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Spectral properties of solutions of hypergeometric-type differential equations

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Abstract

The second-order differential equation $\sigma(x)y'' + \tau(x)y' + \lambda y = 0$ is usually called *equation of hypergeometric type*, provided that σ, τ are polynomials of degree not higher than two and one, respectively, and λ is a constant. Their solutions are commonly known as hypergeometric-type functions (HTFs). In this work, a study of the spectrum of zeros of those HTFs for which $\lambda = -\nu\tau' - \frac{1}{2}\nu(\nu-1)\sigma''$, $\nu \in \mathbb{R}$, and σ, τ are independent of ν, is done within the so-called semiclassical (or WKB) approximation. Specifically, the semiclassical or WKB density of zeros of the HTFs is obtained analytically in a closed way in terms of the coefficients of the differential equation that they satisfy. Applications to the Gaussian and confluent hypergeometric functions as well as to Hermite functions are shown.

Key words: Differential equations; Zeros; Special functions; Semiclassical approximation

1. Introduction

As is well known, many important problems of theoretical and mathematical physics (e.g., the study of the Laplace and Helmholtz equations in curvilinear coordinates by means of the method of separation of variables, or, in quantum mechanics, the nonrelativistic and relativistic equations for the Coulomb potential, the harmonic oscillator,...) lead to the differential equation

$$u'' + \frac{\tilde{\tau}(x)}{\sigma(x)}u' + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}u = 0,$$

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where σ and $\tilde{\sigma}$ are polynomials of degree at most two and $\tilde{\tau}$ is a polynomial of degree one.

As Nikiforov and Uvarov [14] pointed out, to study this differential equation, it is useful to reduce it to a simpler one of the form

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \tag{1}$$

where σ, τ are polynomials of degree at most two and one, respectively, and λ is a constant.

This reduction is done [14] by taking $u = \phi(x)y$, with

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)},$$

$\pi(x)$ being the first-degree polynomial

$$\pi(x) = \tfrac{1}{2}(\sigma' - \tilde{\tau}) \pm \sqrt{\left(\tfrac{1}{2}(\sigma' - \tilde{\tau})\right)^2 - \tilde{\sigma} + k\sigma}\,.$$

Notice that, since π is a polynomial, the constant k is determined by the condition that the expression under the square root sign is the square of a first-degree polynomial.

The differential equation (1) is usually known [14] as an equation of hypergeometric type and its solutions as functions of hypergeometric type (HTFs). Here we will restrict ourselves to the HTF for which

$$\lambda = -\nu\tau' - \tfrac{1}{2}\nu(\nu - 1)\sigma'', \quad \nu \in \mathbb{R}. \tag{2}$$

To this class belong, e.g., the Hermite functions, some particular hypergeometric functions and the classical orthogonal polynomials.

The aim of this work is to study the distribution of zeros of the aforementioned broad class of HTFs defined above. Denoting their zeros by $\{x_{\nu,i}\}_{i=1}^{N}$, this distribution is

$$\rho_N^{(\nu)}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_{\nu,i}). \tag{3}$$

This study will be done within the so-called semiclassical (or WKB) approximation [7,10,14]. As described in Section 2, a recent result [3] based on this method allows us to obtain in an analytical way an approximation (to be called from now on semiclassical or WKB density of zeros) of the density function corresponding to (3) for the solutions of linear second-order differential equations. As a straight consequence of this result we are able to give this "WKB density of zeros" of the HTFs, in terms of the coefficients of the differential equation (1) that they satisfy (see Theorem 2).

Section 3 contains applications of the above general result to some particular cases which will be chosen by taking into account the canonical forms of the HTF. As shown in [14], by inserting certain linear changes of variable, the differential equation (1) can be reduced to four canonical forms, which are Gaussian and confluent hypergeometric functions, Hermite functions and a particular case of Lommel functions which are related with Bessel functions. Here we will consider the first three forms because, in those cases, we will be able to show the goodness of our approximation, since the exact density of zeros of some of them (the classical orthogonal polynomials) is known [8,12,15].

Finally, some concluding remarks are given.

2. The WKB method: density of zeros of the hypergeometric-type functions

The problem of finding uniformly asymptotic solutions of differential equations of the form

$$a_0(x)y'' + a_1(x)y' + \lambda r(x)y = 0,$$

as $\lambda \to +\infty$, can be considered as one of the examples which show the transition from the classical physics of the late nineteenth century to the quantum mechanics of the early twentieth century. The initial work of Wentzel, Kramers and Brillouin on this subject, which was extended later on by Langer [11] and many others, gave rise to a method (usually known as WKB method) to obtain such approximate solutions which are commonly called *semiclassical solutions* [7,10]. This method has been widely used in many problems of mathematical physics, leading to asymptotic formulas for many special functions (e.g., Bessel functions [11], classical orthogonal polynomials [14], etc.).

Here we are going to use this method in order to obtain the WKB density of zeros, which is an analytical approximate expression for the density of zeros of solutions of any linear second-order differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \tag{4}$$

It should be remarked that this approximation is obtained in terms of the coefficients $a_i(x)$, $i = 0, 1, 2$, which characterize the above differential equation. The result is established by means of the following theorem [3].

Theorem 1. *Let S and ϵ be the functions*

$$S(x) = \frac{1}{4a_0^2}\{2a_0(2a_2 - a_1') + a_1(2a_0' - a_1)\}, \tag{5}$$

$$\epsilon(x) = \frac{1}{4[S(x)]^2}\left\{\frac{5[S'(x)]^2}{4S(x)} - S''(x)\right\}. \tag{6}$$

Then, the semiclassical or WKB density of zeros of the solutions of (4) *is given by*

$$\rho_{\mathrm{WKB}}(x) = \frac{1}{\pi}[S(x)]^{1/2}, \quad x \in I \subseteq \mathbb{R}, \tag{7}$$

in every interval I where the function S is positive, provided that the condition $\epsilon(x) \ll 1$ holds.

Proof. Although the proof of this theorem can be found in [3], for completeness we include here a scheme of it.

Firstly, insert in (4) the change of variable

$$y(x) = u(x)\exp\left\{-\frac{1}{2}\int\frac{a_1(x)}{a_0(x)}\,\mathrm{d}x\right\}$$

to obtain its Schrödinger form

$$u'' + S(x)u = 0,$$

where the function S is given by (5). It is then clear that the zeros of the function y exactly coincide with the ones of u. On the other hand, oscillation theory [5] allows us to conclude that when $S(x) < 0$, u has at most one zero. So, to calculate the density function we are interested in, only the case $S(x) > 0$ should be considered.

Then, by taking

$$\phi(x) = \int_{x_0}^{x} [S(t)]^{1/2}\, \mathrm{d}t, \qquad v(x) = [S(x)]^{1/4} u(x)$$

in the Schrödinger form of (4), we obtain the following differential equation for the function $V(\phi)$ $(\equiv v(x))$:

$$V''(\phi) + (1 + \delta(\phi))V(\phi) = 0, \quad \left({}' \equiv \frac{\mathrm{d}}{\mathrm{d}\phi}\right)$$

where $\delta(\phi) \equiv \epsilon(x)$ and ϵ is given by (6).

The application of the WKB method to this equation tells us that if the condition $\delta(\phi) \ll 1$ holds, then its semiclassical solution has the expression

$$V(\phi) \approx C_a \cos \phi + C_b \sin \phi.$$

So, the semiclassical approximation u_{WKB} of the function u is

$$u_{\mathrm{WKB}}(x) = \frac{C_1}{\left[S(x)^{1/4}\right]} \sin[C_2 + \phi(x)],$$

where C_i, $i = 1, 2$, are constants.

Since $\phi(x)$ is a positive increasing function in the interval where $S(x)$ is positive, the zeros of u_{WKB} can be denoted by $x_1 < x_2 < \cdots < x_k < \cdots$. Moreover they satisfy

$$\phi(x_k) = k\pi - C_2, \quad k = 0, 1, 2, \ldots .$$

So,

$$\frac{1}{\pi}\phi(x_k) = k + \mathrm{o}(k).$$

This latter expression allows to consider the function

$$N(x) = \frac{1}{\pi}\phi(x)$$

as the functional extension of the cumulative number of zeros of u_{WKB}. Then, the construction of the semiclassical or WKB density of zeros is completed because, by definition [3], we have

$$\rho_{\mathrm{WKB}}(x) = \frac{\mathrm{d}N(x)}{\mathrm{d}x} = \frac{1}{\pi}[S(x)]^{1/2}. \qquad \square$$

To compare this WKB density of zeros with the true one (see (3)), one has to take into account that, if we denote by ξ the total number of zeros, the function $N(x)/\xi$ gives the proportion of zeros less than or equal to x; so $\rho_{\mathrm{WKB}}(x)/\xi$ is the function which should be used for comparison.

Let us consider now the hypergeometric differential equation (1) with the restriction (2). In this case, the WKB condition $\epsilon(x) \ll 1$ (ϵ given by (6)) is

$$\epsilon(x;\nu)=\frac{P(x;\nu)}{Q(x;\nu)} \ll 1,$$

where $P(x;\nu)$ and $Q(x;\nu)$ are polynomials in x and also in the parameter ν. It is easy to show that, if we consider P and Q as functions of ν, these two polynomials are such that $\deg\{P(x;\nu)\} < \deg\{Q(x;\nu)\}$. So, for the HTFs, the WKB condition holds if we choose ν large enough.

On the other hand, the function S given by (5), which specifies the interval where the WKB density of zeros is defined, has the expression

$$S(x;\nu)=\frac{1}{4[\sigma(x)]^2}R(x;\nu),$$

where R is the following polynomial of second degree in the variable x:

$$R(x;\nu)=2\sigma(x)[2\lambda-\tau'(x)]+\tau(x)[2\sigma'(x)-\tau(x)]. \tag{8}$$

Or, taking into account that σ and τ are polynomials of degree at most two and one, respectively, one has

$$R(x;\nu)=c_2(\nu)x^2+c_1(\nu)x+c_0(\nu), \tag{9}$$

with

$$\begin{aligned}
c_2(\nu) &= -\tau'^2+\tau'\sigma''+2\sigma''\left(-(\nu\tau')-\tfrac{1}{2}(-1+\nu)\nu\sigma''\right),\\
c_1(\nu) &= -2\tau(0)\tau'+2\tau(0)\sigma''+4\sigma'(0)\left(-(\nu\tau')-\tfrac{1}{2}(-1+\nu)\nu\sigma''\right),\\
c_0(\nu) &= -\tau(0)^2+2\tau(0)\sigma'(0)-2\sigma(0)\tau'+4\sigma(0)\left(-(\nu\tau')-\tfrac{1}{2}(-1+\nu)\nu\sigma''(0)\right).
\end{aligned}$$

It is then clear that S is positive in every interval where R is. So, the result stated in Theorem 1 can be summarized for HTFs in the following theorem.

Theorem 2. *The semiclassical or WKB density of zeros of the HTF solutions of* (1) *with the restriction given by* (2) *is*

$$\rho_{\mathrm{WKB}}(x;\nu)=\frac{1}{2\pi\sigma(x)}\sqrt{R(x;\nu)}, \tag{10}$$

in every interval where the function R (given by (8) *or* (9)*) is positive, provided that* $\nu \gg 1$.

From the general expression (10), many particular cases of interest could be deduced. But, before describing them (this will be done in the next section), it should be pointed out that, obviously, the density of zeros (7) is not invariant under any change of variable. So, if we take $x = x(t)$ in the general linear second-order differential equation (4), the WKB density of zeros of the solutions of the new equation cannot be obtained by replacing x by $x(t)$ in (7). However, if the change is linear, say $x(t) = c_1 t + c_2$, then one has the relation

$$\rho_{\mathrm{WKB}}(t) = |c_1| \rho_{\mathrm{WKB}}(x) |_{x=x(t)}, \tag{11}$$

as can be easily deduced.

3. Applications: canonical forms of the hypergeometric-type differential equation

In this section we will apply the general result stated in Theorem 2 to some particular cases of interest. As pointed out in the Introduction, in doing this, the canonical forms of the hypergeometric-type differential equation (1) will be considered.

These canonical forms are obtained from (1) by means of certain linear changes of variable [14] and they are closely related with the concrete degrees of the polynomials σ and τ. For illustration, we consider three of these forms: Gaussian and confluent hypergeometric equations and Hermite equations.

3.1. Gaussian hypergeometric functions

If σ is a second-degree polynomial with two different roots, say $\sigma(x) = (x-a)(b-x)$, the change of variable $x = a + (b-a)t$ transforms (1) into

$$t(1-t)y''(t) + \frac{1}{b-a}\tau[a + (b-a)t]\, y'(t) + \lambda y(t) = 0.$$

Then, it is always possible to choose parameters α, β and γ so that this equation can be written in the form

$$t(1-t)y''(t) + [\gamma - (\alpha + \beta + 1)t]\, y'(t) - \alpha\beta y(t) = 0,$$

which is the Gaussian hypergeometric equation [1], whose solutions are usually represented by $F[\alpha, \beta; \gamma; t]$. In this case, the restriction given by (2) becomes $\alpha = -\nu$, and the application of Theorem 2 gives rise to the following corollary.

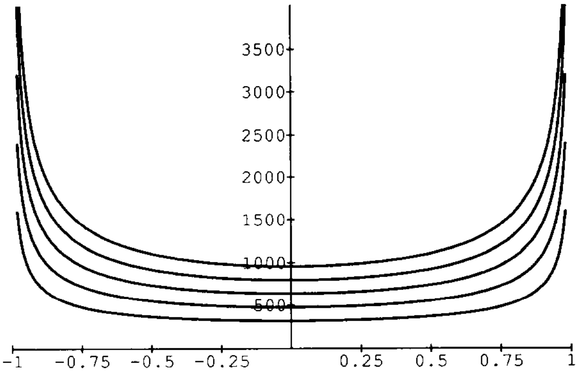
Corollary 3. *The semiclassical or WKB density of zeros of the hypergeometric functions* $F[-\nu, \beta; \gamma; t]$, $\nu \in \mathbb{R}$, *is*

$$\rho_{\mathrm{WKB}}(t; \nu) = \frac{1}{2\pi t(1-t)} \sqrt{R_1(t; \nu)}, \tag{12}$$

where

$$R_1(t; \nu) = \left[1 - (\beta-\nu)^2 - 4\nu\beta\right] t^2 + [2\gamma(\beta - \nu - 1) + 4\nu\beta]t + \gamma(2-\gamma),$$

in every interval where the second-degree polynomial R_1 *is positive, provided that* $\nu \gg 1$.



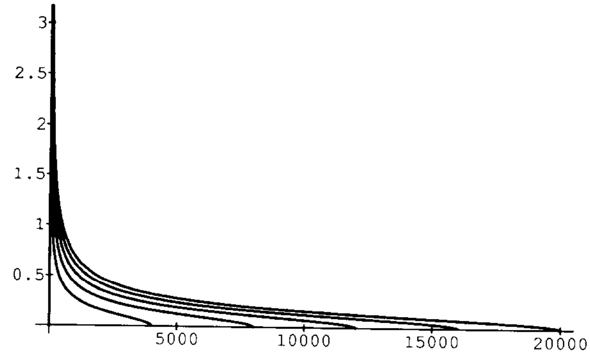


Fig. 1. WKB density of zeros (see (13)) of Legendre polynomials $P_n(x)$ for $n = 1000, 1500, 2000, 2500, 3000$.

Fig. 2. WKB density of zeros (see (16)) of Laguerre polynomials $L_n^{(0)}(x)$ for $n = 1000, 2000, 3000, 4000, 5000$.

As a specific example, let us consider now the Legendre polynomials $\{P_n(x)\}_{n \in \mathbb{N}}$. These polynomials can be expressed in terms of the Gaussian hypergeometric functions [2,14] as

$$P_n(x) = F\left[-n, n+1; 1; \tfrac{1}{2}(1-x)\right].$$

So, putting $\nu = n \in \mathbb{N}$, $\beta = n+1$, $\gamma = 1$ in (12) and taking into account the relation (11) (notice that $P_n(x)$ and the corresponding hypergeometric function are connected by a linear change of variable), we obtain the following WKB density of zeros for the Legendre polynomials:

$$\rho_{\text{WKB}}(x; n) = \frac{1}{\pi(1-x^2)}\sqrt{1+n+n^2-n(n+1)x^2}, \quad x \in (-x_0, x_0), \tag{13}$$

x_0 being $\sqrt{1+1/(n^2+n)}$, provided that $n \gg 1$. The density function (13) is plotted in Fig. 1 for several values of n.

On the other hand, it is known [6,12,15] that in the asymptotic limit ($n \to \infty$) the exact density of zeros of Legendre polynomials is given by the inverted semicircular law

$$\rho(x) = \frac{1}{\pi}(1-x^2)^{-1/2}, \quad x \in (-1, 1). \tag{14}$$

Then, we can compare our approximation with (14) by dividing (13) by n (total number of zeros of the nth-degree Legendre polynomial) and taking that limit in the resulting expression. This gives

$$\lim_{n \to \infty} \frac{\rho_{\text{WKB}}(x; n)}{n} = \frac{1}{\pi}(1-x^2)^{-1/2}, \quad x \in (-1, 1),$$

which exactly coincides with (14). Notice that due to the fact that $\lim_{n \to \infty} x_0 = 1$, we have $x \in (-1, 1)$.

3.2. Confluent hypergeometric functions

Let σ and τ be two polynomials of degree one: $\sigma(x)=(x-a)$, $\tau(x)=\tau' x+\tau(0)$, $\tau'\neq 0$. Taking $x=a-(t/\tau')$ in (1), one has

$$ty''(t)+\tau[a+bt]y'(t)-\frac{\lambda}{\tau'}y(t)=0.$$

Then it is always possible to choose parameters α and γ so that this equation can be written in the form

$$ty''(t)+[\gamma-t]y'(t)-\alpha y(t)=0,$$

which is the confluent hypergeometric equation [1], whose solutions are usually represented by $F[\alpha, \gamma; t]$. In this case, the restriction (2) reduces again to $\alpha=-\nu$, and the application of Theorem 2 gives rise to the following corollary.

Corollary 4. *The semiclassical or WKB density of zeros of the confluent hypergeometric functions* $F[-\nu, \gamma; t]$, $\nu\in\mathbb{R}$, *is*

$$\rho_{\mathrm{WKB}}(t;\nu)=\frac{\sqrt{2\gamma-\gamma^2+(2\gamma+4\nu)t-t^2}}{2\pi t}, \tag{15}$$

in every interval where the polynomial under the square root is positive, provided that $\nu\gg 1$.

A special case of confluent hypergeometric functions are the generalized Laguerre polynomials $\{L_n^{(a)}(t)\}_{n\in\mathbb{N}}$, $a>-1$, [2,14]:

$$L_n^{(a)}(t)=F[-n, a+1; t].$$

So, taking $\nu=n\in\mathbb{N}$ and $\gamma=a+1$ in (15) we get the WKB density of zeros for this polynomial sequence:

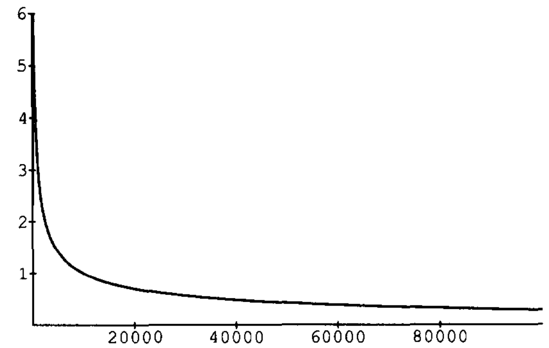
$$\rho_{\mathrm{WKB}}(t;n)=\frac{\sqrt{1-a^2+2(2n+a+1)t-t^2}}{2\pi t},\quad t\in(t_1, t_2),\ n\gg 1, \tag{16}$$

where

$$t_1=(2n+a+1)\left\{1-\sqrt{1+\frac{1-a^2}{(2n+a+1)^2}}\right\},$$

$$t_2=(2n+a+1)\left\{1+\sqrt{1+\frac{1-a^2}{(2n+a+1)^2}}\right\}.$$

The density function (16) is plotted in Fig. 2 for several values of n.



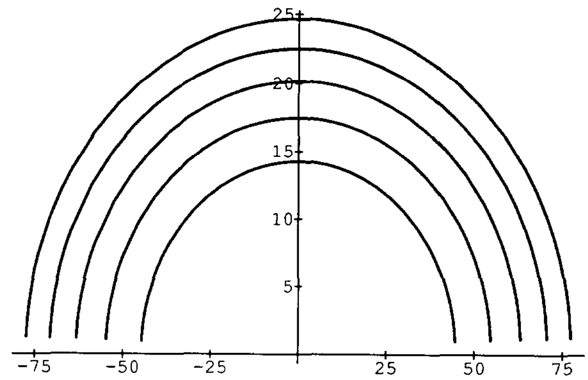


Fig. 3. WKB (Eq. (16)) and random matrices (Eq. (17)) density of zeros of the generalized Laguerre polynomial $L_{10^6}^{(10^3)}$.

Fig. 4. WKB density of zeros (see (18) with $n=\nu$) of Hermite polynomials $H_n(x)$ for $n = 1000, 1500, 2000, 2500, 3000$.

By using a technique based on random matrices, Bronk [4] has also calculated an approximate density of zeros for the generalized Laguerre polynomials. His expression is

$$\rho_1(t;\, n) = \frac{\sqrt{(1-a)^2 + 2(2n+a-1)t - t^2}}{2\pi t}, \quad \frac{a^2}{4n} < x < 4n + 2a, \tag{17}$$

provided that the condition $n \gg a \gg 1$ holds. It is then clear that if this last condition is considered, both expressions (16), (17) are practically the same, as shown in Fig. 3 where both functions have been plotted for $a = 10^3$ and $n = 10^6$.

3.3. Hermite functions

When σ is a constant ($\sigma(x) = 1$) and the degree of τ is exactly one, the hypergeometric-type differential equation can be reduced to the Hermite one

$$y''(t) - 2ty'(t) + 2\alpha y(t) = 0,$$

by means of a linear change of variable. Here condition (2) implies $\alpha = \nu$.

Then, from Theorem 2 one has the following corollary.

Corollary 5. *The semiclassical or WKB density of zeros of the Hermite functions is*

$$\rho_{\mathrm{WKB}}(t;\, \nu) = \frac{1}{\pi}\sqrt{1 + 2\nu - t^2}, \quad t \in \left(-\sqrt{2\nu+1}\,,\, \sqrt{2\nu+1}\right), \tag{18}$$

provided that $\nu \gg 1$.

When $\nu = n \in \mathbb{N}$, the Hermite functions become the Hermite polynomials $\{H_n(t)\}_{n \in \mathbb{N}}$ whose asymptotic density of zeros contracted to the interval $[-1, 1]$ and normalized to unity is known [6,8,9,13,15] and which is given by the semicircular law

$$\rho(x) = \frac{2}{\pi}\sqrt{1-x^2}, \quad x \in [-1, 1]. \tag{19}$$

It is easy to show that our approximation (18) gives rise to the same contracted density of zeros. Just taking $t = (2n+1)^{1/2}x$ in (18), $x \in (-1, 1)$, we obtain

$$\rho(x; n) = \frac{(2n+1)}{\pi}\sqrt{1-x^2}, \quad x \in [-1, 1],$$

and then

$$\lim_{n \to \infty} \frac{\rho_{\mathrm{WKB}}}{n} = \frac{2}{\pi}\sqrt{1-x^2}, \quad x \in [-1, 1],$$

which exactly coincides with (19). Fig. 4, where the WKB density function (18) is plotted for several values of n, shows the abovementioned semicircular behaviour. Notice that in this figure, the lines corresponding to the values of t near the end points of the interval have been omitted, because in that region taking $n \gg 1$ is not enough to ensure that the WKB condition ($\epsilon \ll 1$) given by (6) of Theorem 1 holds. Hence, when t tends to the end points of the interval, the WKB approximation is not a good one.

Of course, many other particular functions can be considered, whose density of zeros can be deduced from the general one given in Theorem 2. More details will be given elsewhere.

4. Concluding remarks

In summary, the WKB method has been used to obtain an analytical, but approximate, expression of a broad class of functions, the so-called hypergeometric-type functions, defined by (1) and (2). The resulting semiclassical or WKB density of zeros is given by (10). Moreover, this spectral density has been explicitly particularized for the Gaussian and confluent hypergeometric functions as well as for the Hermite functions (see (12), (15) and (18), respectively).

Finally, it is important to remark that, in the asymptotic limit, the WKB density of zeros of the Legendre and Hermite polynomials gives the exact density of zeros. To the best of our knowledge, there is no other method which allows to obtain this kind of property starting from the differential equation that the polynomials satisfy.

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