ELSEVIER Journal of Computational and Applied Mathematics 59 (1995) 61–78

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Interlacing properties of zeros of associated polynomials

Franz Peherstorfer[a,*,1], Michael Schmuckenschläger[a,b,c,2]

[a] *Institut für Mathematik, J. Kepler Universität Linz, A-4040 linz, Austria*
[b] *Weizmann Institute of Science, Rehovot, Israel*
[c] *Mathematisches Seminar, Universität Kiel, Germany*

Received 19 May 1993; revised 26 November 1993

Abstract

Let w be a nonnegative integrable weight function on $[-1, 1]$ and let $p_{n+1}(x) = x^{n+1} + \cdots$ be the polynomial of degree $n+1$ orthogonal with respect to w. Furthermore, let $p_n^{(1)}(x) = x^n + \cdots$ denote the polynomials associated with p_{n+1} and $p_n^{(1-x^2)}(x) = x^n + \cdots$ the polynomials orthogonal with respect to the weight function $(1-x^2)w(x)$. In this paper we give necessary and sufficient conditions such that the zeros of $p_n^{(1)}$ and $p_n^{(1-x^2)}$ strictly interlace on $[-1, 1]$ for large n. In particular this problem is studied for the Jacobi weights $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta \in (-1, \infty)$. In this case $p_n^{(1-x^2)} = p'_{n+1}/(n+1)$.

For a large class of parameters, including, e.g. the ultraspherical case $\alpha = \beta$, it is shown that the interlacing property holds for each $n \in \mathbb{N}$. Also a fairly complete description of the parameters for which the interlacing property does not hold is given.

Keywords: Zeros; Interlacing property; Orthogonal polynomials; Functions of the second kind; Associated polynomials; Jacobi weights; Jacobi polynomials; Ultraspherical polynomials

1. Introduction and notation

Let σ be the distribution function of a positive measure (which will also be denoted by σ) on $[-1, 1]$ whose support contains an infinite set of points. Let $p_n(x) = x^n + \cdots$ be the monic polynomial of degree n orthogonal with respect to σ, i.e.

$$\forall j = 0, \dots, n-1 \quad \int_{-1}^{1} x^j p_n(x)\, d\sigma(x) = 0. \tag{1}$$

* Corresponding author.

[1] Supported by the Austrian Fonds zur Förderung der wissenschaftlichen Forschung, Projektnummer P9267-PHY.
[2] Supported in part by BSF and Erwin Schrödinger Auslandsstipendium J0630, J0804.

SSDI 0377-0427(94)00014-R

By $p_n^{(1-x^2)}(x) = x^n + \cdots$ we denote the monic polynomial on $[-1, 1]$ orthogonal with respect to $(1-x^2)\sigma(\mathrm{d}x)$, and, as usual, by $p_n^{(1)}(x) = x^n + \cdots$ the monic associated polynomial (of order one) of p_n defined by

$$p_{n-1}^{(1)}(x) = \frac{1}{d_0}\int_{-1}^{1} \frac{p_n(x) - p_n(t)}{x - t}\,\mathrm{d}\sigma(t), \tag{2}$$

where $d_0 = \sigma(1)$. It is well known that the orthogonal polynomials p_n satisfy a recurrence relation of the form

$$p_n(x) = (x - \alpha_n)p_{n-1}(x) - \lambda_n p_{n-2}(x), \quad p_0(x) := 1, \ p_{-1} := 0, \tag{3}$$

where $\alpha_n \in \mathbb{R}$ and $\lambda_n > 0$ and that the associated polynomials satisfy the shifted recurrence relation

$$p_n^{(1)}(x) = (x - \alpha_{n+1})p_{n-1}^{(1)}(x) - \lambda_{n+1}p_{n-2}^{(1)}(x), \quad p_0^{(1)} := 1, \ p_{-1}^{(1)} := 0. \tag{4}$$

Let us recall (cf. [2, pp. 86–87]) that the associated polynomials $p_n^{(1)}$ are orthogonal with respect to a positive measure $\sigma^{(1)}$ whose support is contained in $[-1, 1]$. To the measure σ on $[-1, 1]$ we assign the measure

$$\mu(\varphi) := \begin{cases} \sigma(1) - \sigma(\cos\varphi) & \text{if } \varphi \in [0, \pi], \\ \sigma(\cos\varphi) - \sigma(1) & \text{if } \varphi \in [-\pi, 0], \end{cases} \tag{5}$$

on the unit circle. Obviously, if σ is absolutely continuous on $[-1, 1]$ then μ is absolutely continuous on $[-\pi, \pi]$ with

$$\mu'(\varphi) = \sigma'(\cos\varphi)\,|\sin\varphi|. \tag{6}$$

If we denote the Stieltjes transform of σ by

$$Q(y, \mathrm{d}\sigma) := \int_{-1}^{1} \frac{1}{y - x}\,\mathrm{d}\sigma(x) \quad \forall y \in \mathbb{C}\backslash[-1, 1], \tag{7}$$

put

$$F(z, \mathrm{d}\mu) := \frac{1}{2\pi c_0}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\varphi} + z}{\mathrm{e}^{\mathrm{i}\varphi} - z}\,\mathrm{d}\mu(\varphi) \quad \text{for } |z| < 1, \tag{8}$$

where $c_0 = (1/2\pi)\int_{-\pi}^{\pi} \mathrm{d}\mu(\varphi)$, and set $z = y - \sqrt{y^2 - 1}$ for $y \in \mathbb{C}\backslash[-1, 1]$, we get the following relation between Q and F (cf. e.g. [4, p. 64]):

$$F(z, \mathrm{d}\mu) = \frac{1}{d_0}\sqrt{y^2 - 1}\,Q(y, \mathrm{d}\sigma). \tag{9}$$

Moreover, assuming that μ is absolutely continuous and $\mu' \in L_p[-\pi, \pi]$, $p > 1$, we have (cf. [12] or [7, pp. 21, 124]):

$$\Re F(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu) := \lim_{r \uparrow 1} \Re F(r\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu) = \frac{1}{c_0}\mu'(\varphi) \quad \text{a.e. on } [-\pi, \pi], \tag{10}$$

$\mathscr{I}F(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu) := \lim_{r\uparrow 1} \mathscr{I}F(r\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu)$ exists a.e. on $[-\pi, \pi]$ and

$$\frac{\mathscr{I}F(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu)}{\sin\varphi} = \frac{1}{\pi c_0}\mathscr{P}\int_{-1}^{1} \frac{\sigma'(t)}{t-x}\,\mathrm{d}t, \tag{11}$$

where $x = \cos\varphi$, $\varphi \in [-\pi, \pi]$ and $\mathscr{P}\int$ denotes the principal value. Hence

$$\mathscr{Q}(x) := \lim_{\varepsilon\downarrow 0} \tfrac{1}{2}(Q(x+\mathrm{i}\varepsilon) + Q(x-\mathrm{i}\varepsilon)) = \mathscr{P}\int \frac{\sigma'(t)}{t-x}\,\mathrm{d}t. \tag{12}$$

Next denote by $P_n(z) = z^n + \cdots$ the polynomial on $[-\pi, \pi]$ orthogonal with respect to $\mathrm{d}\mu$, i.e.

$$\forall k = 0, \ldots, n-1 \quad \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}k\varphi} P_n(\mathrm{e}^{\mathrm{i}\varphi})\,\mathrm{d}\mu(\varphi) = 0. \tag{13}$$

It is well known that the P_n's satisfy a recurrence relation of the following type:

$$P_n(z) = zP_{n-1}(z) - a_{n-1}P_{n-1}^*(z) \quad \forall n \in \mathbb{N}, \tag{14}$$

where $a_n \in (-1, 1)$ and where $P_n^*(z) = z^n P_n(z^{-1})$ denotes the reciprocal polynomial of P_n (the reason that the parameters a_n are real and have absolute value less than one consists in the fact that μ is odd and has an infinite set of increase (cf. [4, Sections 30–31] and [17]). Furthermore, let $\Omega_n(z) = z^n + \cdots$ be defined by the recurrence relation

$$\Omega_n(z) = z\Omega_{n-1}(z) + a_{n-1}\Omega_{n-1}^*(z), \quad n \in \mathbb{N}. \tag{15}$$

Ω_n is called polynomial of second kind, respectively, associated polynomial of P_n. As in the real case the system of associated polynomials Ω_n is orthogonal with respect to a measure $\tilde{\mu}$ which is defined in terms of μ as follows (cf. e.g. [9, Lemma 2] or [1, 14]): There is a unique measure $\tilde{\mu}$ such that for all $|z| < 1$:

$$F(z, \mathrm{d}\tilde{\mu})F(z, \mathrm{d}\mu) = 1, \tag{16}$$

and this measure $\tilde{\mu}$ is the one we are looking for. If the limit functions $\Re F(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu)$ and $\Re 1/F(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu)$ exist a.e. on $[-\pi, \pi]$ and belong to L_p for some $p > 1$, then (cf. e.g. [12, Lemma 2.1] and also [1]) μ and $\tilde{\mu}$ are related by

$$\frac{1}{\tilde{c}_0}\tilde{\mu}'(\varphi) = \Re\frac{1}{F(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu)} = \frac{\mu'(\varphi)}{c_0|F(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{d}\mu)|^2} \quad \text{a.e.} \tag{17}$$

Now let us return to the polynomials p_n. We recall the following relations (cf. [4, Sections 30–31], [5, pp. 90–93] and [17, pp. 294–295]):

$$\begin{aligned} p_n(x) &= 2^{-n+1}\Re(z^{-n+1}P_{2n-1}(z)), \\ p_{n-1}^{(1-x^2)}(x) &= \frac{2^{-n+1}}{\sin\varphi}\mathscr{I}(z^{-n+1}P_{2n-1}(z)) \end{aligned} \tag{18}$$

and

$$p_{n-1}^{(1)}(x) = \frac{2^{-n+1}}{\sin\varphi}\mathscr{I}(z^{-n+1}\Omega_{2n-1}(z)), \tag{19}$$

where $x = \frac{1}{2}(z + z^{-1})$, $z = e^{i\varphi}$ and $\varphi \in [0, \pi]$. Putting

$$\tilde{p}_n(x) = 2^{-n+1}\Re(z^{-n+1}\Omega_{2n-1}(z)), \tag{20}$$

(18) and (19) immediately imply:

$$p_{n-1}^{(1)}(x) = \tilde{p}_{n-1}^{(1-x^2)}(x) \quad \text{and} \quad \tilde{p}_{n-1}^{(1)}(x) = p_{n-1}^{(1-x^2)}(x). \tag{21}$$

Using the relations (21), (17), (10) and (6) we conclude that the polynomials $p_n^{(1)}$ are orthogonal with respect to the weight function:

$$(\sigma^{(1)})'(x) := \tilde{\sigma}'(x)\sqrt{1-x^2} = \frac{\tilde{c}_0}{c_0}\frac{\sigma'(x)}{\sigma'(x)^2 + \mathcal{Q}(x)^2}, \tag{22}$$

if both μ and $\tilde{\mu}$ are absolutely continuous with $\mu', \tilde{\mu}' \in L_p[-\pi, \pi]$ for some $p > 1$. For the determination of the absolutely continuous part of the distribution function $\sigma^{(1)}$ by a completely different approach we refer to [8] and also [6]. Let us denote by $x_{j,n}, x_{j,n}^{(1)}, x_{j,n}, \tilde{x}_{j,n}^{(1)}, j = 1, \ldots, n$ the zeros of $p_n, p_n^{(1)}, \tilde{p}_n, \tilde{p}_n^{(1)}$, where the zeros are arranged in increasing order, i.e. $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$, etc. For the following let us also recall the well-known interlacing property of the zeros of p_n and $p_{n-1}^{(1)}$ (cf. [2]):

$$-1 < x_{1,n} < x_{1,n-1}^{(1)} < \cdots < x_{n-1,n-1}^{(1)} < x_{n,n} < 1. \tag{23}$$

This paper is organized as follows. Stimulated by a conjecture of Ronveaux [8] we will investigate in Section 2 the interlacing property of the zeros of both p_n and $\tilde{p}_n$ and of $p_n^{(1)}$ and $\tilde{p}_n^{(1)} = p_n^{(1-x^2)}$. As it turns out the question whether $x_{j,n} < \tilde{x}_{j,n}$ or $x_{j,n} > \tilde{x}_{j,n}$ holds for large n, only depends on the sign of $\mathcal{Q}$ ($\mathcal{Q}$ is defined in (12)!). In Section 3 we apply this result to Jacobi weights $w_{\alpha,\beta}$, or more precisely to the monic associated Jacobi polynomials $p_{n,\alpha,\beta}^{(1)}$ and to the derivatives $p'_{n+1,\alpha,\beta}$ of the Jacobi polynomials. We get a description of the set of parameters for which Ronveaux's conjecture: If $\forall \alpha + \beta + 1 \overset{(>)}{<} 0$ then

$$\forall n\ \forall j = 1, \ldots, n \quad \tilde{x}_{j,n,\alpha,\beta}^{(1)} \overset{(>)}{<} x_{j,n,\alpha,\beta}^{(1)} \tag{24}$$

does not hold, respectively, holds for large n — recall that by [17] and (21) we have

$$\frac{1}{n+1} p'_{n+1,\alpha,\beta} = p_{n,\alpha,\beta}^{(1-x^2)} = \tilde{p}_{n,\alpha,\beta}^{(1)}. \tag{25}$$

Then by a different method, based on Markov's sufficient condition for the interlacing property of the zeros of polynomials orthogonal with respect to different weights, a set of parameters is described for which (24) holds (Theorem 3.6). The remaining part of this section is devoted to prove the existence of an absolute constant $c \in \mathbb{R}$ such that for all $(\alpha, \beta) \in (c, \infty) \times (-1, -\frac{1}{2})$ (24) holds (Theorem 3.7). Finally a drawing is included showing the set of pairs (α, β) for which (24) holds, respectively does not hold and conjectures on the remaining open cases are given.

2. Sufficient conditions for the interlacing property

Lemma 2.1. *Let $n \in \mathbb{N}$. The zeros $x_{j,n}$, $\tilde{x}_{j,n}$, $x^{(1)}_{j,n-1}$, $\tilde{x}^{(1)}_{j,n-1}$, of p_n, $\tilde{p}_n$, $p^{(1)}_{n-1}$, $\tilde{p}^{(1)}_{n-1}$, satisfy the following relations*:

(1) $\forall j = 1, \dots, n-1, \quad x^{(1)}_{j,n-1}, \tilde{x}^{(1)}_{j,n-1} \in (x_{j,n}, x_{j+1,n}) \cap (\tilde{x}_{j,n}, \tilde{x}_{j+1,n})$,

(2) $\forall j = 1, \dots, n, \quad x_{j,n}, \tilde{x}_{j,n} \in (x_{j-1,n}, x_{j+1,n}) \cap (\tilde{x}_{j-1,n}, \tilde{x}_{j+1,n})$,

where $x_{0,n} := \tilde{x}_{0,n} := -1$ and $x_{n+1,n} := \tilde{x}_{n+1,n} := 1$.

Proof. (1) In view of [4, 5.6] and of (18)–(20) we have

$$\begin{aligned} 0 < \text{const.} &= z^{-(2n-1)}(P_{2n-1}(z)\Omega^*_{2n-1}(z) - \Omega_{2n-1}(z)P^*_{2n-1}(z)) \\ &= \Re(z^{-n+1}P_{2n-1}(z)\overline{z^{-n+1}\Omega_{2n-1}(z)}) \\ &= 2^{2n-2}(p_n(x)\tilde{p}_n(x) + (1-x^2)p^{(1)}_{n-1}(x)\tilde{p}^{(1)}_{n-1}(x)), \end{aligned}$$

where $z = e^{i\varphi}$, $x = \cos\varphi$ and $\varphi \in [0, \pi]$. Considering the last expression at the zeros $\tilde{x}_{j,n}$ of $\tilde{p}_n$ and taking into account (23) with respect to $\tilde{\sigma}$, we get that $x^{(1)}_{j,n-1} \in (\tilde{x}_{j,n}, \tilde{x}_{j+1,n})$. Now the assertion follows from (23). The proof of the second statement follows the same pattern.

(2) If $x_{j,n} \leqslant \tilde{x}_{j-1,n}$ or $x_{j-1,n} \geqslant \tilde{x}_{j,n}$ for some $j \in \{2, \dots, n\}$ then

$$(x_{j-1,n}, x_{j,n}) \cap (\tilde{x}_{j-1,n}, \tilde{x}_{j,n}) = \emptyset,$$

which is a contradiction to part (1). □

Remark. From Lemma 2.1 we also get the more or less known interlacing property (cf. [16]) of the zeros of p_n and $\tilde{p}^{(1)}_{n-1} = p^{(1-x^2)}_{n-1}$ and of $\tilde{p}_n$ and $\tilde{p}^{(1-x^2)}_{n-1}$, i.e.:

$$x_{1,n} < \tilde{x}^{(1)}_{1,n-1} < \cdots < \tilde{x}^{(1)}_{n-1,n-1} < x_{n,n} \quad \text{and} \quad \tilde{x}_{1,n} < x^{(1)}_{1,n-1} < \cdots < x^{(1)}_{n-1,n-1} < \tilde{x}_{n,n}. \tag{26}$$

Now the question arises under which conditions the relations

$$x_{j,n} \overset{(>)}{<} \tilde{x}_{j,n} \qquad x^{(1)}_{j,n-1} \overset{(>)}{<} \tilde{x}^{(1)}_{j,n-1}$$

hold. As the following theorem shows this depends on the sign of $\mathcal{Q}$ only.

Theorem 2.2. *Let σ be absolutely continuous such that*
- $(\log\sigma'(x))/\sqrt{1-x^2} \in L_1[-1, 1]$.
- $\forall x \in [\xi_1, \xi_2] \subset [-1, 1] \quad \sqrt{1-x^2}\sigma'(x) > 0$ *and* $\sigma' \in C^1[\xi_1, \xi_2]$.

Then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all $j \in \{1, \dots, n\}$:

$$x_{j,n} \overset{(>)}{<} \tilde{x}_{j,n} \quad \text{and} \quad x^{(1)}_{j,n-1} \overset{(>)}{<} \tilde{x}^{(1)}_{j,n-1},$$

respectively, if $x_{j,n}$, $\tilde{x}_{j,n}$, $x^{(1)}_{j,n-1}$, $\tilde{x}^{(1)}_{j,n-1}$, *respectively, belong to a compact subinterval of the interval* (ξ_1, ξ_2) *on which* $0 \overset{(>)}{<} \mathscr{Q}$.

Proof. Set $a = \arccos \xi_1$ and $b = \arccos \xi_2$. In view of the assumptions we have (cf. the proof of Theorem 2.1(b) and Remark 2.1(b) in [5]):

$$\lim_{n \to \infty} \frac{\Omega_n^*(e^{i\varphi})}{P_n^*(e^{i\varphi})} = F(e^{i\varphi})$$

uniformly on $[a + \delta, b - \delta]$ for all $\delta > 0$. It follows that there exists an n_0 such that for all $n \geqslant n_0$:

$$\operatorname{sign}\left(\mathscr{I} \frac{\Omega_n^*(e^{i\varphi})}{P_n^*(e^{i\varphi})}\right) = \operatorname{sign}(\mathscr{I} F(e^{i\varphi})) \tag{27}$$

on every compact subset of (a, b) on which $\mathscr{I}F$ does not have a zero. Since

$$\begin{aligned} 0 &\overset{(>)}{<} \mathscr{I}(\Omega_{2n-1}^*(e^{i\varphi})\overline{P_{2n-1}^*(e^{i\varphi})}) \\ &= \Re(z^{-n+1}\Omega_{2n-1}(e^{i\varphi}))\mathscr{I}(z^{-n+1}P_{2n-1}(e^{i\varphi})) - \Re(z^{-n+1}P_{2n-1}(e^{i\varphi}))\mathscr{I}(z^{-n+1}\Omega_{2n-1}(e^{i\varphi})) \\ &= (\tilde{p}_n(x)\tilde{p}^{(1)}_{n-1}(x) - p_n(x)p^{(1)}_{n-1}(x)) \sin \varphi, \end{aligned} \tag{28}$$

we conclude by considering (28) at the zeros of p_n and $p^{(1)}_{n-1}$, respectively, and taking into account (26) and Lemma 2.1 that

$$x_{j,n} \overset{(>)}{<} \tilde{x}_{j,n} \quad \text{and} \quad x^{(1)}_{j,n-1} \overset{(>)}{<} \tilde{x}^{(1)}_{j,n-1},$$

respectively, if (28) holds. The assertion follows from (27) in conjunction with (11) and (12). □

Remark. If $\xi_1 = -1$ and $\xi_2 = 1$, respectively, then the open interval (ξ_1, ξ_2) in Theorem 2.2 can be replaced by the half closed intervals $[-1, \xi_2)$ and $(\xi_1, 1]$, respectively and by $[-1, 1]$ if $-\xi_1 = \xi_2 = 1$.

If the assumptions of Theorem 2.2 are fulfilled and if $0 \overset{(>)}{<} \mathscr{Q}$ on $(x_{j-1,n} - \varepsilon, x_{j+2,n} + \varepsilon)$ for some $\varepsilon > 0$ and sufficiently large n, then Theorem 2.2 and Lemma 2.1 imply:

$$x_{j,n} < \tilde{x}_{j,n} < x^{(1)}_{j,n-1} < \tilde{x}^{(1)}_{j,n-1} < x_{j+1,n} < \tilde{x}_{j+1,n},$$

$$\tilde{x}_{j,n} < x_{j,n} < \tilde{x}^{(1)}_{j,n-1} < x^{(1)}_{j,n-1} < \tilde{x}_{j+1,n} < x_{j+1,n},$$

respectively. Furthermore, let us mention that if σ is absolutely continuous and if σ' is of the form $v(x)/\sqrt{1 - x^2}$ for some continuously differentiable function $v > 0$ on $[-1, 1]$, then we only have to look for the uniquely determined function F, analytic in the open unit disc, which satisfies

$$\forall \varphi \in (0,\pi) \quad \Re F(e^{i\varphi}) = v(\cos \varphi).$$

The asymptotic interlacing behavior of $\tilde{x}_{j,n}$ and $x_{j,n}$ and of $\tilde{x}^{(1)}_{j,n-1}$ and $x^{(1)}_{j,n-1}$, respectively, will then be completely described by the behavior of sign $\mathscr{I}F(e^{i\varphi})$ on $(0, \pi)$. Let us note that the first author succeeded in analyzing the interlacing properties of the zeros of the orthogonal polynomials $p_n(x; w_1)$, respectively $p_n(x; w_2)$ for arbitrary weight functions w_1, w_2 for the Szegő-class by developing the basic idea of Theorem 2.2 in a suitable way (cf. [13]).

3. Interlacing properties of the zeros of the derivatives and of the associated polynomials of Jacobi polynomials

For $\alpha, \beta > -1$ and $x \in (-1, 1)$ let us define

$$Q(\alpha, \beta, x) := \frac{1}{\pi} \mathscr{P} \int_{-1}^{1} \frac{w_{\alpha,\beta}(t)}{t - x} \, dt,$$

where $w_{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta$ is the Jacobi weight. The following nice closed formula for $\mathscr{Q}$ has been given by Grosjean [2, (27)]

$$\mathscr{Q}(\alpha, \beta, x) = \frac{1}{\pi} w_{\alpha,\beta}(x) \left(Z(\alpha, \beta) - c(\alpha, \beta) \int_0^x \frac{1}{w_{\alpha+1,\beta+1}(t)} \, dt \right), \tag{29}$$

where

$$Z(\alpha, \beta) := \int_0^1 \frac{w_{\alpha,\beta}(x) - w_{\beta,\alpha}(x)}{x} \, dx \quad \text{and} \quad c(\alpha, \beta) := 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Using this representation we obtain

Lemma 3.1. *If* $(\alpha, \beta) \in (-\frac{1}{2}, \infty) \times (-\frac{1}{2}, \infty) \cup (-1, -\frac{1}{2}) \times (-1, -\frac{1}{2})$ *then* $\mathscr{Q}$ *has exactly one zero in* $(-1, 1)$, *and for the remaining values of* α *and* β $\mathscr{Q}$ *does not have a zero in* $(-1, 1)$.

Proof. By (29) the function $x \mapsto \mathscr{Q}(\alpha, \beta, x)/w_{\alpha,\beta}(x)$ is strictly increasing if $\alpha + \beta + 1 < 0$ and strictly decreasing if $\alpha + \beta + 1 > 0$. Hence $\mathscr{Q}$ has at most one zero. If $\alpha < 0$ then

$$\begin{aligned}
\lim_{x \uparrow 1} \frac{\mathscr{Q}(\alpha, \beta, x)}{w_{\alpha,\beta}(x)} &= \lim_{x \uparrow 1} \frac{1}{2^\beta \pi (1-x)^\alpha} \mathscr{P} \int_{-1}^{1} \frac{(1-t)^\alpha (1+t)^\beta}{t - x} \, dt \\
&= \frac{2^{\alpha+1}}{\pi} \lim_{x \uparrow 1} \frac{1}{(1-x)^\alpha} \mathscr{P} \int_0^1 \frac{s^\alpha (1-s)^\beta}{1 - x - 2s} \, ds = \frac{1}{\pi} \lim_{y \downarrow 0} y^{-\alpha} \mathscr{P} \int_0^1 \frac{s^\alpha (1-s)^\beta}{y - s} \, ds \\
&= \frac{1}{\pi} \lim_{y \downarrow 0} \mathscr{P} \int_0^{1/y} \frac{t^\alpha (1 - yt)^\beta}{1 - t} \, dt = \frac{1}{\pi} \mathscr{P} \int_0^\infty \frac{t^\alpha}{1 - t} \, dt = \cot(\alpha\pi).
\end{aligned}$$

If $\alpha \geqslant 0$ the limit is $-\infty$. A similar argument shows that

$$\lim_{x \downarrow -1} \frac{\mathscr{Q}(\alpha, \beta, x)}{w_{\alpha,\beta}(x)} = \begin{cases} -\cot(\beta\pi) & \text{if } \beta < 0, \\ \infty & \text{otherwise.} \end{cases} \quad \square \tag{30}$$

As an immediate consequence of (29) and (30) we get the following corollary.

Corollary 3.2. *If* $-1 < \beta < 0$ *then*

$$-\frac{\mathscr{Q}(\alpha,\beta,x)}{w_{\alpha,\beta}(x)} = \cot(\beta\pi) + \frac{c(\alpha,\beta)}{\pi}\int_{-1}^{x}\frac{1}{(1-t)^{\alpha+1}(1+t)^{\beta+1}}\,\mathrm{d}t.$$

Let us recall that $p_{n,\alpha,\beta}(x) = x^n + \cdots$ denotes the monic Jacobi polynomial on $(-1, 1)$ orthogonal with respect to $w_{\alpha,\beta}$, and $p^{(1)}_{n-1,\alpha,\beta}(x) = x^{n-1} + \cdots$ denotes its associated polynomial defined by (2). By [6, 8] (compare (22)) the latter are orthogonal with respect to the weight function

$$w^{(1)}_{\alpha,\beta}(x) = \text{const.}\frac{w_{\alpha,\beta}(x)}{w_{\alpha,\beta}(x)^2 + \mathscr{Q}(\alpha,\beta,x)^2}. \tag{31}$$

$x^{(1)}_{j,n,\alpha,\beta}$ denotes the jth zero of $p^{(1)}_{n,\alpha,\beta}$ and in view of (25) $\tilde{x}^{(1)}_{j,n,\alpha,\beta}$ is the jth zero of $p'_{n+1,\alpha,\beta}$. Let us also note for the following that $(-1)^{n-1}p'_{n,\alpha,\beta}(-x) = p'_{n,\beta,\alpha}(x)$ and $(-1)^n p^{(1)}_{n,\alpha,\beta}(-x) = p^{(1)}_{n,\beta,\alpha}(x)$. Obviously, $\sqrt{1-x^2}w_{\alpha,\beta}$ satisfies the assumption of Theorem 2.2 on any interval $[-1+\varepsilon, 1-\varepsilon]$ for arbitrary $\varepsilon > 0$. Combining Theorem 2.2 and Lemma 3.1 we get immediately a description of those parameters for which conjecture (24) on the zeros of $p^{(1)}_{n,\alpha,\beta}$ and $p'_{n+1,\alpha,\beta}$ does not hold, respectively, holds for large n on compact subintervals of $(-1, 1)$. More precisely we have the following corollary.

Corollary 3.3. (1) *If* $(\alpha+\frac{1}{2})(\beta+\frac{1}{2}) > 0$ *then for* $n > n_0$ *none of the following statements is true*:

$$\forall 1 \leqslant j \leqslant n \quad x^{(1)}_{j,n,\alpha,\beta} > \tilde{x}^{(1)}_{j,n,\alpha,\beta} \quad \textit{or} \quad \forall 1 \leqslant j \leqslant n \quad x^{(1)}_{j,n,\alpha,\beta} < \tilde{x}^{(1)}_{j,n,\alpha,\beta}.$$

(2) *If* $(\alpha,\beta) \in (-1, -\frac{1}{2}) \times (-\frac{1}{2}, \infty)$ *then for all* $n > n_0$:

$$x^{(1)}_{j,n,\alpha,\beta} < \tilde{x}^{(1)}_{j,n,\alpha,\beta} \quad \textit{if}\ x_{j,n,\alpha,\beta}, \tilde{x}_{j,n,\alpha,\beta} \in [-1+\varepsilon, 1-\varepsilon].$$

It is very likely that the interval $[-1+\varepsilon, 1-\varepsilon]$ can be replaced by $[-1, 1]$. Hence Ronveaux's conjecture (24) has to be modified in the following way.

Conjecture 3.4. *For all* $n \in \mathbb{N}$ *and all* $j = 1, \ldots, n$ *we have*

$$x^{(1)}_{j,n,\alpha,\beta} \overset{(>)}{<} \tilde{x}^{(1)}_{j,n,\alpha,\beta} \quad \textit{on the set}\ 0 \overset{(>)}{<} \mathscr{Q}.$$

If we want to have an inequality sign for all zeros, this conjecture transforms into the following.

Conjecture 3.5. *If* $(\alpha,\beta) \in (-1, -\frac{1}{2}) \times (-\frac{1}{2}, \infty)$, *then we have for all* $n \in \mathbb{N}$ *and all* $j = 1, \ldots, n$:

$$x^{(1)}_{j,n,\alpha,\beta} < \tilde{x}^{(1)}_{j,n,\alpha,\beta}.$$

It is also worth mentioning that in contrast to the assumption of Ronveaux [8], the sign of $\alpha+\beta+1$ seems to be secondary since

$$\operatorname{sign}(\alpha+\beta+1) = -\operatorname{sign}(c(\alpha,\beta)) = \operatorname{sign}\mathscr{Q}'(\alpha,\beta),$$

but, as we have seen, the inequalities for the zeros depend only on the sign of $\mathscr{Q}(\alpha,\beta)$. Theorem 2.2 gives us some information for large values of n only. In order to prove Conjectures 3.4 or 3.5 we have to look for a different method. One possible tool is provided by a well-known theorem of Markov (cf. e.g. [17]), which states the following: Suppose we are given two weight functions W and w on $(-1, 1)$, which satisfy

$$\forall x \in (-1, 1) \quad 0 \overset{(>)}{<} \frac{\mathrm{d}}{\mathrm{d}x} \frac{W(x)}{w(x)}, \tag{32}$$

then for all $n \in \mathbb{N}$ and all $j = 1, \ldots, n$: $x_{j,n} \overset{(>)}{<} X_{j,n}$ and $x_{j,n}$ denote the jth zero of the nth orthogonal polynomial with respect to w and W, respectively. If both weight functions are even then

$$\forall x \in (0,1) \quad 0 \overset{(>)}{<} \frac{\mathrm{d}}{\mathrm{d}x} \frac{W(x)}{w(x)} \Rightarrow \forall n \in \mathbb{N}\ \forall j \geqslant \left[\frac{n+1}{2}\right] \ x_{j,n} \overset{(>)}{<} X_{j,n}. \tag{33}$$

In the case under consideration we set $W(x) = (1 - x^2) w_{\alpha,\beta}(x)$ (recall that by (21) $(1 - x^2) w_{\alpha,\beta}(x) = \tilde{w}^{(1)}_{\alpha,\beta}(x)$), $w(x) = w^{(1)}_{\alpha,\beta}(x)$ and obtain from (29) and (31) that (32) is equivalent to

$$\forall x \in (-1, 1) \quad -(\gamma - \delta x) \overset{(>)}{<} X^2(\alpha, \beta, x)(\gamma - \delta x) + \frac{c(\alpha, \beta) X(\alpha, \beta, x)}{\pi w_{\alpha,\beta}(x)}, \tag{34}$$

where

$$\gamma := \beta - \alpha, \quad \delta := \alpha + \beta + 1 \quad \text{and} \quad X(\alpha, \beta, x) := -\frac{\mathcal{Q}_{\alpha,\beta}(x)}{w_{\alpha,\beta}(x)}.$$

By this approach we get the following positive results (note in particular the positive result for the ultraspherical case).

Theorem 3.6. (1) *Let $\alpha > -1$, then for all $n \in \mathbb{N}$ and all $j = 1, \ldots, [\frac{1}{2}n]$:*

$$\tilde{x}^{(1)}_{j,n,\alpha,\alpha} \overset{(>)}{<} x^{(1)}_{j,n,\alpha,\alpha} \quad \text{if } \alpha \overset{(>)}{<} -\tfrac{1}{2}.$$

In view of the symmetry of the zeros the converse inequality holds for all $j > [\frac{1}{2}n]$.

(2) *For all $\alpha > -\frac{1}{2}$, all $n \in \mathbb{N}$ and all $j = 1, \ldots, n$:*

$$x^{(1)}_{j,n,\alpha,-1/2} > \tilde{x}^{(1)}_{j,n,\alpha,-1/2} \quad \text{and} \quad x^{(1)}_{j,n,-1/2,\alpha} < \tilde{x}^{(1)}_{j,n,-1/2,\alpha}.$$

If $\alpha < -\frac{1}{2}$ *the converse inequalities hold.*

(3) *If $\alpha \in (\frac{1}{2}, 1)$ then for all $n \in \mathbb{N}$ and all $j = 1, \ldots, n$:*

$$x^{(1)}_{j,n,\alpha,-\alpha} < \tilde{x}^{(1)}_{j,n,\alpha,-\alpha} \quad \text{and} \quad x^{(1)}_{j,n,-\alpha,\alpha} > \tilde{x}^{(1)}_{j,n,-\alpha,\alpha}.$$

(4) *If $\beta \leqslant -\frac{1}{2}$, $\alpha \geqslant \frac{1}{2}$ and $\alpha + \beta + 1 < 0$ then for all $n \in \mathbb{N}$ and all $j = 1, \ldots, n$:*

$$x^{(1)}_{j,n,\alpha,\beta} < \tilde{x}^{(1)}_{j,n,\alpha,\beta} \quad \text{and} \quad x^{(1)}_{j,n,\beta,\alpha} > \tilde{x}^{(1)}_{j,n,\beta,\alpha}.$$

Proof. (1) In this case $Z(\alpha, \beta) = 0$ and therefore the function X reduces to

$$X = \frac{c(\alpha, \beta)}{\pi} \int_0^x \frac{1}{(1 - t^2)^{\alpha+1}} \, \mathrm{d}t =: \frac{c(\alpha, \beta)}{\pi} I(\alpha, x).$$

Therefore for $\alpha > -\frac{1}{2}$ (34) can be written equivalently

$$\frac{I(\alpha, x)}{x(1-x^2)^\alpha} - (2\alpha+1)I(\alpha, x)^2 < (2\alpha+1)\frac{\pi}{c(\alpha, \beta)}.$$

We will prove that the left-hand side is actually bounded by 1. Define

$$F(\alpha, x) := \frac{I(\alpha, x)}{x(1-x^2)^\alpha} - (2\alpha+1)I(\alpha, x)^2.$$

Since $F(\alpha, 0) = 1$ it is enough to prove that for all $x \in (0, 1)$: $(\partial/\partial x)F(\alpha, x) \leqslant 0$. This is equivalent to

$$-\frac{I(\alpha, x)}{x^2(1-x^2)^\alpha} + \frac{2\alpha I(\alpha, x)}{(1-x^2)^{\alpha+1}} + \frac{1}{x(1-x^2)^{2\alpha+1}} - \frac{(4\alpha+2)I(\alpha, x)}{(1-x^2)^{\alpha+1}} \leqslant 0$$

$$\Leftrightarrow \quad \frac{x}{(1-x^2)^\alpha} \leqslant I(\alpha, x)(1+(2\alpha+1)x^2).$$

Since $\alpha \geqslant -1$ this comes down to

$$f(\alpha, x) := \frac{x}{(1-x^2)^\alpha(1+(2\alpha+1)x^2)} \leqslant I(\alpha, x),$$

which in turn follows from

$$\frac{\partial}{\partial x} f(\alpha, x) \leqslant \frac{1}{(1-x^2)^{\alpha+1}}.$$

Simplifying one gets that this inequality holds as long as for all $x \in (0, 1)$: $-\alpha x^2 \leqslant 1$. This completes the proof of the first assertion.

(2) Since

$$J(\alpha, x) := \int_{-1}^{x} \frac{1}{(1-t)^{1+\alpha}(1+t)^{1/2}}\, dt = 2^{1/2-\alpha} I(\alpha, x),$$

we obtain the inequality

$$\frac{J}{(1-x)^\alpha(1+x)^{1/2}} - J^2(\alpha + \tfrac{1}{2}) \leqslant 4^{-\alpha},$$

which proves part (2). This could also be proved by using well-known relations between $p_{n,\alpha,\pm 1/2}$ and $p_{2n,\alpha,\alpha}$ and $p_{2n+1,\alpha,\alpha}$, respectively (cf. [17, 4.1.5]).

(3) In this case we have

$$X(\alpha, -\alpha, x) = -\cot(\alpha\pi) + \frac{1}{\sin(\alpha\pi)}\left(\frac{1+x}{1-x}\right)^\alpha,$$

and (34) translates to

$$\forall x \in (-1, 1) \quad \left(\frac{1+x}{1-x}\right)^{2\alpha} x - 2\cos(\alpha\pi)\left(\frac{1+x}{1-x}\right)^\alpha (x+\alpha) + x + 2\alpha > 0.$$

Since $\cos(\alpha\pi) \leqslant 0$ the inequality trivally holds if $x \geqslant 0$. If $-\alpha \leqslant x \leqslant 0$ then the left-hand side is bounded from below by

$$x\left(1+\left(\frac{1+x}{1-x}\right)^{2\alpha}\right)+2\alpha \geqslant x\left(1+\frac{1+x}{1-x}\right)+2\alpha \geqslant 2\frac{x+\alpha}{1-x} \geqslant 0.$$

It remains to prove the inequality in the case $-1 \leqslant x \leqslant -\alpha$: Putting $x=-y$, $z=(1-y)/(1+y)$ and $C=-\cos(\alpha\pi)$ we obtain the inequality

$$\forall \tfrac{1}{2} \leqslant \alpha \leqslant y \leqslant 1 \quad \left(z^{\alpha}+C\left(1-\frac{\alpha}{y}\right)\right)^2 \leqslant \frac{2\alpha-y}{y}+C^2\left(1-\frac{\alpha}{y}\right)^2.$$

Since $(a+b)^2 \leqslant 2(a^2+b^2)$ this inequality follows from

$$2z^{2\alpha}+C^2\left(1-\frac{\alpha}{y}\right)^2 \leqslant \frac{2\alpha-y}{y}. \tag{35}$$

For $\frac{1}{2} \leqslant \alpha \leqslant y \leqslant 1$ we have

$$\frac{y(1-y)}{1+y} \leqslant \frac{\alpha(2\alpha-y)}{1+\alpha}.$$

Therefore (35) holds provided

$$\forall \tfrac{1}{2} \leqslant \alpha \leqslant y \leqslant 1 \quad C^2 \leqslant \frac{1-\alpha}{1+\alpha}\frac{y(2\alpha-y)}{(y-\alpha)^2}. \tag{36}$$

Now the right-hand side of this inequality is bounded from below by

$$\frac{y(2\alpha-y)}{1-\alpha^2} \geqslant \frac{2\alpha-1}{1-\alpha^2} \geqslant \frac{\alpha-\frac{1}{2}}{2-\alpha}.$$

Hence (36) immediately follows from $\sin^2(\frac{1}{2}t\pi \leqslant t/(1-t)$, which is true for all $0 \leqslant t \leqslant 1$.

(4) Since $\gamma+\delta<0$ and $\gamma-\delta \leqslant 0$ we conclude that for all $-1<x<1$: $\gamma-\delta x<0$. Also, by Lemma 3.1, $cX(1) \leqslant 0$ and since J is an increasing function of x, the left-hand side of (34) is always negative; on the other hand the right-hand side is positive. □

For Theorem 3.6 (4) and (1) in the case $\alpha < -\frac{1}{2}$, cf. [11].

The remaining part of this section is devoted to the proof of the following.

Theorem 3.7. *There exists an absolute constant $c_0 \in \mathbb{R}$ such that for all $(\alpha, \beta) \in (c_0, \infty) \times (-1, -\frac{1}{2})$, all $n \in \mathbb{N}$ and all $j = 1, \ldots, n$:*

$$x^{(1)}_{j,n,\alpha,\beta} > \tilde{x}^{(1)}_{j,n,\alpha,\beta} \quad \text{and} \quad x^{(1)}_{j,n,\beta,\alpha} < \tilde{x}^{(1)}_{j,n,\beta,\alpha}.$$

In what follows we assume $\beta < -\frac{1}{2}$ and $\alpha+\beta+1>0$; in this case $-\gamma > \delta < 0$ and therefore $\gamma-\delta x<0$ for all $x \in (-1, 1)$, also $X(-1)>0$. Putting $\zeta=-\gamma/\delta>1$ then by Conjecture 3.4 (34) is equivalent to

$$X^2(\alpha,\beta,x)-\frac{c(\alpha,\beta)}{\pi(1+\alpha+\beta)(x+\zeta)w_{\alpha,\beta}(x)}X(\alpha,\beta,x)+1>0. \tag{37}$$

Putting $\eta := -\beta - \frac{1}{2} \in (0, \frac{1}{2})$, $C_0(\eta) := (1/\pi)\cos(\eta\pi)\Gamma(\frac{1}{2} - \eta)$ and

$$\phi(\alpha, \eta, x) := \frac{c(\alpha, \beta)}{\Gamma(1+\beta)} \int_{-1}^{x} \frac{1}{w_{1+\alpha, 1+\beta}(t)} \mathrm{d}t,$$

$$\psi(\alpha, \eta, x) := \phi(\alpha, \eta, x) - \frac{c(\alpha, \beta)}{\Gamma(1+\beta)} \frac{1}{(1+\alpha+\beta)(x+\zeta) w_{\alpha,\beta}(x)},$$

(37) translates to

$$C_0^2 \phi\psi + C_0 \sin(\eta\pi)(\phi + \psi) + 1 > 0. \tag{38}$$

Putting $x = 2y/(1+\alpha) - 1$ it is easily checked that

$$\lim_{\alpha \to \infty} \phi(\alpha, \eta, x) = \int_0^y \mathrm{e}^t t^{\eta - 1/2} \mathrm{d}t =: \phi_0(\eta, y)$$

and

$$\lim_{\alpha \to \infty} \psi(\alpha, \eta, x) = \int_0^y \mathrm{e}^t t^{\eta - 1/2} \mathrm{d}t - \frac{\mathrm{e}^y y^{\eta + 1/2}}{y + \eta} = \tfrac{1}{2} \int_0^y \frac{t - \eta}{(t+\eta)^2} \mathrm{e}^t t^{\eta - 1/2} \,\mathrm{d}t =: \psi_0(\eta, y).$$

Moreover, for all compact subsets K of $\mathbb{R}_0^+$ and all $\varepsilon > 0$ we have

$$\lim_{\alpha \to \infty} \inf_{(\eta, y) \in [0, 1/2] \times K} \phi(\alpha, \eta, y)\psi^-(\alpha, \eta, y) - (1+\varepsilon)\phi_0(\eta, y)\psi_0^-(\eta, y) \geqslant 0, \tag{39}$$

where $\psi^- := \min(\psi, 0)$. In the limit $\alpha \to \infty$ we therefore obtain the following inequality:

$$C_0^2 \phi_0 \psi_0 + C_0 \sin(\eta\pi)(\phi_0 + \psi_0) + 1 > 0. \tag{40}$$

By "blowing up" $y \to \eta x$ this transforms into

$$\tfrac{1}{2}(C_0 \eta^\eta)^2 \Phi\Psi + \tfrac{1}{2} C_0 \eta^\eta \frac{\sin(\eta\pi)}{\sqrt{\eta}} \Psi + C_0 \eta^\eta \sqrt{\eta} \sin(\eta\pi) \Phi + 1 > 0, \tag{41}$$

where

$$\Phi(\eta, x) := \int_0^x t^{\eta - 1/2} \mathrm{e}^{\eta t} \mathrm{d}t \quad \text{and} \quad \Psi(\eta, x) := \int_0^x \frac{t-1}{(t+1)^2} t^{\eta - 1/2} \mathrm{e}^{\eta t} \mathrm{d}t.$$

Putting $K_0(\eta) = C_0(\eta)\eta^\eta$ we get that (41) is equivalent to

$$\forall x > 0 \quad -\Psi \leqslant \frac{2}{K_0} \left(\frac{\cos^2(\eta\pi)}{K_0 \Phi + (\sin(\eta\pi))/\sqrt{\eta}} + \sqrt{\eta} \sin(\eta\pi) \right). \tag{42}$$

Let us make a few simple observations:

- If (42) holds for some $K \geqslant K_0$ then it also holds for K_0 instead of K.
- If (42) holds for some upper bound $\tilde{\Phi}$ of Φ instead of Φ, then it also holds for Φ. This follows immediately from the fact that the left-hand side is a decreasing function of Φ.

- Since $-\Psi$ and Φ are increasing on (0, 1) it suffices to prove (42) for $x \geqslant 1$.
  The following lemma gives some more information than will actually be needed.

Lemma 3.8. *Let $0 < \eta \leqslant \frac{1}{2}$ and let $x(\eta)$ be the zero of the function $-\Psi(\eta, x)$. Then the following holds:*

(1) *For $\eta \leqslant \frac{1}{5}$ we have $x(\eta) \leqslant 1/\eta + \log 1/\eta$.*
(2) *There exists a constant $c > 0$ such that for all η: $c^{-1} \geqslant \eta x(\eta) \geqslant c$.*
(3) *For $\eta \leqslant \frac{2}{5}$ and $x \geqslant 1$ we have*

$$-\Psi(\eta, x) < \frac{2\sqrt{x}}{1+x}(1 - \eta x + \eta \log x).$$

Proof. (1) For $x > 1$ define

$$f(\eta, x) := \int_0^{x-1} \frac{s}{\sqrt{1+s}(2+s)^2} e^{s\eta}(1+s)^\eta \, ds \quad \text{and} \quad g(\eta) := \int_0^1 \frac{1-s}{\sqrt{s}(1+s)^2} e^{s\eta} s^\eta \, ds.$$

Then $-\Psi(\eta, x) = -e^\eta f(x-1, \eta) + g(\eta)$. Since all partial derivatives of f with respect to η are positive, we obtain by Taylor's theorem

$$\begin{aligned} f(\eta, x) &\geqslant f(0, x) + \frac{\partial}{\partial \eta} f(0, x) \\ &= 1 - \frac{2\sqrt{1+x}}{2+x} + \eta\left(-4 + \frac{2\sqrt{1+x}}{2+x}(4 + x - \log(1+x))\right). \end{aligned}$$

Since the third derivative of g is negative on the whole interval $(0, \frac{1}{2})$ we get

$$g(\eta) \leqslant g(0) + g'(0)\eta + \tfrac{1}{2}g''(0)\eta^2 = 1 - 3\eta + \tfrac{43}{6}\eta^2.$$

Combining these with the estimate $e^\eta \geqslant 1 + \eta + \frac{1}{2}\eta^2$ we conclude that $-e^\eta f(\eta, x-1) + g(\eta)$ is bounded from above by

$$\frac{2\sqrt{x}}{1+x}(1 - \eta x + \eta \log x - 2\eta) + \eta^2\left(\frac{32}{3} - \frac{2\sqrt{x}}{1+x}(\tfrac{5}{2} + x - \log x)\right). \tag{43}$$

For $x \geqslant x_0$ the coefficient of η^2 is smaller or equal than zero and for $x \geqslant 1/\eta + \log 1/\eta$ the first term is bounded from above by

$$\eta\left(\log\left(1 + \eta \log \frac{1}{\eta}\right) - 2\right) \leqslant \eta(\log(1 + e^{-1}) - 2) < 0.$$

Hence $x(\eta) \leqslant 1/\eta + \log 1/\eta$. Setting $a = \frac{1}{2}\log(1 + e^{-1})$ and $b = 1 - a$ we can determine x_0 to be the zero of the function

$$\frac{32}{3} - \frac{2\sqrt{x}}{1+x}(\tfrac{5}{2} + x - \log x + 2bx);$$

i.e. $x_0 \leqslant 5$. Thus, if $\eta \leqslant \frac{1}{5}$ then $\frac{32}{3} - 2\sqrt{x}/(1+x)(\frac{5}{2} + x - \log x + 2b/\eta)$ is negative for all $x \geqslant 5$.

(2) The upper bound is a consequence of part (1). As for the lower we get by Kimball's inequality:

$$\psi_0(n, y) \leqslant e^y y^{\eta - 1/2}\left(\frac{1 - e^{-y}}{\eta + \frac{1}{2}} - \frac{1}{y + \eta}\right).$$

This implies $x(\eta)\eta \geqslant \log 2$.

(3) We note that (43) can be written as

$$\frac{2\sqrt{x}}{1 + x}(1 - \eta x + \eta \log x) + \eta^2\left(\frac{32}{3} - \frac{2\sqrt{x}}{1 + x}\left(\frac{5}{2} + x - \log x + \frac{2}{\eta}\right)\right)$$

and for $\eta \leqslant \frac{2}{9}$ the coefficient of η^2 is negative for all $x \geqslant 1$. □

Remark. Actually $\eta x(\eta)$ converges to the first positive zero of the function

$$\int_0^x t^{-1/2} e^t \, dt - e^x x^{-1/2},$$

which is easily shown to be smaller than 1.

Lemma 3.9. *For all $x > 0$ and all $\eta \in [0, \frac{1}{2}]$ the following inequalities hold:*

$$\Phi(\eta, x) \leqslant 2x^{1/2 - \eta} e^{\eta x}(1 - e^{-x/2})^{2\eta} \quad \textit{and} \quad -\Psi(\eta, 1) \leqslant (2 - \sqrt{e})^{2\eta}.$$

Proof. First of all we consider the two extremal cases $\eta = 0$, i.e. $\beta = -\frac{1}{2}$ and $\eta = \frac{1}{2}$, i.e. $\beta = -1$; though the second case is excluded (42) is still defined!

- If $\eta = 0$ then $\Phi = 2\sqrt{x}$, $-\Psi = 2\sqrt{x}/(1 + x)$ (and (42) holds provided $K \leqslant 1/\sqrt{2}$).
- If $\eta = \frac{1}{2}$ then $\Phi = 2(e^{x/2} - 1)$ and $-\Psi = 2 - 2e^{x/2}/(1 + x)$, (in this case (42) holds provided $K \leqslant \sqrt{2}/(\sqrt{e} - 1)$.

By Hölder's inequality both the function $t \mapsto \Phi(t, x)$ and $t \mapsto -\Psi(t, 1)$ are log-convex. Hence the assertions follows by considering the extremal cases. □

Proof of Theorem 3.7. First we will prove the following claim.

Claim. (42) *holds for*

$$K(\eta) := e^{-1/2\cos^2(\eta\pi) + 1/4\eta^2} \quad \textit{and} \quad \tilde{\Phi}(\eta, x) := 2x^{1/2 - \eta} e^{\eta x}$$

instead of K_0 and Φ (by Lemma 3.9, $\tilde{\Phi}$ is an upper bound for Φ!).

A lower bound for the expression on the right-hand side of (42) can be obtained as follows: the elementary inequality $e^x \geqslant 1 + x$ and the inequality between the geometric and the arithmetic mean imply:

$$\frac{2}{K}\left(\frac{\cos^2(\eta\pi)}{K\tilde{\Phi}+\sin(\eta\pi)/\sqrt{\eta}}+\sqrt{\eta}\sin(\eta\pi)\right)\geqslant\frac{2}{K^2\tilde{\Phi}}\left(\exp\left(-\frac{\sin(\eta\pi)}{K\tilde{\Phi}\sqrt{\eta}}\right)\cos^2(\eta\pi)+\frac{K\tilde{\Phi}\sqrt{\eta}}{\sin(\eta\pi)}\sin^2(\eta\pi)\right)$$

$$\geqslant\frac{2}{K^2\tilde{\Phi}}\left(\exp\left(-\frac{\sin(\eta\pi)}{K\tilde{\Phi}\sqrt{\eta}}\right)\right)^{\cos^2(\eta\pi)}\left(\frac{K\tilde{\Phi}\sqrt{\eta}}{\sin(\eta\pi)}\right)^{\sin^2(\eta\pi)}\geqslant\frac{2}{K^2\tilde{\Phi}}\exp\left(\sin^2(\eta\pi)-\frac{\sin(\eta\pi)}{K\tilde{\Phi}\sqrt{\eta}}\right).$$

Thus (42) holds if

$$-\Psi\tilde{\Phi}\leqslant\frac{2}{K^2}\exp\left(\sin^2(\eta\pi)-\frac{\sin(\eta\pi)}{K\tilde{\Phi}\sqrt{\eta}}\right). \tag{44}$$

For $\eta\leqslant\frac{2}{9}$ and $x\geqslant 1$ we get by Lemma 3.8:

$$-\Psi(\eta,x)\tilde{\Phi}(\eta,x)\leqslant 2x^{1/2-\eta}e^{\eta x}\frac{2\sqrt{x}}{1+x}(1-\eta x+\eta\log x)=\frac{4x}{1+x}x^{-\eta}e^{\eta x}(1-\eta x+\eta\log x).$$

Since for all $t\geqslant 0$: $(1-t)e^t\leqslant e^{-t^2/2}$ we conclude that the above expression is bounded by

$$\frac{4x}{1+x}e^{-\eta^2(x-\log x)^2/2}.$$

Combining this with the estimate (44) we see that the inequality

$$\sin^2(\eta\pi)+\tfrac{1}{2}\eta^2(x-\log x)^2+\log\left(\frac{1}{2K^2}\right)+\log\left(1+\frac{1}{x}\right)-\frac{\sin(\eta\pi)}{2K\sqrt{\eta}}x^{-1/2+\eta}e^{-\eta x}\geqslant 0$$

implies (42) for all $x\geqslant 1$ and all $\eta\leqslant\frac{2}{9}$. Since $\eta\geqslant 0$ and $x\geqslant 1$ we have

$$x^{-1/2+\eta}e^{-\eta x}\leqslant x^{-1/2}e^{-\eta};$$

though its enough to prove

$$\log\left(\frac{e^{\sin^2(\eta\pi)+\eta^2/2}}{2K^2}\right)+\log\left(1+\frac{1}{x}\right)-\frac{\sin(\eta\pi)}{2K\sqrt{\eta}}x^{-1/2}e^{-\eta}\geqslant 0. \tag{45}$$

In order to show this we first notice that for all $\eta\leqslant\frac{1}{4}$

$$A(\eta):=\frac{\sin(\eta\pi)}{2\sqrt{\eta}}e^{-\eta+(\cos^2\eta\pi)/2-\eta^{2/4}}$$

is smaller than 1. Now define

$$f_\eta(x):=\log\left(\frac{e}{2}\left(1+\frac{1}{x}\right)\right)-\frac{A(\eta)}{\sqrt{x}}.$$

If f_η has a local minimum at y, then $2/(1+y) = A(\eta)/\sqrt{y}$. The value of f_η at the point y is therefore given by

$$\log\left(\frac{e}{2}\left(1+\frac{1}{y}\right)\right) - \frac{2}{1+y} =: h(y).$$

Since $f_\eta(1) = 1 - A(\eta) > 0$ and $f_\eta(\infty) > 0$ it suffices to prove that for all $y \geqslant 1$: $h(y) \geqslant 0$. But $h(1) = 0$ and $h'(y) \geqslant 0$ for all $y \geqslant 1$. Thus finishing the proof of our claim in the case $\eta \leqslant \frac{2}{9}$. But by Lemma 3.9, (42) trivially holds for

$$K = \frac{2\sqrt{\eta}\sin(\eta\pi)}{(2-\sqrt{e})^{2\eta}},$$

and for $\eta \geqslant \frac{2}{9}$ this is greater than the value of $K(\eta)$ in our claim. It can be checked easily that $K(\eta)$ is strictly greater than $K_0(\eta)$; this is important for the following argument.

It remains to prove (38) for sufficiently large values of α. This inequality holds trivially if $\psi \geqslant 0$ and on compact subsets of $[1, \infty) \times [0, \frac{1}{2}]$, $\psi(\alpha, \eta, 2y/(1+\alpha) - 1)$ converges uniformly to $\psi_0(\eta, y)$ as α converges to ∞; by Lemma 3.8 there exist absolute constants c_1 and c_2 such that for all $y \geqslant c_2$: $\psi_0(y) \geqslant c_1$. Differentiation of $\psi(\alpha, \eta, x)$ with respect to x yields another constant c_3 such that for all $y \geqslant c_3$:

$$D_3\psi\left(\alpha, \eta, \frac{2y}{1+\alpha} - 1\right) > 0.$$

By uniform convergence we conclude that there exist constants c_4 and c_5 such that for all $y \geqslant c_4$, $\alpha \geqslant c_5$ and all $0 \leqslant \eta \leqslant \frac{1}{2}$:

$$\psi\left(\alpha, \eta, \frac{2y}{1+\alpha} - 1\right) > 0.$$

Since K is strictly greater than K_0 the theorem follows from (39). □

To conclude this paper we summarize the results in Fig. 1 (the right one illustrates our conjecture). The meaning of the symbols is declared by

$$+:\quad \forall n\ \forall j \leqslant n:\ \tilde{x}^{(1)}_{j,n,\alpha,\beta} < x^{(1)}_{j,n,\alpha,\beta},$$

$$-:\quad \forall n\ \forall j \leqslant n:\ \tilde{x}^{(1)}_{j,n,\alpha,\beta} > x^{(1)}_{j,n,\alpha,\beta},$$

$$0:\quad \exists n_0\ \forall n > n_0\ \exists j, k \leqslant n:\ \tilde{x}^{(1)}_{j,n,\alpha,\beta} < x^{(1)}_{j,n,\alpha,\beta} \quad \text{and} \quad \tilde{x}^{(1)}_{k,n,\alpha,\beta} > x^{(1)}_{k,n,\alpha,\beta}.$$

Let us also recall our general Conjecture 3.4.

Conjecture. *For all $n \in \mathbb{N}$ and all $j = 1, \ldots, n$ we have*

$$x^{(1)}_{j,n,\alpha,\beta} \overset{(>)}{<} \tilde{x}^{(1)}_{j,n,\alpha,\beta} \quad \text{on the set } 0 \overset{(>)}{<} \mathcal{Q}_{\alpha,\beta}.$$

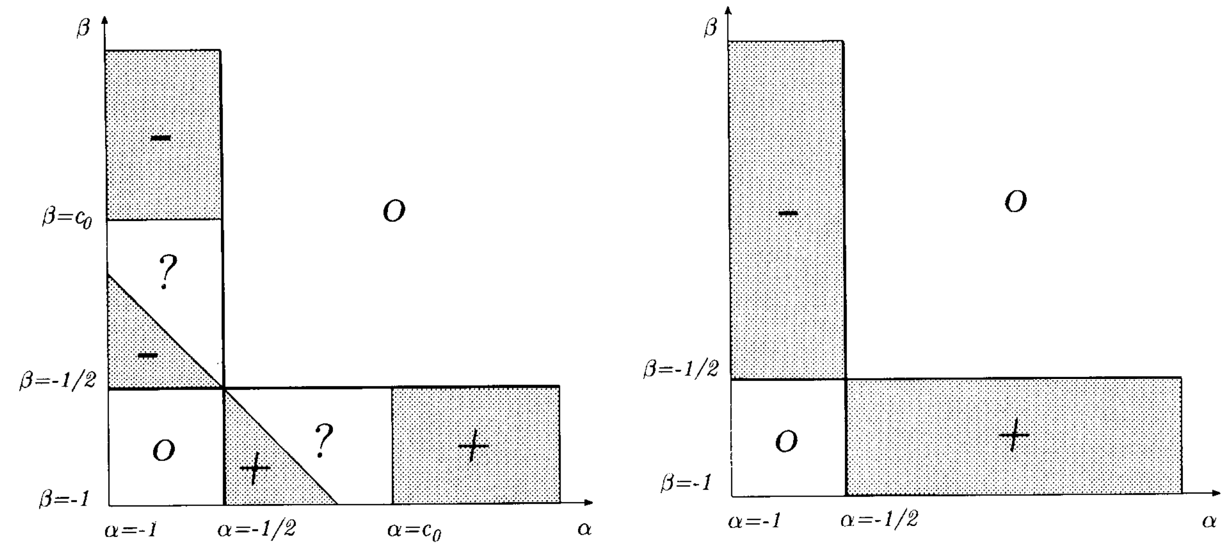

Fig. 1.

Added in proof. As one of the referees informed us, A. Elbert and A. Laforgia [3] also proved Ronveaux's conjecture (24) in the ultraspherical case (compare Theorem 3.6(1) of this paper). Yet we have not seen their manuscript nor do we know about their method of proof.

References

[1] M. Alfaro, J.J. Guadalupe, M.P. Alfaro and L. Vigil, Correspondance entre suites de polynômes orthogonaux et fonctions de la boule unité de $H_0^\infty(D)$, in: C. Breszinski et al., Eds., *Polynômes Orthogonaux et Applications*, Lecture Notes in Mathematics, Vol. 1171 (Springer, Berlin 1985) 158–163.

[2] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York 1978).

[3] A. Elbert and A. Laforgia, On the zeros of associated polynomials of classical orthogonal polynomials, manuscript.

[4] Ya.L. Geronimus, Polynomials orthogonal on a circle and their applications, *Zap. Naucno-Issled. Inst. Math. Mekh. Kharkov. Mat. Obshch.* **19** (1962) 35–120 (English translation); *Amer. Math. Soc. Transl.* **3** (1962) 1–78.

[5] Ya.L. Geronimus, Orthogonal polynomials, *Amer. Math. Soc. Transl.* **108**(2) (1977) 37–130.

[6] C.C. Grosjean, The weight functions, generating functions and miscellaneous properties of the sequences of orthogonal polynomials of the second kind associated with the Jacobi and the Gegenbauer polynomials, *J. Comput. Appl. Math.* **16** (1986) 259–307.

[7] P. Koosis, *Introduction to H_p-Spaces*, London Math. Soc. Lecture Note Ser. 40 (Cambridge Univ. Press, Cambridge, 1980).

[8] P. Nevai, A new class of orthogonal polynomials, *Proc. Amer. Math. Soc.* **91** (1984) 409–415.

[9] F. Peherstorfer, Orthogonal polynomials in L_1 approximation, *J. Approx. Theory* **52** (1988) 241–268.

[10] F. Peherstorfer, On the asymptotic behavior of functions of the second kind and Stieltjes polynomials and on the Gauss–Kronrod quadrature formulas, *J. Approx. Theory* **70** (1992) 156–190.

[11] F. Peherstorfer, A partial solution to a problem, in: W. Van Assche, Ed., *Open problems*, *J. Comput. Appl. Math.* **48** (1993) 241–243.

[12] F. Peherstorfer, A special class of polynomials orthogonal on the unit circle including the associated polynomials, *Constr. Approx.*, to appear.

[13] F. Peherstorfer, Asymptotic interlacing properties of zeros of orthogonal polynomials, manuscript.
[14] E.A. Rakhmanov, On the asymptotics of the ratio of orthogonal polynomials, II, *Math. USSR-Sb.* **46** (1983) 105–117.
[15] A. Ronveaux, Zeros of associated versus zeros of derivatives of classical orthogonal polynomials, in: C. Brezinski et al., Eds., *Orthogonal Polynomials and their Applications* (Baltzer, Basel, 1991).
[16] J. Shohat, On mechanical quadratures, in particular with positive coefficients, *Trans. Amer. Math. Soc.* **42** (1937) 461–496.
[17] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Vol. 23 (Amer Math. Soc., Providence, RI, 4th ed., 1975).