ELSEVIER Journal of Computational and Applied Mathematics 56 (1994) 321–329

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Uniform convergence of optimal order quadrature rules for Cauchy principal value integrals

Kai Diethelm*

Institut für Mathematik, Universität Hildesheim, Marienburger Platz 22, D-31141 Hildesheim, Germany

Received 22 January 1993; revised 18 May 1993

Abstract

For the numerical evaluation of Cauchy principal value integrals of the form $⨍_{-1}^{1} f(x)/(x-\lambda)\,dx$, $\lambda \in (-1,1)$, $f \in C^s[-1,1]$, we consider a quadrature method based on spline interpolation of odd degree $2k+1$, $k \in \mathbb{N}_0$. We show that these rules converge uniformly for $\lambda \in (-1,1)$. In particular, we calculate the exact order of magnitude of the error and show that it is equal to the order of the optimal remainder in the class of functions with bounded sth derivative if $s \in \{2k+1, 2k+2\}$. Finally, we compare the rule to the well-known quadrature rule of Elliott and Paget which only converges pointwise.

Keywords: Cauchy principal value integrals; Quadrature formula; Spline interpolation; Optimal order of convergence; Peano constants; Uniform convergence

1. Introduction

In this paper, we consider a particular method for the numerical evaluation of the Cauchy principal value integral

$$I[f;\lambda] := ⨍_{-1}^{1} \frac{f(x)}{x-\lambda}\,dx \tag{1}$$

$$:= \lim_{\varepsilon \to 0+} \left(\int_{-1}^{\lambda-\varepsilon} \frac{f(x)}{x-\lambda}\,dx + \int_{\lambda+\varepsilon}^{1} \frac{f(x)}{x-\lambda}\,dx \right), \tag{2}$$

where $\lambda \in (-1,1)$ and $f \in C^s[-1,1]$, $s \geqslant 1$. (These hypotheses ensure the existence of $I[f;\lambda]$.)

For the approximation of these integrals, we will use quadrature formulae Q_n of the form

$$Q_n[f;\lambda] := \sum_{\nu=1}^{n} a_\nu(\lambda) f(x_\nu), \tag{3}$$

* E-mail: diethelm@informatik.uni-hildesheim.de.

SSDI 0377-0427(93)E0205-Z

where, in particular, the weights a_ν depend upon λ but the nodes x_ν do not. A sequence $(Q_n)_{n=1}^{\infty}$ of quadrature formulae will be called a quadrature rule.

Cauchy principal value integrals appear in numerous practical applications, for example in various integral equations which originate from such fields as aerodynamics or fluid mechanics [7,9]. For the numerical solution of these equations it is very important to have a quadrature rule where for every suitable function f the remainder

$$R_n[f;\lambda] := I[f;\lambda] - Q_n[f;\lambda] \tag{4}$$

converges to zero uniformly for all $\lambda \in (-1,1)$. We will show a very simple necessary condition for uniform convergence in Section 4 below from which it follows that a number of frequently considered quadrature rules do not have the uniform convergence property (although their pointwise convergence is well known).

Before we state our results, we will introduce some notation. For every $\mu \in \mathbb{N}_0$, we define $p_\mu(x) := x^\mu$. If L is a linear functional on $C^1[-1,1]$, we define its degree of exactness $\deg(L)$ by

$$\deg(L) = d :\Leftrightarrow \begin{cases} L[p_\mu] = 0 & \forall \mu \in \{0,1,\ldots,d\} \\ L[p_{d+1}] \neq 0. \end{cases}$$

Our main tool for the comparison of the quality of quadrature formulae is the Peano constant of order s of R_n defined as

$$\varrho_s(R_n[\cdot;\lambda]) := \sup\{|R_n[f;\lambda]| \mid f \in \mathscr{K}_s\}, \tag{5}$$

where $\mathscr{K}_s := \{f \mid f^{(s-1)}$ is absolutely continuous and $\|f^{(s)}\| \leqslant 1\}$. Here and in the following, $\|f\|$ denotes the usual sup-norm of f.

It is well known that $\varrho_s(R_n[\cdot;\lambda]) < \infty$ if and only if $\deg(R_n[\cdot;\lambda]) \geqslant s-1$.

As an immediate consequence of (5), we have the following estimate for the quadrature error:

$$|R_n[f;\lambda]| \leqslant \|f^{(s)}\| \, \varrho_s(R_n[\cdot;\lambda]) \tag{6}$$

if $\deg(R_n[\cdot;\lambda]) \geqslant s-1$, and the value $\varrho_s(R_n[\cdot;\lambda])$ is the smallest possible in this inequality.

This explains why Peano constants are frequently used as a quality measure for quadrature formulae. The theory of Peano constants is also a very important part of the classical numerical integration theory, see e.g. [2].

Finally, we introduce the interpolation operator our quadrature rule is based upon. Let $s \in \mathbb{N}_0$ and, for every $n \in \mathbb{N}$, $n \geqslant 2s+2$, define $x_{\nu,n} := -1 + (2\nu/n)$ $(\nu = 0,1,\ldots,n)$. Then, for every $f \in C[-1,1]$, we define $\operatorname{intpol}_{n+1}^{2s+1}[f]$ to be the uniquely determined spline of degree $2s+1$ with knots $x_{s+1}, x_{s+2}, \ldots, x_{n-s-1}$ which interpolates f at the points $x_0, x_1, \ldots, x_n$. These splines are called splines with not-a-knot-end condition (see [8] or [4] for the case $s=1$). To obtain the quadrature formula, we simply set

$$Q_{n+1}^{2s+1}[f;\lambda] := I[\operatorname{intpol}_{n+1}^{2s+1}[f];\lambda]. \tag{7}$$

The case $s=0$ has been investigated in [10]; Dagnino and Santi [4] have previously considered the case $s=1$. In particular, Strauß [10] and Dagnino and Santi [4] have given explicit expressions for the weights of the quadrature formulae or a numerically stable method for their evaluation so that we do not have to consider this here.

In our Theorem 2.1 below, we have been able to improve Dagnino and Santi's statement on the order of convergence [4, Theorem 2], and our Theorems 3.1 and 3.4 below show that this new estimate is unimprovable. Statements on the general case have been set up in [8], whose results are also improved and completed by our Theorems 2.1, 3.1 and 3.4.

2. Error estimates for the quadrature rule

In this section, we will deduce upper bounds for some of the Peano constants of the quadrature rules Q_{n+1}^{2s+1}. First, we note that by definition we have

$$\deg(R_{n+1}^{2s+1}[\cdot;\lambda]) \geqslant 2s+1$$

and hence Q_{n+1}^{2s+1} has got Peano constants at least up to the order $2s+2$. For some particular values of λ, the degree of exactness may be even higher.

The main statement of this section is as follows.

Theorem 2.1. *For $s \in \mathbb{N}_0$, $n \geqslant 2s+2$ and $j \in \{0,1\}$ we have*

$$\varrho_{2s+2-j}(R_{n+1}^{2s+1}[\cdot;\lambda]) = \mathrm{O}(n^{-2s-2+j}\ln n). \tag{8}$$

In particular, the O*-term holds uniformly for all $\lambda \in (-1,1)$.*

For the proof, we need the following lemma.

Lemma 2.2. *For $j,k \in \{0,1\}$ and $f \in \mathscr{K}_{2s+2-j}$, we have*

$$\|(f - \operatorname{intpol}_{n+1}^{2s+1}[f])^{(k)}\| \leqslant c_{j,k,s}\, n^{-2s-2+j+k}, \tag{9}$$

where $c_{j,k,s}$ depends on j, k and s only.

For the case $s=0$, i.e. piecewise linear interpolation, this is a well-known property. The case $s=1$ has been shown in [1, (1.7), (1.8)], and the proof of the general case can be found in [5, Theorem 3] (see also [8]).

Proof of Theorem 2.1. Because of the symmetry of the problem, we can restrict ourselves to the case $\lambda \geqslant 0$. Let $f \in \mathscr{K}_{2s+2-j}$ and $r := f - \operatorname{intpol}_{n+1}^{2s+1}[f]$. We have to consider two cases depending on the distance between the singularity λ and the end point of the integration interval:

(1) $\lambda + (1/n) \leqslant 1$:

$$|R_{n+1}^{2s+1}[f;\lambda]| \leqslant \underbrace{\int_{-1}^{\lambda-1/n} \left|\frac{r(x)}{x-\lambda}\right| \mathrm{d}x}_{=:R_1} + \underbrace{\left| \int\!\!\!\!\!\!- _{\lambda-1/n}^{\lambda+1/n} \frac{r(x)}{x-\lambda}\mathrm{d}x \right|}_{=:R_2} + \underbrace{\int_{\lambda+1/n}^{1} \left|\frac{r(x)}{x-\lambda}\right| \mathrm{d}x}_{=:R_3}.$$

Here, we have:

$$R_1 \leqslant \|r\| \int_{-1}^{\lambda - 1/n} \frac{\mathrm{d}x}{\lambda - x} = \|r\| \ln(n(1+\lambda)) \leqslant c_{j,0,s} n^{-2s-2+j}(\ln 2 + \ln n),$$

$$R_2 \leqslant \left| \int_{\lambda - 1/n}^{\lambda + 1/n} \frac{r(x) - r(\lambda)}{x - \lambda} \mathrm{d}x \right| \leqslant \|r'\| \int_{\lambda - 1/n}^{\lambda + 1/n} \mathrm{d}x \leqslant 2c_{j,1,s} n^{-2s-2+j},$$

$$R_3 \leqslant \|r\| \int_{\lambda + 1/n}^{1} \frac{\mathrm{d}x}{x - \lambda} \leqslant c_{j,0,s} n^{-2s-2+j} \ln((1-\lambda)n) \leqslant c_{j,0,s} n^{-2s-2+j} \ln n$$

and it follows

$$|R_{n+1}^{2s+1}[f;\lambda]| \leqslant \frac{2c_{j,1,s} + c_{j,0,s}\ln 2 + 2c_{j,0,s}\ln n}{n^{2s+2-j}}. \tag{10}$$

(2) $\lambda + (1/n) > 1$:

$$|R_{n+1}^{2s+1}[f;\lambda]| \leqslant R_1 + \underbrace{\left| \int_{\lambda - 1/n}^{2\lambda - 1} \frac{r(x)}{x - \lambda} \mathrm{d}x \right|}_{=:R_4} + \underbrace{\left| \int_{2\lambda - 1}^{1} \frac{r(x)}{x - \lambda} \mathrm{d}x \right|}_{=:R_5}$$

with R_1 taken from part (1). In this case, we have:

$$\begin{aligned} R_4 &\leqslant \int_{\lambda - 1/n}^{2\lambda - 1} \left| \frac{r(x) - r(\lambda)}{x - \lambda} \right| \mathrm{d}x + |r(\lambda)| \left| \int_{\lambda - 1/n}^{2\lambda - 1} \frac{\mathrm{d}x}{x - \lambda} \right| \\ &\leqslant \|r'\| \left(\lambda - 1 + \frac{1}{n} \right) + |r(\lambda) - r(1)| \ln \frac{1/n}{1 - \lambda} \\ &\leqslant c_{j,1,s} n^{-2s-2+j} - \|r'\| (1-\lambda)\ln(1-\lambda) \\ &\leqslant c_{j,1,s} n^{-2s-2+j}(1 + \ln n), \end{aligned}$$

where, for the last inequality, we can use the fact that the function g given by $g(t) = -(1-t)\ln(1-t)$ decreases monotonously in $[1-(1/\mathrm{e}), 1]$ and consequently in $[1-(1/n), 1]$ if $n \geqslant 3$. Hence, $g(\lambda) \leqslant g(1-(1/n)) = (1/n)\ln n$ for $n \geqslant 3$.

$$R_5 \leqslant \int_{2\lambda - 1}^{1} \left| \frac{r(x) - r(\lambda)}{x - \lambda} \right| \mathrm{d}x \leqslant (2 - 2\lambda)\|r'\| \leqslant 2c_{j,1,s} n^{-2s-2+j}.$$

Putting this together, we have, for $n \geqslant 3$,

$$R_{n+1}^{2s+1}[f;\lambda] \leqslant \frac{c_{j,0,s}\ln 2 + 3c_{j,1,s} + (c_{j,0,s} + c_{j,1,s})\ln n}{n^{2s+2-j}}. \tag{11}$$

The statement of the theorem follows readily from (10) and (11). □

Remark. It has been shown in [1] that, for the special case $s = 1$, the statement of Lemma 2.2 also holds for $j \in \{2, 3\}$. Using this and proceeding exactly as in the proof of Theorem 2.1, one can also show that for $j \in \{0, 1, 2, 3\}$,

$$\varrho_{4-j}(R_{n+1}^3[\cdot;\lambda]) = \mathrm{O}(n^{-4+j}\ln n)$$

holds uniformly for $\lambda \in (-1, 1)$, i.e. for $(Q_{n+1}^3)_{n=1}^{\infty}$ not only the third and fourth but also the first and second Peano constants are of optimal order.

3. A lower bound for Peano constants

Having shown the order of magnitude of the Peano constants, we will now show that there do not exist any quadrature rules whose Peano constants are of a smaller order.

Theorem 3.1. *Let $s \in \mathbb{N}$ be fixed. There exists a positive constant c_s depending on s only such that the following statement holds:*

Let Q_{n+1} be a quadrature formula with nodes $-1 \leqslant x_0 < x_1 < \cdots < x_n \leqslant 1$, and let R_{n+1} be its remainder. Then, there exist $\lambda \in (-1, 1)$ and $f \in \mathscr{K}_s$ such that

$$|R_{n+1}[f;\lambda]| \geqslant c_s n^{-s} \ln n. \tag{12}$$

This immediately yields the following corollary.

Corollary 3.2. *For every quadrature formula Q_n, we have*

$$\sup_{\lambda \in (-1,1)} \varrho_s(R_n[\cdot;\lambda]) \geqslant c_s n^{-s} \ln n,$$

with a constant c_s which is independent of Q_n.

Adding a further hypothesis on the quadrature formula, we can obtain even more.

Definition 3.3. Let a quadrature rule $(Q_n)_{n=1}^{\infty}$ be given with nodes $-1 =: x_{0,n} < x_{1,n} < \cdots < x_{n,n} := 1$. We say that the quadrature rule is of *almost equidistant type* if there exist absolute constants $A > 0$ and $B > 0$ such that for all $n \in \mathbb{N}$ and all $\nu \in \{0, 1, \ldots, n-1\}$ we have

$$\frac{A}{n} \leqslant x_{\nu+1,n} - x_{\nu,n} \leqslant \frac{B}{n}.$$

Theorem 3.4. *Let $s \in \mathbb{N}$ be fixed. There exists a positive constant d_s depending on s only such that the following statement holds:*

If $(Q_n)_{n=1}^{\infty}$ is a quadrature rule of almost equidistant type, then for all $\lambda \in (-1, 1)$ we have

$$\varrho_s(R_n[\cdot;\lambda]) \geqslant \frac{A^{s+1}}{B} d_s n^{-s}(\ln n - \ln 3B) \tag{13}$$

where A and B are as in Definition 3.3.

Corollary 3.5. *Under the hypotheses of Theorem* 3.4, *we have*

$$\inf_{\lambda\in(-1,1)} \varrho_s(R_n[\cdot;\lambda]) \geqslant \tilde{d}_s n^{-s}\ln n.$$

For the proof of Theorem 3.1, we need the following lemma directly deduced from [10, Lemma 2].

Lemma 3.6. *Let* $-1 \leqslant x_0 < x_1 < \cdots < x_n \leqslant 1$ *be given. For* $0 \leqslant j \leqslant 2n+2$ *define* $t_j := -1 + j/(n+1)$, *and for* $1 \leqslant j \leqslant 2n+2$ *set* $\Delta_j := (t_{j-1}, t_j)$. *Then, the following statement holds:*

There exist m intervals $\{\Delta_i^* \mid i = 1,2,\ldots,m\} \subseteq \{\Delta_j \mid j = 1,2,\ldots,2n+2\}$ *and* $\xi \in [-1,1]$ *with the following properties:*

(1) $x_k \notin \Delta_i^*$ *for* $k = 0,1,\ldots,n$ *and* $i = 1,2,\ldots,m$,

(2) $m \geqslant \frac{1}{6}(n+1)$ *and*

(3) $0 \leqslant t - \xi < 3i/(n+1)$ *for* $t \in \Delta_i^*$ *and* $i = 1,2,\ldots,\lfloor \frac{1}{6}(n+1)\rfloor$.

Proof of Theorem 3.1. Let B_s be the basic spline with knots $0,1,\ldots,s+1$, and define $M_s := \operatorname{ess\,sup}_{x\in[0,s+1]} |B_s^{(s)}(x)|$, the derivative being understood in the generalized sense. We have to show the existence of λ and f with the desired properties. Let us first consider the number ξ from Lemma 3.6. We will assume $\xi > -1$, the case $\xi = -1$ requires only minor modifications. Now, set $\lambda := \xi$ and define (using the notation of Lemma 3.6)

$$f(x) := \begin{cases} \dfrac{B_s((s+3)(n+1)(x-t_{j-1})-1)}{M_s(n+1)^s(s+3)^s} & \text{for } x\in[t_{j-1},t_j] \text{ if } t_{j-1}\geqslant\lambda, \\ & \quad \text{and } (t_{j-1},t_j)\in\{\Delta_i^*\}, \\ 0 & \text{else.} \end{cases}$$

Using this definition, we have $f \in \mathscr{K}_s$ and $f(x_\nu) = 0$ for all ν. Consequently, $Q_{n+1}[f;\lambda] = 0$. Using Lemma 3.6, parts (2) and (3), we have

$$\begin{aligned} |R_{n+1}[f;\lambda]| &= |I[f;\lambda] - Q_{n+1}[f;\lambda]| \\ &= \sum_{j=1}^{m} \int_{\Delta_j^*} \frac{f(x)}{x-\lambda}\,dx \geqslant \frac{n+1}{3} \sum_{j=1}^{\lfloor (n+1)/6\rfloor} \frac{1}{j}\int_{\Delta_j^*} f(x)\,dx \\ &= \frac{1}{3M_s(s+3)^{s+1}(s+1)!\,(n+1)^s} \sum_{j=1}^{\lfloor (n+1)/6\rfloor} \frac{1}{j} \\ &\geqslant \frac{\ln\lfloor \frac{1}{6}(n+7)\rfloor}{3M_s(s+3)^{s+1}(s+1)!\,(n+1)^s} \end{aligned}$$

which is the statement of the theorem. □

Remark. Strauß [10] has claimed this for the cases $s = 1$ and $s = 2$, but there is an error in his proof because he constructs a function φ_0 which is not in $\mathscr{K}_s$ as he claims.

The proof of Theorem 3.4 can be done in a very similar manner using the additional assumption on the nodes instead of Lemma 3.6.

Remark. The function f defined in the proof of Theorem 3.1 also fulfills $f'(x_\nu) = 0$ for all ν and $f^{(j)}(\lambda) = 0$ for all j. Hence, the statement of Theorem 3.1 (and the statement of Theorem 3.4) still holds if the definition of the quadrature formula is extended to include derivatives and a free node of arbitrary order, i.e. if formulae of the type

$$Q_n[f;\lambda] = \sum_{\nu=1}^{n} (a_\nu(\lambda) f(x_\nu) + b_\nu(\lambda) f'(x_\nu)) + \sum_{\nu=1}^{k} c_\nu(\lambda) f^{(\nu)}(\lambda) \tag{14}$$

are used rather than those from (3).

4. Uniform convergence

We have now seen that there exists a uniformly convergent optimal order quadrature rule for Cauchy principal value integrals. The uniform convergence property is very important in the solution of Cauchy-type integral equations, but not all quadrature rules have got this property. As a counterexample, we will now consider the rule (Q_n^{EP}) of Elliott and Paget [6] which is characterized by its nodes being the zeroes of the nth Legendre polynomial and its degree of exactness being $\geqslant n-1$. It is well known that for this rule, we also have

$$\varrho_s(R_n^{\mathrm{EP}}[\cdot;\lambda]) = \mathrm{O}(n^{-s}\ln n)$$

for every fixed s (see [3, Theorem 2.3]), but this only holds pointwise and not uniformly. This can easily be seen from the following statement.

Theorem 4.1. *Let Q_n be a quadrature formula which does not have a node at the point $x = -1$. Then, for all $s \in \mathbb{N}$,*

$$\lim_{\lambda \to -1} \varrho_s(R_n[\cdot;\lambda]) = \infty.$$

Remark. For reasons of symmetry, an analoguous result holds if $x = +1$ is not a node of Q_n.

Corollary 4.2. *If for every $n \in \mathbb{N}$ one of the points ± 1 is not a node of Q_n, then $\varrho_s(R_n[\cdot;\lambda])$ does not converge to zero uniformly for $\lambda \in (-1, 1)$.*

Proof of Theorem 4.1. Let x_1 be the smallest node of Q_n and let $\delta := x_1 + 1$. Define

$$f(x) := \begin{cases} 0 & \text{if } x \geqslant x_1, \\ \left(\dfrac{2\delta}{s+1}\right)^s \dfrac{1}{M_s} B_s\left(\dfrac{s+1}{2\delta}(x+1+\delta)\right) & \text{if } x < x_1, \end{cases}$$

where B_s and M_s are defined as in the proof of Theorem 3.1. Obviously, $Q_n[f;\lambda] = 0$ and $f \in \mathscr{K}_s$. Thus,

$$\begin{aligned}\varrho_s(R_n[\cdot;\lambda]) &\geqslant |R_n[f;\lambda]| = |I[f;\lambda]| \\ &= \left|\int_{-1}^{x_1} \frac{f(x)}{x-\lambda}\,\mathrm{d}x\right| = \left|\int_{-1}^{x_1} \frac{f(x)-f(\lambda)}{x-\lambda}\,\mathrm{d}x - f(\lambda)\ln\left|\frac{1+\lambda}{|x_1-\lambda|}\right|\right| \\ &\geqslant f(\lambda)\left|\ln\frac{1+\lambda}{|\lambda-x_1|}\right| - \int_{-1}^{x_1}\left|\frac{f(x)-f(\lambda)}{x-\lambda}\right|\mathrm{d}x \\ &\geqslant f(\lambda)\left|\ln\frac{1+\lambda}{|\lambda-x_1|}\right| - \delta\,\|f'\|.\end{aligned}$$

Hence,

$$\lim_{\lambda\to-1}\varrho_s(R_n[\cdot;\lambda]) \geqslant \lim_{\lambda\to-1}\left(f(\lambda)\left|\ln\frac{1+\lambda}{|\lambda-x_1|}\right| - \delta\,\|f'\|\right) = \infty$$

since

$$f(-1) = \left(\frac{2\delta}{s+1}\right)^s \frac{1}{M_s} B_s\left(\frac{s+1}{2}\right) > 0. \qquad \square$$

In order to illustrate this behaviour, using suitably adapted methods from standard Peano kernel theory, we have computed some Peano constants for the Elliott–Paget rule as well as for Q^1_{n+1} (the simplest rule of those under consideration in this paper) and different values of λ. For the computation, we have used the MATHEMATICA software package [11] which guarantees the required accuracy. The results are given in Tables 1 and 2.

Table 1
$\varrho_1(R^1_m[\cdot;\lambda])$ for $m \in \{5, 50, 150\}$ and $\lambda \in \{0, 0.5, 0.9, 0.999, 0.999999\}$

λ	5 nodes	50 nodes	150 nodes
0	0.906205689	0.106280442	0.042522716
0.5	0.871377865	0.095630046	0.039022727
0.9	0.665255668	0.084301498	0.035949679
0.999	0.541480654	0.071257879	0.028124487
0.999999	0.539359575	0.069588031	0.026618531

Table 2
$\varrho_1(R^{\mathrm{EP}}_n[\cdot;\lambda])$ for $n \in \{5, 50, 150\}$ and $\lambda \in \{0, 0.5, 0.9, 0.999, 0.999999\}$

λ	5 nodes	50 nodes	150 nodes
0	0.712324122	0.148469902	0.061214677
0.5	0.997568219	0.221382703	0.057030160
0.9	1.121191341	0.084807772	0.063762076
0.999	15.007892409	0.347592935	0.042810338
0.999999	31.800253774	28.777635487	28.029703245

Acknowledgements

I am indebted to Professor Braß for many valuable hints during the research that led to this paper. I would also like to thank the referee for his helpful comments.

References

[1] R.K. Beatson, On the convergence of some cubic spline interpolation schemes, *SIAM J. Numer. Anal.* **23** (1986) 903–912.
[2] H. Braß, *Quadraturverfahren* (Vandenhoeck and Ruprecht, Göttingen, 1977).
[3] G. Criscuolo and G. Mastroianni, On the convergence of an interpolatory product rule for evaluating Cauchy principal value integrals, *Math. Comp.* **48** (1987) 725–735.
[4] C. Dagnino and E. Santi, Spline product quadrature rules for Cauchy singular integrals, *J. Comput. Appl. Math.* **33** (1990) 133–140.
[5] J. Domsta, Approximation by spline interpolating bases, *Studia Math.* **LVIII** (1976) 223–237.
[6] D. Elliott and D.F. Paget, Gauss type quadrature rules for Cauchy principal value integrals, *Math. Comp.* **33** (1979) 301–309.
[7] A. Gerasoulis, Piecewise polynomial quadratures for Cauchy singular integrals, *SIAM J. Numer. Anal.* **23** (1986) 891–902.
[8] A. Palamara Orsi, Spline approximation for Cauchy principal value integrals, *J. Comput. Appl. Math.* **30** (1990) 191–201.
[9] C.E. Stewart, On the numerical evaluation of singular integrals of Cauchy type, *J. Soc. Indust. Appl. Math.* **8** (1960) 342–353.
[10] R. Strauß, Eine Interpolationsquadratur für Cauchy-Hauptwertintegrale, *Rostock. Math. Kolloq.* **22** (1983) 57–66.
[11] S. Wolfram, *Mathematica—A System for Doing Mathematics by Computer* (Addison-Wesley, Redwood City, 1988).