} + K^2_{0,N}a_i\alpha_i\right)P_i + 2R_i\right] + \left(2 + \frac{h_i - h_{i+1}}{3}P_i\right)K^1_{i,0,N},\\
K^1_{i,0,N} &= a_i\alpha_id_N - a_{i+1}\alpha_{i+1}d_0 - \frac{1}{6}(2h_i\alpha_i + h_{i+1}\alpha_{i+1})M_N\\
&\quad - \frac{1}{6}(h_i\alpha_i + 2h_{i+1}\alpha_{i+1})M_0 + \frac{1}{\ell}(a_{i+1}\alpha_{i+1} - a_i\alpha_i)(y_N - y_0).
\end{aligned}$$

But an explicit form of M_{i-1} can be obtained in terms of y_{i-1} and y_i by eliminating M_i from (3.11) (by replacing i by $i-1$) and (3.12) as

$$A_{7,i}M_{i-1} = A_{8,i}y_{i-1} - A_{9,i}y_i + A_{10,i}, \tag{3.15}$$

where

$$\begin{aligned}
A_{7,i} &= \left(1 - \frac{h_i}{3}P_{i-1}\right)\left(1 + \frac{h_i}{3}P_i\right) + \frac{h_i^2}{36}P_{i-1}P_i,\\
A_{8,i} &= \left(1 + \frac{h_i}{3}P_i\right)\left(\frac{P_{i-1}}{h_i} - Q_{i-1}\right) + \frac{P_iP_{i-1}}{6},\\
A_{9,i} &= \left(1 + \frac{h_i}{3}P_i\right)\frac{P_{i-1}}{h_i} + \frac{h_iP_{i-1}}{6}\left(\frac{P_i}{h_i} + Q_i\right),\\
A_{10,i} &= \left[\left(1 + \frac{h_iP_i}{3}\right)K^1_{0,N} + \frac{h_i}{6}K^2_{0,N}P_i\right]a_i\alpha_iP_{i-1} + \left(1 + \frac{h_i}{3}P_i\right)R_{i-1} + \frac{h_iP_{i-1}}{6}R_i.
\end{aligned}$$

Similarly, an explicit form of M_{i+1} can be obtained in terms of y_i and y_{i+1} by eliminating M_i from (3.11) (by replacing i by $i+1$) and (3.12) as

$$A_{11,i}M_{i+1} = A_{12,i}y_i + A_{13,i}y_{i+1} + A_{14,i}, \tag{3.16}$$

where

$$
\begin{aligned}
A_{11,i} &= \Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big(1+\frac{h_{i+1}}{3}P_{i+1}\Big)+\frac{h_{i+1}^2}{36}P_iP_{i+1},\\
A_{12,i} &= \Big(1-\frac{h_{i+1}}{3}P_i\Big)\frac{P_{i+1}}{h_{i+1}}-\frac{h_{i+1}P_{i+1}}{6}\Big(\frac{P_i}{h_{i+1}}-Q_{i+1}\Big),\\
A_{13,i} &= \frac{P_iP_{i+1}}{6}-\Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big(\frac{P_{i+1}}{h_{i+1}}-Q_{i+1}\Big),\\
A_{14,i} &= \Big[\Big(1-\frac{h_{i+1}P_i}{3}\Big)K^1_{0,N}-\frac{h_{i+1}}{6}K^2_{0,N}P_i\Big]a_{i+1}\alpha_{i+1}P_{i+1}+\Big(1-\frac{h_{i+1}}{3}P_i\Big)R_{i+1}\\
&\quad+\frac{h_{i+1}P_{i+1}}{6}R_i.
\end{aligned}
$$

Substituting M_{i-1} and M_{i+1} in (3.14), we obtain the following equations in terms of y_{i-1}, y_i and y_{i+1}:

$$
\Lambda_{1,i}y_{i+1}+\Lambda_{2,i}y_i+\Lambda_{3,i}y_{i-1}=\Lambda_{4,i}R_{i+1}+\Lambda_{5,i}R_i+\Lambda_{6,i}R_{i-1}+\Lambda_{7,i},\quad i\in\mathbb{N}_{N-1}, \tag{3.17}
$$

where

$$
\begin{aligned}
\Lambda_{1,i} &= \Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big(1+\frac{h_i}{3}P_i\Big)\Big[P_{i-1}-\frac{h_i}{3}Q_{i-1}-\frac{2}{h_i}\Big]A_{11,i},\\
\Lambda_{2,i} &= \Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big(1+\frac{h_i}{3}P_i\Big)\Big[\ \frac{1}{h_{n+1}}\Big(1+\frac{h_{i+1}P_{i+1}}{2}\Big)A_{7,i}+\frac{1}{h_i}(1-\frac{h_{i-1}P_{i-1}}{2}A_{11,i})\\
&\quad-\frac{1}{3}C_iQ_i\ \Big],\\
\Lambda_{3,i} &= -\Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big(1+\frac{h_i}{3}P_i\Big)\Big[P_{i+1}+\frac{h_{i+1}}{3}Q_{i+1}+\frac{2}{h_{i+1}}\Big]A_{7,i},\\
\Lambda_{4,i} &= -\frac{h_i}{3}\Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big(1+\frac{h_i}{3}P_i\Big)A_{11,i},\\
\Lambda_{5,i} &= \frac{h_{i+1}^2P_{i+1}}{18}\Big(1+\frac{h_iP_i}{3}\Big)A_{7,i}-\frac{h_i^2P_{i-1}}{18}\Big(1-\frac{h_{i+1}P(i)}{3}\Big)A_{11,i},\\
\Lambda_{6,i} &= -\frac{h_{i+1}}{3}\Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big(1+\frac{h_i}{3}P_i\Big)A_{7,i},\\
\Lambda_{7,i} &= \Big[K^1_{0,N}a_{i+1}\alpha_{i+1}+K^2_{0,N}a_i\alpha_i+\Big(2-\frac{h_{i+1}-h_i}{3}P_i\Big)K^1_{i,0,N}\Big]A_{7,i}A_{11,i}\\
&\quad-\frac{h_i}{3}\Big(1-\frac{h_{i+1}}{3}P_i\Big)\Big[\Big(1+\frac{h_i}{3}P_i\Big)K^1_{0,N}+\frac{h_i}{6}K^2_{0,N}P_i\Big]a_i\alpha_iP_{i-1}A_{11,i}\\
&\quad-\frac{h_{i+1}}{3}\Big(1+\frac{h_i}{3}P_i\Big)\Big[\Big(1-\frac{h_{i+1}}{3}P_i\Big)K^1_{0,N}-\frac{h_{i+1}}{6}K^2_{0,N}P_i\Big]a_{i+1}\alpha_{i+1}P_{i+1}A_{7,i}\\
C_i &= (h_i+h_{i+1})-\frac{3h_i+4h_{i+1}}{12}h_iP_{i-1}+\frac{3h_{i+1}+4h_i}{12}h(i+1)P_{i+1}\\
&\quad-\frac{h_ih_{i+1}}{12}(h_i+h_{i+1})P_{i-1}P_{i+1}.
\end{aligned}
$$

Solving (3.17) in conjunction with the boundary conditions in (3.9), we compute the values of $y_j (j\in\mathbb{N}_N\cup\{0\})$. Using (3.15) and (3.16), we will get the values of all moments $M_i,\ i\in\mathbb{N}_N\cup\{0\}$. Finally, the cubic spline fractal solution to the BVP is obtained from (3.1).

Remark 3.1. When the scaling factors are taken zeros and the partition points are equally spaced then the spline FIF (3.1) reduces to the classical spline function as discussed in Albasiny et al. [11].

Remark 3.2. When we select a uniform partition ($h_i = h$ gives $a_i = a = \frac{1}{N}$) and equal scaling factors $\alpha_i = \alpha, i \in \mathbb{N}_N$, then (3.17) reduces to the following recurrence relation:

$$
\begin{aligned}
\frac{a}{a^2h}(y_{i-1}-2y_i+y_{i+1})-\frac{h}{6}(P_{i-1}y_{i-1}+4P_iy_i+P_{i+1}y_{i+1})\\
=\frac{h}{6}(R_{i-1}+4R_i+R_{i+1})+K_0,\quad i\in\mathbb{N}_{N-1}.
\end{aligned}
$$

This clearly corresponds to the central finite-difference representation:

$$
\delta^2y_i+a^2h^2\Big(1+\frac{\delta^2}{6}\Big)P_iy_i=a^2h^2\Big(1+\frac{\delta^2}{6}\Big)R_i+K_0,\quad i\in\mathbb{N}_{N-1},
$$

where $K_0 = a^2h\alpha(d_N - d_0) + \frac{ah\alpha}{2}(P_0y_0 + P_Ny_N) - \frac{ah\alpha}{2}(R_0 + R_N)$, and the central difference is defined as $\delta y_i = f(y_i + \frac{h}{2}) - f(y_i - \frac{h}{2})$. Similarly considering $P(x)$ as a constant P (say), then it is clear that $A_{7,i} = A_{11,i} = C_i = \frac{a^2h^2P^2}{12}$ (a cancellation term throughout (3.13)) and (3.13) reduces to the following finite-difference representation:

$$\delta^2 y_i + \frac{ahP}{2}(y_{i+1} - y_{i-1}) + a^2h^2\left(1 + \frac{\delta^2}{6}\right)Q_i y_i = a^2h^2\left(1 + \frac{\delta^2}{6}\right)R_i + K_{i,0,N}^2, \quad i \in \mathbb{N}_{N-1}.$$

By solving the above algebraic equations, the unknowns $y_j, j \in \mathbb{N}_N \cup \{0\}$, can be computed. Finally, using these y_j and M_j, we get the analytical fractal solution of BVP from (3.1).

Remark 3.3. When the first derivative is absent i.e., $Q(x) = 0$, one can use the above procedure with $Q_j = 0$ for all $j \in \mathbb{N}_N \cup \{0\}$. Otherwise, one can use a simpler way to calculate the moments as described in the following:
In this case, (3.8) reduces to

$$y''(x) + P(x)y(x) = R(x). \tag{3.18}$$

Discretizing (3.18), we get

$$M_i = R_i - P_i y_i \quad \text{for } i = 0, 1, 2, \ldots, N, \tag{3.19}$$

where $M_i = y''(x_i),\ R_i = R(x_i),\ P_i = P(x_i),\ y_i = y(x_i)$ for $i = 0, 1, 2, \ldots, N$.
Using (3.19) in (3.5), we have

$$\begin{aligned}
&\frac{6 + h_i^2 P_{i-1}}{6h_i} y_{i-1} + \left[\frac{P_i}{3} - \frac{1}{h_i h_{i+1}}\right](h_i + h_{i+1}) y_i + \frac{6 + h_{i+1}^2 P_{i+1}}{6h_{i+1}} y_{i+1} = \frac{h_i + h_{i+1}}{3} R_i \\
&\frac{h_i}{6} R_{i-1} + \frac{h_{i+1}}{6} R_{i+1} + \Big\{ a_i\alpha_i d_N - a_{i+1}\alpha_{i+1} d_0 - \frac{1}{6}(2h_i\alpha_i + h_{i+1}\alpha_{i+1})(R_N - P_N y_N) \\
&\quad - \frac{1}{6}(h_i\alpha_i + 2h_{i+1}\alpha_{i+1})(R_0 - P_0 y_0) + \frac{1}{\ell}(a_{i+1}\alpha_{i+1} - a_i\alpha_i)(y_N - y_0) \Big\}, \quad i \in \mathbb{N}_{N-1}.
\end{aligned} \tag{3.20}$$

From (3.6) and (3.19), we obtain

$$\begin{aligned}
&\left(\frac{1}{h_1} - \frac{a_1\alpha_1}{\ell} + \frac{h_1(1-\alpha_1)P_0}{3}\right) y_0 - \frac{6 + h_1^2 P_1}{6h_1} y_1 + \frac{6a_1 + \ell h_1 P_N}{6\ell}\alpha_1 y_N \\
&\qquad = \frac{h_1}{3}(1-\alpha_1)R_0 - \frac{h_1}{6}(R_1 - \alpha_1 R_N) - (1 - a_1\alpha_1)d_0.
\end{aligned} \tag{3.21}$$

Similarly, from (3.7) and (3.19), we get

$$\begin{aligned}
&\frac{6a_N - \ell h_N P_N}{6\ell}\alpha_N y_0 - \frac{6 - h_N^2 P_{N-1}}{6h_N} y_{N-1} + \left(\frac{1}{h_N} - \frac{a_N\alpha_N}{\ell} + \frac{h_N(1-\alpha_N)P_N}{3}\right) y_N \\
&\qquad = \frac{h_N}{3}(1-\alpha_N)R_N + \frac{h_N}{6}(R_{N-1} - \alpha_N R_0) + (1 - a_N\alpha_N)d_N.
\end{aligned} \tag{3.22}$$

Since $y_0 = y(x_0) = \beta^*$ and $y_N = y(x_N) = \gamma^*$ are available, we need to compute $(N+1)$ unknowns $y_1, \ldots, y_{N-1}$, d_0 and d_N from the system (3.20)–(3.22). Once this system is solved, using the values with $M_0, \ldots, M_N$ in (3.1), we obtain the desired spline FIF solution to the BVP.

4. Error analysis

In this section, we investigate the truncation error corresponding to the proposed method. Assume that the partition $\triangle := x_0 < x_1 < x_2 < \cdots < x_N$ of the domain interval $I := [x_0, x_N]$ is equally spaced. That is $h_i = x_i - x_{i-1} = h \ \forall\ i \in \mathbb{N}_N$, where $h = \frac{x_N - x_0}{N}$, and $a_i = a == \frac{1}{N} \forall i \in \mathbb{N}_N$. Then (3.5)–(3.7) can be written as:

$$\begin{aligned}
\frac{h}{6}M_{i-1} + \frac{2h}{3}M_i + \frac{h}{6}M_{i+1} &= \frac{1}{h}y_{i-1} - \left(\frac{1}{h} + \frac{1}{h}\right)y_i + \frac{1}{h}y_{i+1} - \Big\{ a\alpha_i d_N - a\alpha_{i+1} d_0 - \frac{1}{6}(2h\alpha_i \\
&\quad + h\alpha_{i+1})M_N - \frac{1}{6}(h\alpha_i + 2h\alpha_{i+1})M_0 + \frac{1}{\ell}(a\alpha_{i+1} - a\alpha_i)(y_N - y_0) \Big\} \\
&\quad \text{for } i = 1, 2, \ldots, N-1.
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
d_0 &= a\alpha_1 d_0 - \frac{h}{2}(1-\alpha_1)M_0 + \frac{h}{6}((1-\alpha_1)M_0 - M_1 + \alpha_1 M_N) \\
&\quad + \frac{a}{\ell}\left(\frac{y_1 - y_0}{a^2} - \alpha_1(y_N - y_0)\right),
\end{aligned}$$

$$\Rightarrow (1-a\alpha_1)d_0 = \frac{y_1-y_0}{h} - \frac{a\alpha_1}{\ell}(y_N-y_0) - \frac{h}{3}(1-\alpha_1)M_0 - \frac{h}{6}(M_1-\alpha_1 M_N). \quad (4.2)$$

and

$$\begin{aligned} d_N &= a\alpha_N d_N + \frac{h}{2}((1-\alpha_N)M_N) + \frac{h}{6}\Big[M_{N-1} - M_N + \alpha_N(M_N \\ &\quad - M_0)\Big] + \frac{a}{\ell}\Big(\frac{y_N-y_{N-1}}{a^2} - \alpha_N(y_N-y_0)\Big), \\ \Rightarrow (1-a\alpha_N)d_N &= \frac{y_N-y_{N-1}}{h} - \frac{a\alpha_N}{\ell}(y_N-y_0) + \frac{h}{3}(1-\alpha_N)M_N \\ &\quad + \frac{h}{6}(M_{N-1}-\alpha_N M_0). \end{aligned} \quad (4.3)$$

Assume that $y \in \mathcal{C}^4[x_0, x_N]$. Let us consider the Taylor series expansion of $y(x_i)$ and $y(x_{i+1})$ around the point x_{i-1} as

$$y(x_i) = y(x_{i-1}) + hy'(x_{i-1}) + \frac{h^2}{2!}y''(x_{i-1}) + \frac{h^3}{3!}y'''(x_{i-1}) + \mathcal{O}(h^4),$$

and

$$y(x_{i+1}) = y(x_{i-1}) + 2hy'(x_{i-1}) + 2h^2y''(x_{i-1}) + \frac{4h^3}{3}y'''(x_{i-1}) + \mathcal{O}(h^4).$$

It is clear that

$$\frac{-y(x_{i+1}) + 4y(x_i) - 3y(x_{i-1})}{2h} = y'(x_{i-1}) - \frac{h^2}{3}y'''(x_{i-1}) + \mathcal{O}(h^3).$$

Using the Taylor series expansion of $y(x_{i-1})$ and $y(x_{i+1})$ around the point x_i, we obtain the following:

$$\frac{y(x_{i+1}) - y(x_{i-1})}{2h} = y'(x_i) + \frac{h^2}{3!}y'''(x_i) + \mathcal{O}(h^4).$$

Similarly using the Taylor series expansion of $y(x_{i-1})$ and $y(x_i)$ around the point x_{i+1}, we obtain the following:

$$\frac{3y(x_{i+1}) - 4y(x_i) + y(x_{i-1})}{2h} = y'(x_{i+1}) - \frac{h^2}{3}y'''(x_{i+1}) + \mathcal{O}(h^3).$$

Hence, the following second order approximations to d_i can be used.

$$d_i = \frac{y_{i+1}-y_{i-1}}{2h}, \qquad d_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, \qquad d_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}.$$

Substituting $M_i = R_i - P_i d_i - Q_i y_i$ and d_i for $i = 0, 1, N$ in (4.1), we get

$$\begin{aligned} &\Big[-3h(1-a\alpha_1) + h^2(1-\alpha_1)P_0 + \frac{h^2P_1}{6} + 2h - \frac{2h^3\alpha_1}{\ell^2} - \frac{2h^3(1-\alpha_1)Q_0}{3}\Big]y_0 + \Big[4h(1-a\alpha_1) \\ &- \frac{4h^2(1-\alpha_1)P_0}{3} - 2h - \frac{h^3Q_1}{3}\Big]y_1 + \Big[-h(1-a\alpha_1) + \frac{h^2(1-\alpha_1)P_0}{3} - \frac{h^2P_1}{6}\Big]y_2 + \frac{h^2\alpha_1P_N}{6}y_{N-2} \\ &- \frac{2h^2\alpha_1P_N}{3}y_{N-1} + h^2\alpha_1\Big[\frac{P_N}{2} + \frac{2h\alpha_1}{\ell^2} + \frac{h\alpha_1Q_N}{3}\Big]y_N = -\frac{2h^3(1-\alpha_1)}{3}R_0 - \frac{h^3}{3}R_1 + \frac{h^3\alpha_1}{3}R_N. \end{aligned} \quad (4.4)$$

Replacing $M_i = R_i - P_i d_i - Q_i y_i$ and d_i for $i = 0, N-1, N$ in (4.2), we deduce

$$\begin{aligned} &\Big[3h(1-a\alpha_N) + h^2(1-\alpha_N)P_N + \frac{h^2P_{N-1}}{6} - 2h + \frac{2h^3\alpha_N}{\ell^2} - \frac{2h^3(1-\alpha_N)Q_N}{3}\Big]y_N + \Big[-4h(1-a\alpha_N) \\ &- \frac{4h^2(1-\alpha_N)P_N}{3} + 2h + \frac{h^3Q_{N-1}}{3}\Big]y_{N-1} + \Big[h(1-a\alpha_N) + \frac{h^2(1-\alpha_N)P_N}{3} - \frac{h^2P_{N-1}}{6} + 2h + \frac{h^3Q_{N-1}}{3}\Big]y_{N-2} \\ &- \frac{h^2\alpha_NP_0}{6}y_2 + \frac{2h^2\alpha_NP_0}{3}y_1 - h^2\alpha_N\Big[\frac{P_0}{2} + \frac{2h\alpha_N}{\ell^2} + \frac{h\alpha_NQ_0}{3}\Big]y_0 = \frac{2h^3(1-\alpha_N)}{3}R_N + \frac{h^3}{3}R_{N-1} - \frac{h^3\alpha_N}{3}R_0. \end{aligned} \quad (4.5)$$

Similarly, substituting $M_j = R_j - P_j d_j - Q_j y_j$ and d_j for $j = 0, N, i-1, i, i+1$ in (4.3), we have

$$\begin{aligned} &\Big[6 + h^2Q_{i-1} - \frac{3hP_{i-1}}{2} - 2hP_i + \frac{hP_{i+1}}{2}\Big]y_{i-1} + \Big[4h^2Q_i - 12 + 2hP_{i-1} - 2hP_i - 2hP_{i+1}\Big]y_i \\ &\Big[6 + h^Q_{i+1} - \frac{hP_{i-1}}{2} + 2hP_i + \frac{3hP_{i+1}}{2}\Big]y_{i+1} + \Big[h^2(\alpha_i + 2\alpha_{i+1}Q_0 - \frac{6h^2(\alpha_{i+1}-\alpha_i)}{\ell^2}) - \frac{9h\alpha_{i+1}}{\ell} \end{aligned}$$

$$+\frac{3h(\alpha_i+2\alpha_{i+1})P_0}{2}\Big]y_0+\Big[\frac{12h\alpha_{i+1}}{\ell}+2h(\alpha_i+2\alpha_{i+1})P_0\Big]y_1-\Big[\frac{3h\alpha_{i+1}}{\ell}+\frac{h(\alpha_i+2\alpha_{i+1})}{2}P_0\Big]y_2$$
$$-\Big[\frac{3h\alpha_i}{\ell}+\frac{h(2\alpha_i+\alpha_{i+1})}{2}P_N\Big]y_{N-2}+\Big[\frac{12h\alpha_i}{\ell}+2h(2\alpha_i+\alpha_{i+1})P_N\Big]y_{N-1}-\Big[\frac{9h\alpha_i}{\ell}+\frac{3h(2\alpha_i+\alpha_{i+1})P_N}{2}\Big]$$
$$=h^2R_{i-1}+4h^2R_i+h^2R_{i+1}+h^2(\alpha_i+2\alpha_{i+1})R_0-h^2(2\alpha_i+\alpha_{i+1})R_N. \tag{4.6}$$

Now let $|\alpha_i|<a^2=\frac{h^2}{\ell^2}$. We get the truncation error $T_0(h)$ associated with the equation given in (4.4) as

$$T_0(h)=\Big[-3h(1-a\alpha_1)+h^2(1-\alpha_1)P_0+\frac{h^2P_1}{6}+2h-\frac{2h^3\alpha_1}{\ell^2}-\frac{2h^3(1-\alpha_1)Q_0}{3}\Big]y_0+\Big[4h(1-a\alpha_1)$$
$$-\frac{4h^2(1-\alpha_1)P_0}{3}-2h-\frac{h^3Q_1}{3}\Big]y_1+\Big[-h(1-a\alpha_1)+\frac{h^2(1-\alpha_1)P_0}{3}-\frac{h^2P_1}{6}\Big]y_2+\frac{h^2\alpha_1P_N}{6}y_{N-2}$$
$$-\frac{2h^2\alpha_1P_N}{3}y_{N-1}+h^2\alpha_1\Big[\frac{P_N}{2}+\frac{2h}{\ell^2}+\frac{hQ_N}{3}\Big]y_N+\frac{2h^3(1-\alpha_1)}{3}R_0+\frac{h^3}{3}R_1-\frac{h^3\alpha_1}{3}R_N. \tag{4.7}$$

Replacing $R_i=y''(x_i)+P_iy'(x_i)+Q_iy_i$ for $i=0,1,N$ in (4.7) and after simplifying, we obtain

$$T_0(h)=\frac{2h^3\alpha_1}{\ell^2}(y_N-y_0)+\alpha_1h^2P_N\Big[\frac{y_{N-2}+3y_N-4y_{N-1}}{6}-\frac{h}{3}y'(x_N)\Big]+\frac{2h^3(1-\alpha_1)}{3}y''(x_0)$$
$$-\frac{h^3}{3}\alpha_1y''(x_N)+\Big[h(3a\alpha_1-1)+h^2(1-\alpha_1)P_0+\frac{h^2P_1}{6}\Big]y_0+\Big[2h(1-2a\alpha_1)$$
$$-\frac{4h^2(1-\alpha_1)P_0}{3}\Big]y_1+\Big[-\frac{h^2P_1}{6}-h(1-a\alpha_1)+\frac{h^2(1-\alpha_1)P_0}{3}\Big]y_2$$
$$+\frac{2h^3(1-\alpha_1)P_0}{3}y'(x_0)+\frac{h^3}{3}\Big[y''(x_1)+P_0y'(x_1)\Big]. \tag{4.8}$$

Using the Taylor series expansion for $y(x_1),y(x_2),y'(x_1),y''(x_1)$ about the point x_0 and using

$$\frac{y_{N-2}+3y_N-4y_{N-1}}{2h}=y'(x_N)-\frac{h^2}{3}y'''(x_N)+\cdots.$$

in (4.8), we obtain

$$T_0(h)=\frac{2h^3\alpha_1}{\ell^2}(y_N-y_0)+\alpha_1h^2P_N\Big[-\frac{h^3}{9}y'''(x_N)+\cdots\Big]-\frac{h^3\alpha_1}{3}y''(x_N)$$
$$+h^3\Big[\frac{P_0-P_1}{3}-\frac{2\alpha_1}{\ell}\Big]y'(x_0)+h^4\Big[\frac{P_0-P_1}{3}-\frac{2\alpha_1}{\ell}\Big]y''(x_0)+\cdots$$

Clearly, we can write $|T_0(h)|\le C_0h^3$, where C_0 is a constant. Thus, $T_0(h)=\mathcal{O}(h^3)$ as $h\to 0$. Since $|\alpha_i|<a^2=\frac{h^2}{\ell^2}$, it is possible to get $T_0(h)=\mathcal{O}(h^5)$ as $h\to 0$ under the assumption $|P_0-P_1|=\mathcal{O}(h^2)$.
We get the truncation error $T_N(h)$ associated with the equation given in (4.5) as

$$T_N(h)=\Big[3h(1-a\alpha_N)+h^2(1-\alpha_N)P_N+\frac{h^2P_{N-1}}{6}-2h+\frac{2h^3\alpha_N}{\ell^2}-\frac{2h^3(1-\alpha_N)Q_N}{3}\Big]y_N$$
$$+\Big[-4h(1-a\alpha_N)-\frac{4h^2(1-\alpha_N)P_N}{3}+2h+\frac{h^3Q_{N-1}}{3}\Big]y_{N-1}+\Big[h(1-a\alpha_N)$$
$$+\frac{h^2(1-\alpha_N)P_N}{3}-\frac{h^2P_{N-1}}{6}\Big]y_{N-2}-\frac{h^2\alpha_NP_0}{6}y_2+\frac{2h^2\alpha_NP_0}{3}y_1+h^2\alpha_N\Big[\frac{P_0}{2}-\frac{2h}{\ell^2}$$
$$-\frac{hQ_N}{3}\Big]y_0-\frac{2h^3(1-\alpha_N)}{3}R_N-\frac{h^3}{3}R_{N-1}+\frac{h^3\alpha_N}{3}R_0. \tag{4.9}$$

Substituting $R_i=y''(x_i)+P_iy'(x_i)+Q_iy_i$ for $i=0,N-1,N$ in (4.7) and after simplifying, we obtain

$$T_N(h)=\frac{2h^3\alpha_N}{\ell^2}(y_N-y_0)+\alpha_Nh^2P_0\Big[\frac{-y_2-3y_0+4y_1}{6}+\frac{h}{3}y'(x_0)\Big]-\frac{2h^3(1-\alpha_N)}{3}y''(x_N)$$
$$+\frac{h^3}{3}\alpha_Ny''(x_0)+\Big[-h(3a\alpha_N-1)+h^2(1-\alpha_N)P_N+\frac{h^2P_{N-1}}{6}\Big]y_N+\Big[-2h(1-2a\alpha_1)$$

$$-\frac{4h^2(1-\alpha_N)P_N}{3}\Big]y_{N-1}+\Big[-\frac{h^2P_{N-1}}{6}+h(1-a\alpha_N)+\frac{h^2(1-\alpha_N)P_N}{3}\Big]y_{N-2}$$
$$-\frac{2h^3(1-\alpha_N)}{3}y'(x_N)-\frac{h^3}{3}\Big[y''(x_{N-1})+P_Ny'(x_{N-1})\Big]. \tag{4.10}$$

Using the Taylor series expansion for $y(x_{N-1}), y(x_N), y'(x_{N-1}), y''(x_{N-1})$ about the point x_N and using

$$\frac{y_2+3y_0-4y_1}{2h}=y'(x_0)-\frac{h^2}{3}y'''(x_0)+\cdots.$$

in (4.10), we obtain

$$T_N(h)=\frac{2h^3\alpha_N}{\ell^2}(y_N-y_0)+\alpha_Nh^2P_0\Big[\frac{h^3}{9}y'''(x_0)-\cdots\Big]+\frac{h^3\alpha_N}{3}y''(x_0)$$
$$h^3\Big[\frac{P_{N-1}-P_N}{3}-\frac{2\alpha_N}{\ell}\Big]y'(x_N)+h^4\Big[\frac{P_{N-1}-P_N}{3}-\frac{2\alpha_N}{\ell}\Big]y''(x_N)+\cdots \tag{4.11}$$

From (4.11), we can write $|T_N(h)|\le C_Nh^3$, where C_N is a suitable constant. Thus, $T_N(h)=\mathcal{O}(h^3)$ as $h\to 0$. Since $|\alpha_i|<a^2=\frac{h^2}{\ell^2}$, it is possible to get $T_N(h)=\mathcal{O}(h^5)$ as $h\to 0$ under the assumption $|P_{N-1}-P_N|=\mathcal{O}(h^2)$.
Further, we can write the truncation error $T_i(h)$ associated with the equation given in (4.6) as

$$T_i(h)=\Big[12+2h^2Q_{i-1}-3hP_{i-1}-4h+hP_{i+1}\Big]y_{i-1}+\Big[8h^2Q_i-24+4hP_{i-1}\Big]y_i+\Big[12$$
$$+2h^2Q_{i-1}-hP_{i-1}+4h+3hP_{i+1}\Big]y_{i+1}+\Big[2h^2(\alpha_i+2\alpha_{i+1})Q_0-\frac{12h^2}{\ell^2}(\alpha_{i+1}-\alpha_i)$$
$$-\frac{18h\alpha_{i+1}}{\ell}-3h(\alpha_i+2\alpha_{i+1})P_0\Big]y_0+\Big[\frac{24h\alpha_{i+1}}{\ell}+4h(\alpha_i+2\alpha_{i+1})P_0\Big]y_1+\Big[-\frac{6h\alpha_{i+1}}{\ell}$$
$$-2h(\alpha_i+2\alpha_{i+1})P_0\Big]y_2+\Big[-\frac{6h\alpha_i}{\ell}-2h(2\alpha_i+\alpha_{i+1})P_N\Big]y_{N-2}+\Big[4h(2\alpha_i+\alpha_{i+1})P_N$$
$$+\frac{24h\alpha_i}{\ell}\Big]y_{N-1}+\Big[-2h^2(2\alpha_i+\alpha_{i+1})Q_N+\frac{12h^2}{\ell^2}(\alpha_{i+1}-\alpha_i)-3h(2\alpha_i+\alpha_{i+1})P_N$$
$$-\frac{18h\alpha_i}{\ell}\Big]y_N-2h^2R_{i-1}-8h^2R_i-2h^2R_{i+1}-2h^2(\alpha_i+2\alpha_{i+1})R_0+2h^2(2\alpha_i+\alpha_{i+1})R_N. \tag{4.12}$$

Replacing $R_j=y''(x_j)+P_jy'(x_j)+Q_jy_j$ for $j=0,i-1,i,i+1,N$ and using the Taylor series expansions of $y(x_i)$ and $y(x_{i+1})$ around the point x_{i-1}, we obtain the following:

$$T_i(h)=4h^3(1-2P_i)y''(x_{i-1})+h^3\Big[12-\frac{8h}{3}-\frac{2hP_{i-1}}{3}-\frac{4h^2P_i}{3}+\frac{10hP_{i+1}}{3}\Big]y'''(x_0)+\cdots$$
$$+\Big[2h^2(\alpha_i+2\alpha_{i+1})Q_0-\frac{12h^2}{\ell^2}(\alpha_{i+1}-\alpha_i)-\frac{18h\alpha_{i+1}}{\ell}-3h(\alpha_i+2\alpha_{i+1})P_0\Big]y_0$$
$$+\Big[4h(\alpha_i+2\alpha_{i+1})P_0+\frac{24h\alpha_{i+1}}{\ell}\Big]y_1+\Big[-\frac{6h\alpha_{i+1}}{\ell}-2h(\alpha_i+2\alpha_{i+1})P_0\Big]y_2$$
$$+\Big[-2h(2\alpha_i+\alpha_{i+1})P_N-\frac{6h\alpha_i}{\ell}\Big]y_{N-2}+\Big[4h(2\alpha_i+\alpha_{i+1})P_N+\frac{24h\alpha_i}{\ell}\Big]y_{N-1}$$
$$+\Big[-2h^2(2\alpha_i+\alpha_{i+1})Q_N+\frac{12h^2}{\ell^2}(\alpha_{i+1}-\alpha_i)-3h(2\alpha_i+\alpha_{i+1})P_N-\frac{18h\alpha_i}{\ell}\Big]y_N$$

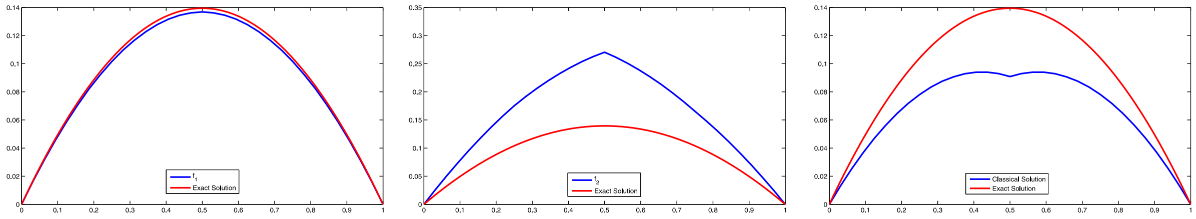
Since $|\alpha_i|<a^2=\frac{h^2}{\ell^2}$, we have $|T_i(h)|\le C_ih^3$, where C_i is a constant. Thus, $T_i(h)=\mathcal{O}(h^3)$ as $h\to 0$ for $i=1,2,\ldots,N-1$. Consequently, any approximation solution will converge to the exact solution as $h\to 0$.

5. Numerical results and discussion

First, we consider the case (a) when y' is absent. Here, we try to get a smooth fractal solution of the following BVP:

Example 5.1. Consider the two point boundary value problem

$$y''+y+1=0,\quad y(0)=y(1)=0, \tag{5.1}$$



(a) Exact solution S, fractal solution f_1 (b) Exact solution S, fractal solution f_2 (c) Exact solution S, classical solution P

Fig. 1. Solutions of BVP with different scaling factors.

whose analytical (or exact) solution is

$$y(x) = \cos x + \tan(1/2)\sin x - 1.$$

Divide the interval [0, 1] into two equal sub-intervals ($N = 2$) as [0, 1/2] and [1/2, 1] with $x_0 = 0$, $x_1 = 1/2$ and $x_2 = 1$. It is clear that $h_1 = h_2 = h = 1/2$, $\ell = 1$ and $a = 1/2$. Assume the scaling factors $\alpha_i = 0.1 = \alpha$, $i = 1, 2$. Here $P(x) = 1$, $Q(x) = 0$, and $R(x) = -1$. Thus, $P_j = 1$, $Q_j = 0$, and $R_j = -1$ for $j = 0, 1, 2$. From (3.17), we will get the following equation for $i = 1$:

$$\alpha(d_2 - d_0) + 8y_1 - \frac{2}{3}y_1 = -\alpha + 1 \Rightarrow y_1 = \frac{3\alpha}{22}(d_0 - d_2 - 1) + \frac{3}{22}. \tag{5.2}$$

From (3.6) and (3.7), we find the following equations respectively:

$$\begin{aligned} d_0 &= f'(0) = \frac{\alpha}{88}(69d_0 - 25d_2 - 49) + \frac{47}{88}, \\ d_2 &= f'(1) = \frac{\alpha}{88}(69d_2 - 25d_0 + 49) - \frac{47}{88}. \end{aligned} \tag{5.3}$$

Solving the above system (5.2)–(5.3), we compute $d_0 = -d_2 = 0.5356$ and $y_1 = y(1/2) = 0.136904 \equiv 3/22$. Now, we will compute the moments using the above values with $y_0 = y_2 = 0$.

$$M_0 = R_0 - P_0 y_0 \Rightarrow M_0 = -1,$$

$$M_2 = R_2 - P_2 y_2 \Rightarrow M_2 = -1,$$

$$M_1 = R_1 - P_1 y_1 \Rightarrow M_1 = -1.1081.$$

These moments are then used in (3.1) to write the functional equation of a cubic spline FIF (say f_1) in the following:

$$f_1(L_i(x)) = \begin{cases} \frac{1}{40}f_1(x) - \frac{229}{40000}x^3 - \frac{81}{160}x^2 + \frac{1}{80}x & \text{for } x \in [0, 1/2], \\ \frac{1}{40}f_1(x) + \frac{229}{40000}x^3 - \frac{5187}{40000}x^2 - \frac{269}{20000}x + \frac{687}{5000}, & \text{for } x \in [1/2, 1]. \end{cases}$$

The iteration of this functional equation gives us the graph of the desired cubic spline FIF in Fig. 1(a). By assuming different values to scaling factors, say $\alpha_1 = 1/8$ and $\alpha_2 = -1/7$, we obtain the following system of equations with the unknown variables y_1, d_0, d_2 directly from (3.20)–(3.22):

$$\begin{aligned} &y_1 + 0.0195d_0 + 0.0170d_3 = 0.2740, \\ &y_1 - 0.45d_0 = 0.1050, \\ &y_1 + 0.5450d_2 = 0.1491. \end{aligned}$$

The solution of the above system gives $y_1 = 0.2705$, $d_0 = 0.3678$ and $d_2 = -0.2172$. Thus, we get $M_0 = -1$, $M_1 = -1.2705$ and $M_2 = -1$. Using these moments in (3.1), we obtain another cubic spline FIF (say f_2), see Fig. 1(b), and its functional equation is

$$f_2(L_i(x)) = \begin{cases} \frac{1}{32}f_2(x) - 0.0113x^3 - 0.1094x^2 + 0.3911x, & \text{for } x \in [0, 1/2], \\ -\frac{1}{28}f_2(x) + 0.0113x^3 - 0.1767x^2 - 0.0105x + 0.2705, & \text{for } x \in [1/2, 1]. \end{cases}$$

Note that the scaling factors can be chosen from the range $(-a^2, a^2) = (-0.25, 0.25)$. We have shown f_2 to demonstrate the possible of getting a large class of solutions by fractal methodology. By setting all scaling factors to zero, we obtain the classical cubic spline interpolant P, see Fig. 1(c). From Fig. 1, it is easy to observe that the fractal cubic spline f_1 is a better approximant in comparison with the classical cubic spline solution.

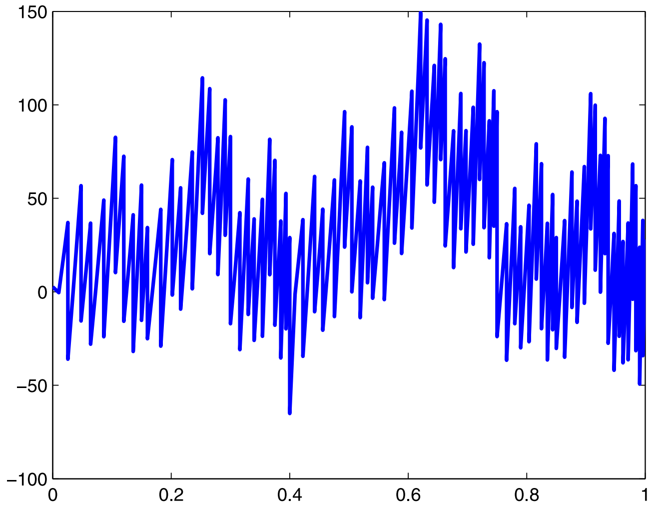


Fig. 2. Nowhere differentiable curve $R(x)$.

Table 1
Parameters for cubic spline fractal solutions of BVP (5.4).

Fig.	Scaling vector, end derivatives and moments
3a	$\alpha = (0.8, 0.8, 0.8)$, $d_0 = 9.4232$, $d_3 = 19.4085$ $M_0 = 1.9987$, $M_1 = -65.0177$, $M_2 = 93.8429$, $M_3 = 4.9980$
3b	$\alpha = (0.8, 0.8, 0.8)$, $d_0 = 9.4232$, $d_3 = 19.4085$ $M_0 = 1.7445$, $M_1 = -64.8296$, $M_2 = 93.8429$, $M_3 = 4.9667$
3c	$\alpha = (0.35, 0.3, 0.25)$, $d_0 = 6.8810$, $d_3 = 19.3810$ $M_0 = 9.1776$, $M_1 = -54.8010$, $M_2 = 99.3891$, $M_3 = 35.9852$
3d	$\alpha = (0.8, 0.8, 0.8)$, $d_0 = 9.4232$, $d_3 = 19.4085$ $M_0 = -45.1382$, $M_1 = -28.7917$, $M_2 = 74.3697$, $M_3 = 62.6486$
3e	$\alpha = (0.35, 0.3, 0.25)$, $d_0 = 6.8810$, $d_3 = 19.3810$ $M_0 = 9.1776$, $M_1 = -54.8010$, $M_2 = 99.3891$, $M_3 = 35.9852$
3f	$\alpha = (0, 0, 0)$, $d_0 = 9.4232$, $d_3 = 19.4085$ $M_0 = -28.3648$, $M_1 = -47.1184$, $M_2 = 93.5364$, $M_3 = 42.1338$

Example 5.2. Consider the following two-point BVP

$$y'' + P(x)y' + Q(x)y = R(x), \quad y(0) = 1,\ y(1) = 2, \tag{5.4}$$

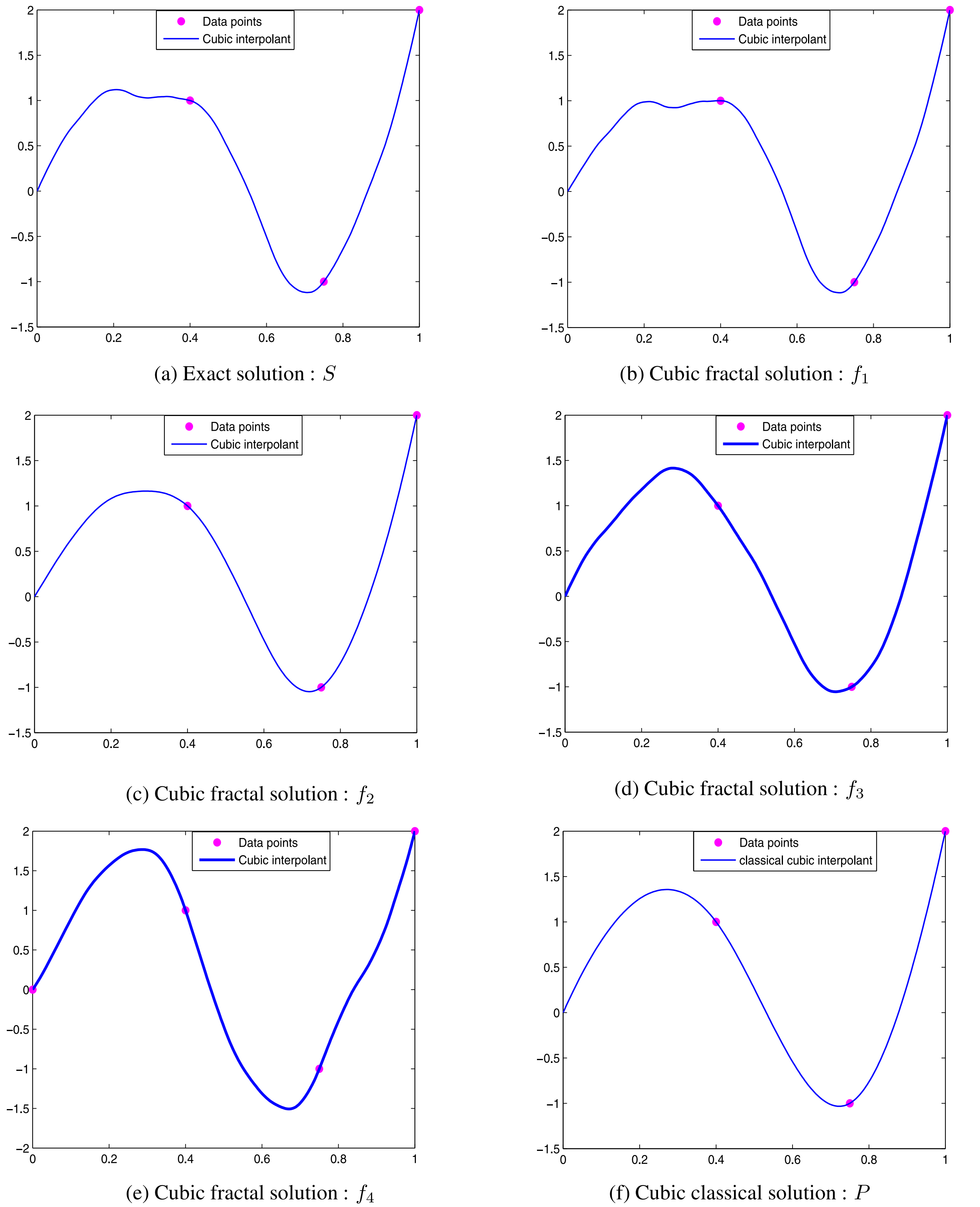
where $P(x) = x^2 + 0.1$, $Q(x) = \sin x + 0.1$ and $R(x)$ is the nowhere differentiable function as shown in Fig. 2.

Divide the interval [0, 1] into subintervals as $[x_0, x_1]$, $[x_1, x_2]$ and $[x_2, x_3]$, where $x_0 = 0$, $x_1 = 0.4$, $x_2 = 0.75$ and $x_3 = 1$. From the graph, it is found that $R_0 = 2.6868$, $R_1 = -65.0670$, $R_2 = 96.3384$ and $R_3 = 27.9332$ at the knot points. Thus from (3.6)–(3.7) and (3.11)–(3.17), we obtain a system of equations with the unknown variables y_1, y_2, d_0 and d_3. Assuming the scaling factors as $\alpha_i = 0.8$, $i \in \mathbb{N}_3$, after some simple algebraic computations, we obtain the values $d_0 = 9.4232$, $d_3 = 19.4085$ and the moments $M_0 = 1.7445$, $M_1 = -64.8296$, $M_2 = 93.8429$ and $M_3 = 4.9667$. Using these values, we have calculated $y_1 = 1$ and $y_2 = -1$.

The exact solution S of the BVP is given in Fig. 3a. An approximated fractal cubic spline solution using moments is as shown in Fig. 3b and is defined as follows:

$$f_1(L_i(x)) = \begin{cases} \frac{16}{125}f_1(x) - \frac{4669}{2500}x^3 + \frac{4}{125}x^2 + \frac{129}{50}x, & \text{for } x \in [0, 0.4], \\ \frac{49}{500}f_1(x) + \frac{5491}{2500}x^3 - \frac{2059}{500}x^2 + \frac{32}{625}x - \frac{110}{625}, & \text{for } x \in [0.4, 0.75], \\ \frac{1}{20}f_1(x) - \frac{235}{250}x^3 + \frac{715}{250}x^2 - \frac{91}{100}x + \frac{91}{100}, & \text{for } x \in [0.75, 1]. \end{cases}$$

Similarly, for different choices of scaling factors α_i, $i = 1, 2, 3$, we obtain different values for d_0, d_3, M_0, M_1, M_2 and M_3 which are listed in Table 1. With respect to these values, we obtain various fractal cubic spline solutions, and they are presented in Figs. 3c–e. Setting all scaling factors to 0, we obtain the classical cubic spline solution P defined through moments as shown in Fig. 3f. A comparison between the exact solution, 4 different cubic fractal solutions and the classical solution are illustrated in Fig. 4. The uniform errors between the exact solution of the BVP with various fractal cubic spline solutions are listed in Table 2. It is easy to observe from Fig. 4 that the cubic fractal spline solution f_1 is the closest approximated solution among the proposed five solutions to the given BVP, whose non-homogeneous term is nowhere differentiable function in the given domain.



(a) Exact solution : S

(b) Cubic fractal solution : f_1

(c) Cubic fractal solution : f_2

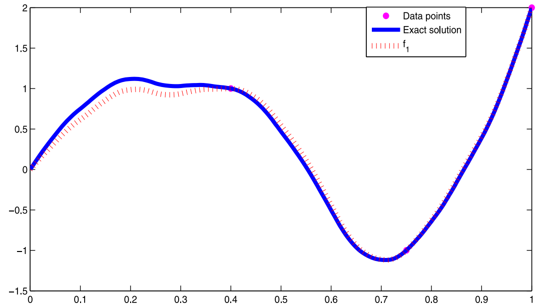
(d) Cubic fractal solution : f_3

(e) Cubic fractal solution : f_4

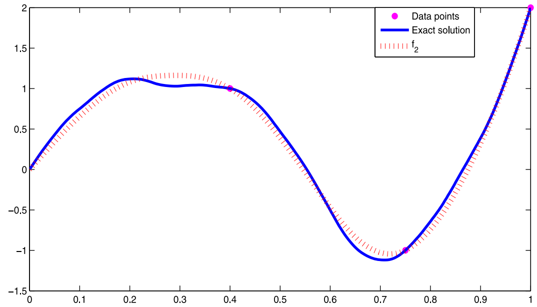
(f) Cubic classical solution : P

Fig. 3. Solutions of BVP with different scaling factors.

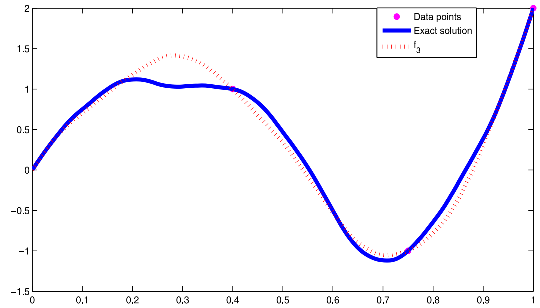
Remark 5.1. Note that the solution of BVP (3.8)–(3.9) is not unique if $y(x)$ is not infinitely differentiable. When we assume that y'' is not differentiable, it is possible to get a wide variety of solutions by fractal methodology due to the flexibility offered by the scaling factors. The flexibility in the choice of fractal interpolants can be harvested to elect a



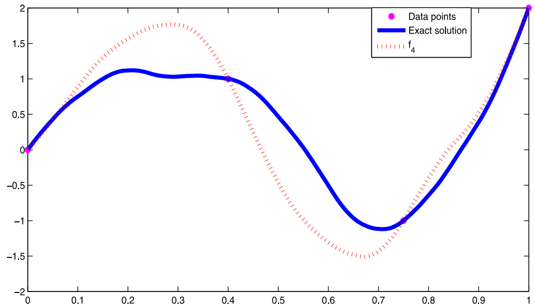
(a) S, f_1



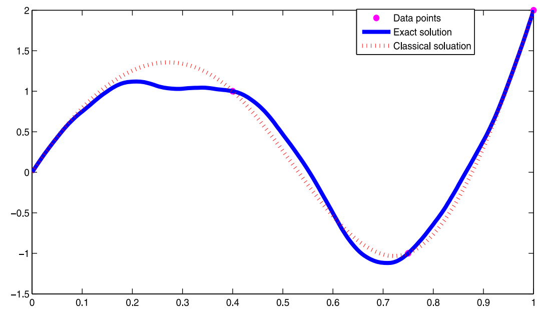
(b) S, f_2



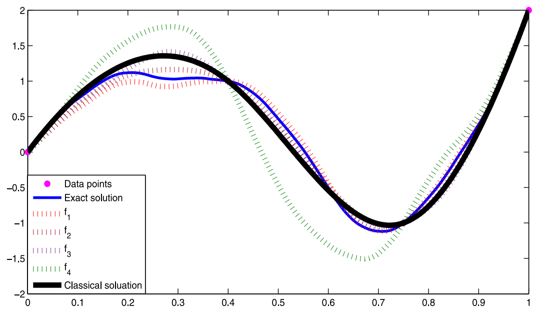
(c) S, f_3



(d) S, f_4



(e) S, P



(f) S, f_1, f_2, f_3, f_4, P

Fig. 4. Approximation of exact solution of BVP (5.4) with the proposed cubic spline FIFs.

Table 2
Uniform error between the exact solution and cubic spline fractal solutions.

$\|S - f_1\|_\infty$	0.1401
$\|S - f_2\|_\infty$	0.1445
$\|S - f_3\|_\infty$	0.3846
$\|S - f_4\|_\infty$	1.03
$\|S - C\|_\infty$	0.3232

suitable interpolant by concentrating some desirable features such as smoothness, fairness, fractality in the derivative, and threshold error between $R(x)$ and its approximation. The question on "optimum curve" can be addressed based on Levkovich's work [23]. Here the problem is for given functions $P(x)$, $Q(x)$, $R(x)$, find the IFS parameters of y such that $y'' + Py' + Q = R$ on $[x_0, x_N]$. In Levkovich's work, contraction affine mappings generating a given function is obtained based on the connection between the maxima skeleton of wavelet transform of the function and positions of the fixed points of the affine mappings. One can adapt a similar procedure to find the connection between the strongest singularities of $R(x)$ and the 2nd derivative of the fixed points of the cubic spline IFS. This will give an idea to fix the position of grid points on $[x_0, x_N]$. Then, the optimal curve can be defined as the fractal curve produced by a specific set of parameters that minimizes a suitable numerical quantity assigned to all possible curves obtained by the scheme ensuring that the error $\|y'' + Py' + Q - R\|$ is minimal. Such a constrained optimization problem may be solved by means of a differential evolution optimization algorithm/genetic algorithm to choose the scaling parameters for construction of the desired cubic spline FIF.

6. Summary

In this work, we have discussed the solutions of two-point BVPs by using cubic spline FIFs through moments in a deterministic manner even if the non-homogeneous differential equation involves a continuous function which is nowhere differentiable. Using the continuity condition at the grid points of the cubic spline fractal and the end point conditions by the derivatives, we have computed the values at the nodes $y_1, y_2, \ldots, y_{N-1}$ from a tridiagonal system. Then, the moments are computed in terms of $y_i,\ i \in \mathbb{N}_N \cup \{0\}$ to obtain the desired cubic spline fractal solution. Selecting the scaling factors to be zero and equal interpolating knot sequences, the proposed method coincides with the solution developed in [11]. Hence, our method is more general than the classical cubic spline solution of the second order ordinary BVP. The truncation error of the proposed method is $\mathcal{O}(h^3)$ as $h \to 0$.

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