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Uniform convergence of arbitrary order on nonuniform meshes for a singularly perturbed boundary value problem

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Abstract

We consider the numerical solution of a singularly perturbed linear self-adjoint boundary value problem. Assuming that the coefficients of the differential equation are smooth, we construct and analyze finite difference methods that converge both with high order and uniformly with respect to the singular perturbation parameter. The analysis is done on a locally quasiuniform mesh, which permits its extension to the case of adaptive meshes which may be used to improve the solution. Numerical examples are presented to demonstrate the effectiveness of the method and its low computational cost. The convergence obtained in practice satisfies the theoretical predictions.

Keywords: Singular perturbation; Arbitrary order; Irregular meshes; Uniform convergence

1. Introduction

In this paper we construct and analyze numerical approximations obtained from new exponentially fitted finite difference schemes applied to the singularly perturbed boundary value problem

$$L_\varepsilon u \equiv -\varepsilon^2 u''(x) + b^2(x)u(x) = f(x), \quad 0 < x < 1, \tag{1.1a}$$

$$u(0) = A, \qquad u(1) = B, \tag{1.1b}$$

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where $\varepsilon \in (0, 1]$ is a small perturbation parameter, A and B are given constants, b and f are smooth functions and we assume that

$$b(x) \geqslant b > 0. \tag{1.2}$$

It is known that if (1.2) holds, then problem (1.1) has a unique solution (e.g. [6]).

Numerical treatment of problem (1.1) has been widespread in recent years, for example [6, 11, 13, 17, 19]. The need to resolve the boundary layers has motivated the use of nonuniform meshes, where the majority of the mesh points are placed in fast transition zones. Examples of this are contained in [12, 18], where finite difference schemes on fixed nonuniform meshes are considered. Another interesting scheme is to construct methods starting with arbitrary initial meshes and then improving these with the aid of adaptive techniques. A further possibility is to construct exponentially fitted discretizations adapted to the singularly perturbed problem. The main contribution of this paper lies in the construction of methods of this latter types (ε-uniformly convergent methods), whose solutions converge independent of the perturbation parameter to the theoretical solution, with arbitrarily high order and low computational cost.

We know of only a few papers concerned with the construction of such high order ε-uniformly convergent methods. In [19] methods of order $\leqslant 3$ on nonuniform meshes are analyzed, but the technique used is difficult to generalize to other cases. [1, 5] establish ε-uniform convergence of second order for the El-Mistikawy–Werle method, and in [2, 3] methods of OCI-type are shown to give very good results. [12] analyses a fourth-order ε-uniform method for a semilinear problem with specially designed nonuniform meshes. In [21] Shishkin obtains theoretical results similar to ours using different methods of proof and without numerical examples to test the methods in practice. [9] constructs a family of schemes (exponentially fitted HODIE schemes) that are ε-uniformly convergent for a non-self-adjoint problem with nonzero convective term. He shows moreover that the arbitrary order of the method enables accurate results to be obtained on a uniform mesh. In [20] Sakai and Usman use simple B-splines to construct a finite difference method of order two for a non-self-adjoint problem, which requires less computational effort.

In this paper we construct and analyze ε-uniformly convergent methods on locally quasiuniform meshes for a self-adjoint problem without a convective term, and we validate our theoretical results by sample computations. The method of proof is similar to that in [9], but it differs fundamentally from it in that we are able to establish that our methods are not only ε-uniformly stable (as in Gartland) but are also ε-uniformly accurate (which is not possible in general for non-self-adjoint problems). This enables us to simplify greatly the proofs compared with those of [21, 9]. Indeed, we can make use of an obvious modification of the standard convergence theorem, namely that ε-uniform stability and ε-uniform accuracy of order p imply ε-uniform convergence of order p. Furthermore, we are able to establish the ε-uniform convergence without using the results of [24], which are required by Gartland's proof for the non-self-adjoint case.

Throughout the paper C denotes a generic positive constant, independent of both the perturbation parameter ε and the mesh parameter. Also, for any $U = (u_0, \ldots, u_N) \in \mathbb{R}^{N+1}$, we define the discrete L^∞ norm

$$\|U\|_\infty = \max_{0 \leqslant i \leqslant N} |u_i|. \tag{1.3}$$

2. The continuous problem

In this section we obtain a decomposition of the solution $u(x, \varepsilon)$ of problem (1.1), which is important in the later analysis of the ε-uniform convergence of our finite difference schemes.

Theorem 2.1. *Let s be a positive integer. Then for ε sufficiently small and b and f sufficiently smooth functions, the solution $u(x, \varepsilon)$ of* (1.1) *admits the representation*

$$u(x,\varepsilon) = A(x,\varepsilon) + B_1(x,\varepsilon)\exp\left(\frac{-1}{\varepsilon}\int_x^1 b(t)\,\mathrm{d}t\right) + B_2(x,\varepsilon)\exp\left(\frac{-1}{\varepsilon}\int_0^x b(t)\,\mathrm{d}t\right) \tag{2.1}$$

where A, B_1, B_2, are their derivatives up to order s, are bounded independently of the perturbation parameter ε.

Proof. We give an outline of the proof (see [4] for details). Let $v(x, \varepsilon)$ and $w(x, \varepsilon)$ be the solutions of the problems

$$-\varepsilon^2 v''(x) + b^2(x)v(x) = f(x), \quad 0 < x < 1,$$

$$-\varepsilon^2 w''(x) + b^2(x)w(x) = 0, \quad 0 < x < 1,$$

$$w(0) = A - v(0,\varepsilon), \qquad w(1) = B - v(1,\varepsilon),$$

respectively. Then, it is obvious that

$$u(x,\varepsilon) = v(x,\varepsilon) + w(x,\varepsilon).$$

If we write $v(x, \varepsilon)$ in the form

$$v(x,\varepsilon) = v_0(x) + \varepsilon v_1(x) + \cdots + \varepsilon^{s-1}v_{s-1} + \varepsilon^s V_s(x,\varepsilon)$$

and take

$$y(x,\varepsilon) = \exp\left(\frac{-1}{\varepsilon}\int_x^1 b(t)\,\mathrm{d}t\right)[y_0(x) + \varepsilon y_1(x) + \cdots + \varepsilon^{s-1}y_{s-1} + \varepsilon^s Y_s(x,\varepsilon)],$$

$$z(x,\varepsilon) = \exp\left(\frac{-1}{\varepsilon}\int_0^x b(t)\,\mathrm{d}t\right)[z_0(x) + \varepsilon z_1(x) + \cdots + \varepsilon^{s-1}z_{s-1} + \varepsilon^s Z_s(x,\varepsilon)],$$

to be two linearly independent solutions of the differential equation $L_\varepsilon w = 0$, where the functions for $j = 1, \ldots, s-1$, $v_j(x)$, $y_j(x)$, $z_j(x)$, and $V_s(x,\varepsilon)$, $Y_s(x,\varepsilon)$ and $Z_s(x,\varepsilon)$ are chosen appropriately, then we can prove that for $k = 0, \ldots, s$

$$|v_j^{(k)}(x)| \leqslant C, \quad |y_j^{(k)}(x)| \leqslant C, \quad |z_j^{(k)}(x)| \leqslant C, \qquad j = 0, \ldots, s-1$$

and

$$|V_s^{(k)}(x,\varepsilon)| = \mathrm{O}(\varepsilon^{-k}), \qquad |Y_s^{(k)}(x,\varepsilon)| = \mathrm{O}(\varepsilon^{-k}), \qquad |Z_s^{(k)}(x,\varepsilon)| = \mathrm{O}(\varepsilon^{-k}).$$

The result follows when we impose the boundary conditions. □

The decomposition given in the last theorem leads to the following inequality.

Corollary 2.2. *Let $\Omega = (0, 1)$ and k be a nonnegative integer which depends on the smoothness of the coefficients of differential equation* (1.1a). *Assume that $u \in C^k(\bar{\Omega})$. Then, for all $x \in \Omega$ and $\varepsilon \in (0, 1]$*

$$|u^{(k)}(x, \varepsilon)| \leqslant C(1 + \varepsilon^{-k}(\mathrm{e}^{-bx/\varepsilon} + \mathrm{e}^{-b(1-x)/\varepsilon}))$$

where C is independent of ε.

This inequality is used in the analysis of the local truncation error of the finite difference schemes (see for example [12]).

3. Construction and properties of the finite difference scheme

We partition $[0, 1]$ with the mesh points $0 = x_0 < x_1 < \cdots < x_N = 1$ taking $h_i = x_i - x_{i-1}$, $1 \leqslant i \leqslant N$, $\tilde{h}_i = (h_i + h_{i+1})/2$, $1 \leqslant i \leqslant N-1$, $H = \max_i h_i$, $\alpha_i = h_{i+1}/h_i$, $1 \leqslant i \leqslant N-1$, $\rho_i = h_i/\varepsilon$, $1 \leqslant i \leqslant N$. We assume that this mesh is locally quasiuniform, i.e. that there is a constant $\lambda > 0$ such that

$$\lambda^{-1} \leqslant \alpha_i \leqslant \lambda, \quad 1 \leqslant i \leqslant N-1. \tag{3.1}$$

For problem (1.1) we consider the discretization

$$L_{\varepsilon,h}(U_h)_j \equiv r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = \sum_{i=1}^{I} q_j^i f(\xi_j^i), \quad 1 \leqslant j \leqslant N-1, \tag{3.2a}$$

$$U_0 = A, \qquad U_N = B, \tag{3.2b}$$

where the r's and q's are determined later and the ξ_j^i are chosen to be distinct points of the interval $[x_{j-1}, x_{j+1}]$, called HODIE points [16].

Schemes of type (3.2), called exponentially fitted HODIE schemes, were first suggested in [9] for a singular perturbation problem with positive convective term on a uniform mesh and were later generalized in [4] to locally quansiuniform meshes for the same problem.

The coefficients of (3.2) are determined by imposing conditions that permit us to analyse easily the local truncation error and then the ε-uniform convergence of the scheme. Thus we require that the local truncation error vanishes on a set of functions that we choose according to the splitting given in (2.1), i.e. on a set of functions of polynomial and exponential type.

Let $p \geqslant 1$ be a positive integer. We construct our scheme by requiring it to be exact for each function in the set

$$\{1, x, \ldots, x^{p-1}, E(x), xE(x), \ldots, x^{p-1}E(x), e(x), xe(x), \ldots, x^{p-1}e(x)\}, \tag{3.3}$$

where

$$E(x) = \exp\left(\frac{1}{\varepsilon}\int_0^x b(t)\,\mathrm{d}t\right), \quad e(x) = 1/E(x), \tag{3.4}$$

and subject to the normalization condition

$$\sum_{i=1}^{3p-2} q_j^i = 1, \quad 1 \leqslant j \leqslant N-1. \tag{3.5}$$

These conditions lead to a local linear algebraic system for the r's and q's, once the auxiliary ξ_j^i are chosen. In our case we take $\xi_j^r < \xi_j^s$ if $r < s$ and

$$\xi_j^1 = x_{j-1}, \quad \xi_j^{3p/2} = x_j, \quad \xi_j^{3p-2} = x_{j+1}, \quad \text{if } p \text{ is even,} \tag{3.6a}$$

$$\xi_j^1 = x_{j-1}, \quad \xi_j^{(3p-1)/2} = x_j, \quad \xi_j^{3p-2} = x_{j+1}, \quad \text{if } p \text{ is odd,} \tag{3.6b}$$

$$|\xi_j^p - x_j| = \xi_j^{2p}. \tag{3.6c}$$

Remark 3.1. Condition (3.6c) is used later in estimates of the r's and q's. We conjecture however that it is not a necessary condition. This is consistent with the observation that the numerical results do not depend on it.

Theorem 3.2. *Let p be a positive integer. Suppose that the functions b and f are sufficiently smooth, that the local truncation error is zero for the functions* (3.3), (3.4), *and that the normalization condition* (3.5) *and the distribution requirements* (3.6) *hold. Then for all H sufficiently small and independent of ε, the finite difference scheme* (3.2) *is uniquely determined.*

Moreover, for $1 \leqslant j \leqslant N-1$ the coefficients satisfy the following inequalities

$$0 < r_j^- + r_j^c + r_j^+ \leqslant C, \tag{3.7a}$$

$$r_j^- < 0, \qquad r_j^+ < 0, \tag{3.7b}$$

and the inequalities

$$|r_j^-| \leqslant \frac{C\varepsilon}{\rho_j \bar{h}_j}, \quad |r_j^+| \leqslant \frac{C\varepsilon}{\rho_{j+1} \bar{h}_j}, \quad |q_j^i| \leqslant C, \qquad \text{if } \rho_j \leqslant 1, \tag{3.8a}$$

$$|r_j^- e(-h_j)| \leqslant C, \qquad |r_j^+ E(-h_j)| \leqslant C, \tag{3.8b}$$

and

$$|r_j^- e(h_{j+1})| \leqslant C, \qquad |r_j^+ E(h_{j+1})| \leqslant C,$$

$$|q_j^i e(\xi_j^i - x_j)| \leqslant C\rho_j, \quad |q_j^i E(\xi_j^i - x_j)| \leqslant C\rho_{j+1}, \qquad \text{if } \rho_j \geqslant 1,$$

for $1 \leqslant i \leqslant 3p-2$.

Proof. For simplicity we transform the interval $[x_{j-1}, x_{j+1}]$ to $[-h_j, h_{j+1}]$. Let p be even (analogously for p odd) and $\xi_j^1 = -h_j$, $\xi_j^{3p/2} = 0, \xi_j^{3p-2} = h_{j+1}$. For simplicity we drop the subscript j. In the proof we use the following notation

$$E_i = E(\xi^i), \qquad e_i = e(\xi^i), \qquad b_i = b^2(\xi^i), \qquad b_i' = b'(\xi^i), \qquad \eta_i = \frac{\xi^i}{h_j}$$

for $1 \leqslant i \leqslant 3p-2$. Then the linear algebraic system for the r's and q's in (3.2) is

$$\begin{aligned}
&r^- + r^c + r^+ - \textstyle\sum q^i b_i = 0,\\
&E_1 r^- + r^c + E_{3p-2} r^+ - \textstyle\sum q^i E_i(-b_i'\varepsilon) = 0,\\
&e_1 r^- + r^c + e_{3p-2} r^+ - \textstyle\sum q^i e_i(b_i'\varepsilon) = 0,\\
&\textstyle\sum q^i = 1,\\
&(-1)^k r^- + \alpha_j^k r^+ - \textstyle\sum q^i(-k(k-1)\rho_j^{-2}\eta_i^{k-2} + b_i\eta_i^k) = 0, \quad k = 1, \dots, p-1, \qquad (3.9a)\\
&(-1)^n \rho_j E_1 r^- + \alpha_j^n \rho_j E_{3p-2} r^+\\
&\quad - \textstyle\sum q^i E_i(-n(n-1)\rho_j^{-1}\eta_i^{n-2} - (2nb_i\eta_i^{n-1} + b_i'\eta_i^{n-1}\xi_i)) = 0, \quad n = 1, \dots, p-1,\\
&(-1)^m \rho_j e_1 r^- + \alpha_j^m \rho_j e_{3p-2} r^+\\
&\quad - \textstyle\sum q^i E_i(-m(m-1)\rho_j^{-1}\eta_i^{m-2} - (2mb_i\eta_i^{m-1} + b_i'\eta_i^{m-1}\xi_i)) = 0, \quad m = 1, \dots, p-1,
\end{aligned}$$

which we can write in the form

$$Qq = b, \tag{3.9b}$$

where

$$q = (r^-, r^c, r^+, q^1, \dots, q^{3p-2})^t, \quad b = (0,0,0,1,0, \dots, 0)^t. \tag{3.9c}$$

We have to prove that this system has a unique solution for any value of $\rho_j, j = 1, \dots, N$. To do this we divide the proof into three parts, according in each case to the size of ρ_j.

Case 1: $\rho_j \to \infty$. From (3.4) we know that $E_i \to \infty$ and $e_i \to 0$, for $i = \frac{3}{2}p + 1, \dots, 3p-2$ and also $e_i \to \infty$ and $E_i \to 0$, for $i = 1, \dots, \frac{3}{2}p - 1$. The technique in [9] now requires us to change variables and to eliminate the unbounded coefficients. But we would then have both positive and negative exponentials. Furthermore, the resulting system would have less equations than unknowns and so its solution would not be unique. It is necessary therefore to analyze this case more carefully.

Using the first three equations of (3.9), it can be shown that

$$r^- = -(T^1(E_{3p-2} - e_{3p-2}) + T^2(e_{3p-2} - 1) + T^3(1 - E_{3p-2}))/D, \tag{3.10a}$$

$$r^+ = -(T^1(e_1 - E_1) + T^2(1 - e_1) + T^3(E_1 - 1))/D, \tag{3.10b}$$

where

$$T^1 = \textstyle\sum q^i b_i, \qquad T^2 = \sum q^i E_i(-b_i'\varepsilon), \qquad T^3 = \sum q^i e_i(b_i'\varepsilon) \tag{3.11a}$$

and

$$D = (E_{3p-2} - 1)(e_1 - 1) + (1 - E_1)(e_{3p-2} - 1). \tag{3.11b}$$

Substituting these values into the other equations we obtain the system

$$\bar{Q}\bar{q} = \bar{b}, \tag{3.12}$$

where

$$\bar{q} = (q^1, \dots, q^{3p-2})^t, \qquad \bar{b} = (1,0, \dots, 0)^t \tag{3.13a}$$

and the matrix $\bar{Q}$ is given by

$$\begin{pmatrix} 1 & \cdots & 1 \\ b_0\eta_1 + O(h_j) + O(\rho_j^{-1}) & \cdots & b_0\eta_{3p-2} + O(h_j) + O(\rho_j^{-1}) \\ \vdots & \vdots & \vdots \\ b_0\eta_1^{p-1} + O(h_j) + O(\rho_j^{-1}) & \cdots & b_0\eta_{3p-2}^{p-1} + O(h_j) + O(\rho_j^{-1}) \\ (E_1 A_{1,1} + B_{1,1})/E_{3p-2} & \cdots & (E_{3p-2}A_{3p-2,1} + B_{3p-2,1})/E_{3p-2} \\ \vdots & \vdots & \vdots \\ (E_1 A_{1,p-1} + B_{1,p-1})/E_{2p} & \cdots & (E_{3p-2}A_{3p-2,p-1} + B_{3p-2,p-1})/E_{2p} \\ (e_1 C_{1,1} + D_{1,1})/e_1 & \cdots & (e_{3p-2}C_{3p-2,1} + D_{3p-2,1})/e_1 \\ \vdots & \vdots & \vdots \\ (e_1 C_{1,p-1} + D_{1,p-1})/e_{p-1} & \cdots & (e_{3p-2}C_{3p-2,p-1} + D_{3p-2,p-1})/e_{p-1} \end{pmatrix} \tag{3.13b}$$

with

$$A_{i,n} = 2nb_0\eta_i^{n-1} + O(h_j) + O(\rho_j^{-1}), \quad n = 1, \dots, p-1,$$

$$B_{i,n} = -\alpha_j^n b_i \rho_j + O(h_j) + O(\rho_j^{-1}), \quad n = 1, \dots, p-1,$$

$$C_{i,m} = 2mb_0\eta_i^{m-1} + O(h_j) + O(\rho_j^{-1}), \quad m = 1, \dots, p-1,$$

$$D_{i,m} = (-1)^{m-1} b_i \rho_j + O(h_j) + O(\rho_j^{-1}), \quad m = 1, \dots, p-1.$$

We now prove that for h_j sufficiently small and ρ_j sufficiently large, we have $\det \bar{Q} \neq 0$. Writing

$$\bar{Q} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix},$$

where Q_{12} is of order p and Q_{23} and Q_{31} are of order $(p-1)$, and using the form of the matrix $\bar{Q}$ given in (3.13b), we see that

$$\det \bar{Q} = \det Q_{12} \det Q_{23} \det Q_{31} + O(h_j) + O(\rho_j^{-1}).$$

Furthermore

$$Q_{12} = Q_{12}^* + L, \qquad Q_{23} = Q_{23}^* + M, \qquad Q_{31} = Q_{31}^* + N,$$

where

$$Q_{12}^* = \begin{pmatrix} 1 & \cdots & 1 \\ b_0\eta_p & \cdots & b_0\eta_{2p-1} \\ \vdots & \ddots & \vdots \\ b_0\eta_p^{p-1} & \cdots & b_0\eta_{2p-1}^{p-1} \end{pmatrix},$$

$$Q_{23}^* = \begin{pmatrix} 2b_0 & \cdots & 2b_0 \\ 4b_0\eta_{2p} & \cdots & 4b_0\eta_{3p-2} \\ \vdots & \ddots & \vdots \\ 2(p-1)b_0\eta_{2p}^{p-2} & \cdots & 2(p-1)b_0\eta_{3p-2}^{p-2} \end{pmatrix},$$

$$Q_{31}^* = \begin{pmatrix} 2b_0 & \cdots & 2b_0 \\ 4b_0\eta_1 & \cdots & 4b_0\eta_{p-1} \\ \vdots & \ddots & \vdots \\ 2(p-1)b_0\eta_1^{p-2} & \cdots & 2(p-1)b_0\eta_{p-1}^{p-2} \end{pmatrix}$$

and $L = \mathrm{O}(h_j) + \mathrm{O}(\rho_j^{-1})$, $M = \mathrm{O}(h_j) + \mathrm{O}(\rho_j^{-1})$, $N = \mathrm{O}(h_j) + \mathrm{O}(\rho_j^{-1})$.

Since Q_{12}^*, Q_{23}^* and Q_{31}^* are Vandermonde matrices, it follows that $\det \bar{Q} \neq 0$, which completes the proof in this case.

Case 2: $\rho_j \to 0$. In this case we have $E_i \to 1$ and $e_i \to 1$, for $i = 1, \ldots, 3p-2$. In the limit the resulting system is consistent but it is also singular. Despite this the coefficients have finite limits. To see this we use the same technique as in [9]. We choose the following set of functions equivalent to (3.2)

$$\{1, \ldots, x^{p-1}, x^p(1+\psi_1), \ldots, x^{2p-1}(1+\psi_p), x^{2p}(1+\phi_1), \ldots, x^{3p-1}(1+\phi_p)\}, \tag{3.14}$$

where

$$\begin{aligned} \psi_1(x) &= \frac{1}{d_p}(d_{p+1}x + d_{p+2}x^2 + \cdots), \\ &\vdots \\ \psi_p(x) &= \frac{1}{d_{2p}}(d_{2p+1}x + d_{2p+2}x^2 + \cdots), \\ \phi_1(x) &= \frac{1}{f_{2p+1}}(f_{2p+2}x + f_{2p+3}x^2 + \cdots), \\ &\vdots \\ \phi_p(x) &= \frac{1}{f_{3p}}(f_{3p+1}x + f_{3p+2}x^2 + \cdots). \end{aligned} \tag{3.15}$$

Here d_i and f_i are the coefficients of the Taylor expansions of $E(x)$ and $e(x)$, respectively. Imposing conditions (3.15) on the system (3.14), the matrix of the linear algebraic system is given in this case by

$$Q = A_1 + M,$$

where $M = \mathrm{O}(\rho_j)$ and A_1 is the matrix associated with the system

$$\{1, x, \dots, x^{3p-1}\}.$$

In [5] Doedel shows that A_1 is nonsingular. It follows that, for ρ_j sufficiently small, Q is also nonsingular.

Case 3: $0 < 1/C \leqslant \rho_j \leqslant C < \infty$, *where* C *is any sufficiently large positive constant.* We make the following change of variables:

$$(\tilde{r}^-, \tilde{r}^c, \tilde{r}^+)^t = \left(\frac{r^-}{h_j}, r^c, \frac{r^+}{h_{j+1}}\right)^t.$$

The resulting system is

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} r^* \\ q^* \end{pmatrix} = \begin{pmatrix} b_1^* \\ b_2^* \end{pmatrix},$$

where

$$r^* = (\tilde{r}^-, \tilde{r}^c, \tilde{r}^+)^t, \qquad q^* = (q^1, \dots, q^{3p-2})^t,$$

$$b_1^* = \left(0, \frac{-b_0}{h_j}, \frac{-b_0}{h_j}\right)^t, \quad b_2^* = (1, \dots, 0)^t.$$

Furthermore, $A_{12} = \mathrm{O}(1)$, $A_{21} = \mathrm{O}(h_j)$, A_{22} is a matrix which is row equivalent to an interpolation matrix for the Chebyshev system

$$\{1, \dots, x^{p-1}, \mathrm{e}^{b_0 x/\varepsilon}, \dots, x^{p-2}, \mathrm{e}^{b_0 x/\varepsilon}, \mathrm{e}^{-b_0 x/\varepsilon}, \dots, x^{p-2}, \mathrm{e}^{-b_0 x/\varepsilon}\}$$

and thus it is nonsingular. Finally we can write

$$A_{11} = \begin{pmatrix} 0 & 1 & 0 \\ E_1 - 1 & 0 & \alpha_j(E_{3p-2} - 1) \\ e_1 - 1 & 0 & \alpha_j(e_{3p-2} - 1) \end{pmatrix} + \mathrm{O}(h_j)$$

and it can be shown easily that A_{11} is also a nonsingular matrix.

We have shown that for all h_j sufficiently small, the matrix Q of system (3.9) is nonsingular as required. We now show that inequalities (3.7) and (3.8) also hold in all three cases. We first establish the bounds on the q_j^i, $i = 1, \dots, 3p-2$. In cases 2 and 3 it is obvious that $|q_j^i| \leqslant C$. We now analyze what happens in case 1. We do this only for q_j^1. The proof is similar for the other quantities.

Examining system (3.9), we see that it suffices to consider only the dominant terms in the corresponding minor of the determinant. It is then a straightforward, though tedious, calculation to show that

$$|q_j^1| \leqslant \frac{C}{e_1}\frac{e_p}{E_{2p}} \leqslant \frac{C}{e_1}$$

using hypothesis (3.6c) on the choice of the auxiliary points ξ_j^i.

We next establish bounds on the coefficients r_j^+, r_j^- and r_j^c. In case 1, using the expressions (3.10) for r_j^- and r_j^+, we easily obtain

$$r_j^- = -\frac{b_0(1+\mathrm{O}(h_j))}{e_1}\left(1+\mathrm{O}\left(\frac{1}{e_1}\right)\right),$$

$$r_j^+ = -\frac{b_0(1+\mathrm{O}(h_j))}{E_{3p-2}}\left(1+\mathrm{O}\left(\frac{1}{E_{3p-2}}\right)\right),$$

and (3.8b) follows. In case 2, using

$$E_i = 1 + b_0\frac{\xi^i}{\varepsilon} + (b_0 + b_0'\varepsilon)\frac{(\xi^i)^2}{\varepsilon^2} + \mathrm{O}(\rho_j^3),$$

$$e_i = 1 - b_0\frac{\xi^i}{\varepsilon} + (b_0 - b_0'\varepsilon)\frac{(\xi^i)^2}{\varepsilon^2} + \mathrm{O}(\rho_j^3),$$

we deduce that

$$r^- = -\frac{\varepsilon}{\rho_j\bar{h}_j}(1+\mathrm{O}(h_j))(1+\mathrm{O}(\rho_j)),$$

$$r^+ = -\frac{\varepsilon}{\rho_{j+1}\bar{h}_j}(1+\mathrm{O}(h_{j+1}))(1+\mathrm{O}(\rho_{j+1})),$$

and (3.8a) follows. Finally, in case 3, from

$$E(x) = \mathrm{e}^{b_0x/\varepsilon}(1+\mathrm{O}(x)), \qquad e(x) = \mathrm{e}^{-b_0x/\varepsilon}(1+\mathrm{O}(x)),$$

we obtain

$$r^- = -b_0(1+\mathrm{O}(h_j))\frac{\mathrm{e}^{b_0\rho_{j+1}} - \mathrm{e}^{-b_0\rho_{j+1}}}{D_1},$$

$$r^+ = -b_0(1+\mathrm{O}(h_j))\frac{\mathrm{e}^{b_0\rho_j} - \mathrm{e}^{-b_0\rho_j}}{D_1},$$

where

$$D_1 = (\mathrm{e}^{b_0\rho_{j+1}} - 1)(\mathrm{e}^{b_0\rho_j} - 1) + (1 - \mathrm{e}^{-b_0\rho_j})(\mathrm{e}^{-b_0\rho_{j+1}} - 1) > 0,$$

which gives the required results in this case.

Lastly,

$$r^- + r^c + r^+ = \sum q^i b_i = \sum q^i b_0(1+\mathrm{O}(h_j)) > 0,$$

$$r^- + r^c + r^+ = \sum q^i b_i \leqslant \|b\|_\infty \sum q^i \leqslant C,$$

and (3.7a) follows. This completes the proof of Theorem 3.2. $\square$.

4. Uniform convergence of the finite difference scheme

In the previous section we proved that the finite difference scheme (3.2) has a unique solution. We prove now that it is ε-uniformly convergent in the discrete L^∞ norm (1.3). To do this, we use the ε-uniform accuracy and the ε-uniform stability of the method.

In the following definitions u denotes the solution of the boundary problem (1.1) and U the solution of the finite difference scheme (3.2).

Definition 4.1. A finite difference operator $L_{h,\varepsilon}$ determined by the coefficients r_j^-, r_j^c and r_j^+, is of *positive type* if

$$r_j^- < 0, \qquad r_j^+ < 0, \qquad r_j^- + r_j^c + r_j^+ \geqslant 0.$$

Definition 4.2. A finite difference operator $L_{h,\varepsilon}$ satisfies a *discrete maximum principle* if the inequalities $U_0 \geqslant 0$, $U_N \geqslant 0$ and $L_{h,\varepsilon}(U_h)_j \geqslant 0$, $1 \leqslant j \leqslant N-1$, imply that $U_j \geqslant 0$, $0 \leqslant j \leqslant N$.

Definition 4.3. Let τ denote the local truncation error for a finite difference operator $L_{h,\varepsilon}$. Then $L_{h,\varepsilon}$ is ε-*uniformly accurate* of order p with respect to the discrete norm $\| \, \|^\star$, if there are some constants C_1 and H_0 independent of h and ε such that for all $0 < H < H_0$

$$\|\tau\|^\star \leqslant C_1 H^p.$$

Definition 4.4. A finite difference operator $L_{h,\varepsilon}$ is ε-*uniformly stable* with respect to the discrete norms $\| \, \|$ and $\| \, \|^\star$, if there is a constant C_2 independent of ε such that

$$\|v_h\| \leqslant C_2 \| L_{h,\varepsilon}(v_h) \|^\star$$

for all mesh functions v_h in the domain of $L_{h,\varepsilon}$.

Theorem 4.5. *Let $\| \, \|$ and $\| \, \|^\star$ be two discrete norms. Let $L_{h,\varepsilon}$ be a finite difference operator which is ε-uniformly stable with respect to those norms and ε-uniformly accurate of order p with respect to the $\| \, \|^\star$ norm. Then, the finite difference scheme*

$$L_{h,\varepsilon} U_h = f_h$$

is ε-uniformly convergent of order p with respect to the $\| \, \|$ norm.

Proof. The proof is an obvious modification of the standard proof that accuracy of order p and stability are sufficient conditions for convergence of order p. □

We know that the local truncation error for the finite difference operator (3.2) at the node x_j is given by

$$\tau_j = L_{h,\varepsilon}(u(x_j,\varepsilon)) - \sum_{i=1}^{3p-2} q_j^i L_\varepsilon(u(x_j,\varepsilon)).$$

We establish in the following lemma that τ_j is ε-uniformly accurate of order $p-1$ in the discrete L^∞ norm.

Lemma 4.6. *Let $p \geqslant 2$ be a positive integer and $u(x, \varepsilon)$ the theoretical solution of* (1.1). *Then the local truncation error satisfies*

$$|\tau_j| \leqslant C h_j^{p-1} \max\{\varepsilon, h_j\} \left(1 + \exp\left(\frac{-1}{\varepsilon}\int_0^{x_j} b(t)\mathrm{d}t\right) + \exp\left(\frac{-1}{\varepsilon}\int_{x_j}^{1} b(t)\mathrm{d}t\right)\right), \tag{4.1}$$

where C is a positive constant independent of ε and the mesh parameter.

Proof. The decomposition (2.1) of the solution $u(x, \varepsilon)$ of problem (1.1) and the choice of the set of functions (3.3), (3.4) permit us to write

$$\begin{aligned} |\tau_j| \leqslant\, & C|\tau_j[(x - x_j)^p]| \\ & + C\exp\left(\frac{-1}{\varepsilon}\int_0^{x_j} b(t)\mathrm{d}t\right)\left|\tau_j\left((x - x_j)^p \exp\left(\frac{-1}{\varepsilon}\int_{x_j}^{x} b(t)\mathrm{d}t\right)\right)\right| \\ & + C\exp\left(\frac{-1}{\varepsilon}\int_{x_j}^{1} b(t)\mathrm{d}t\right)\left|\tau_j\left((x - x_j)^p \exp\left(\frac{1}{\varepsilon}\int_{x_j}^{x} b(t)\mathrm{d}t\right)\right)\right|. \end{aligned} \tag{4.2}$$

We now examine the local truncation error for each of the three terms on the right-hand side. We distinguish two cases depending on ρ_j, and we transform the interval $[x_{j-1}, x_{j+1}]$ to $[-h_j, h_{j+1}]$.

Case A: $\rho_j \leqslant 1$. A direct examination of the local truncation error will not yield inequality (4.1), so we must proceed in a more careful way. Using Taylor's expansion for the exponential function, we have

$$E(x) = 1 + d_1 x + d_2 x^2 + \cdots + d_p x^p + d'_{p+1} x^{p+1}$$

with $d_i = \mathrm{O}(\varepsilon^{-i})$, $i = 1, \ldots, p$ and $d'_{p+1} = \mathrm{O}(\varepsilon^{-(p+1)})$. It follows that

$$x^p = \frac{1}{d_p}[E(x) - (1 + d_1 x + \cdots + d_{p-1}x^{p-1} + d'_{p+1}x^{p+1})]$$

and from the construction of the finite difference scheme we deduce that

$$\begin{aligned} |\tau_j(x^p)| &\leqslant \frac{|d'_{p+1}|}{|d_p|}|\tau_j(x^{p+1})| \leqslant \frac{C}{\varepsilon}[h_j^{p+1}|r_j^-| + h_{j+1}^{p+1}|r_j^+|] \\ &\quad + C\sum|q_j^i|\,[\varepsilon^2|\xi^i|^{p-1} + |\xi^i|^{p+1}] \\ &\leqslant C h_j^{p-1}\varepsilon, \end{aligned} \tag{4.3a}$$

where we have used the bounds in (3.7a). Then it is easy to see that

$$x^p(E(x) + e(x)) = 2x^p + d'_1 x^{p+1}$$

and so

$$|\tau_j[x^p(E(x) + e(x))]| \leqslant |\tau_j(x^p)| \leqslant C h_j^{p-1}\varepsilon. \tag{4.3b}$$

Case B: $\rho_j \geqslant 1$. In this case using (3.8b) it follows that

$$\begin{aligned} |\tau_j(x^p)| &\leqslant [h_j^p|r_j^-| + h_{j+1}^p|r_j^+|] + C\sum|q_j^i|\,[\varepsilon^2|\xi^i|^{p-2} + |\xi^i|^p] \\ &\leqslant C h_j^p \end{aligned} \tag{4.4a}$$

and by using $\rho_j \geqslant 1$

$$
\begin{aligned}
&|\tau_j[x^p(E(x) + e(x))]| \\
&\quad \leqslant [h_j^p(E_1 + e_1)|r_j^-| + h_{j+1}^p(E_{3p-2} + e_{3p-2})|r_j^+|] \\
&\qquad + C\textstyle\sum(E_i + e_i)|q_j^i|[\varepsilon^2|\xi^i|^{p-2} + \varepsilon(|\xi^i|^{p-1} + |\xi^i|^p)] \\
&\quad \leqslant Ch_j^p.
\end{aligned} \tag{4.4b}
$$

Then (4.2)–(4.4) give the required result. □

Remark 4.7. We note that when $p = 1$, the last theorem does not imply the ε-uniform accuracy of the difference scheme. In this case, however, we compute the coefficients of the method explicitly and use them to prove the result [4].

Using the inequalities of Theorem 3.2 the following lemma is immediate.

Lemma 4.8. *Assuming the hypotheses of Theorem* 3.2, *the finite difference scheme* (3.2) (i) *is of positive type,* (ii) *satisfies a discrete maximum principle and* (iii) *is ε-uniformly stable in the discrete L^∞ norm.*

Theorem 4.9. *Let $u(x, \varepsilon)$ be the exact solution of problem* (1.1) *and u_h its restriction to the mesh. Let $U_h = (U_0, \dots, U_N)^t \in \mathbb{R}^{N+1}$ be the solution of the finite difference scheme* (3.2). *Then, there exists a positive constant C, independent of ε and the mesh parameter, such that for $p \geqslant 2$*

$$\|u_h - U_h\| \leqslant CH^{p-1}\max\{\varepsilon, H\} \tag{4.5}$$

and so the difference scheme is ε-uniformly convergent.

Proof. This follows at once from Lemma 4.6 and Lemma 4.8. □

Remark 4.10. In practice, we normally take $\varepsilon \leqslant H$ and the ε-uniform convergence of order p of the difference method is given by (4.5).

An alternative approach to the proof of ε-uniform convergence is the comparison function technique [14]. This approach involves using the comparison functions

$$\phi_{1,j}(\beta) = 1, \qquad \phi_{2,j}(\beta) = \mathrm{e}^{-\beta x_j/\varepsilon}, \qquad \phi_{3,j}(\beta) = \mathrm{e}^{\beta x_j/\varepsilon},$$

where β is a positive constant, satisfying $\beta < \min_x b^2(x)$. We then obtain estimates of $L_{h,\varepsilon}(\phi_{k,j}(\beta))$, for $k = 1, 2, 3$, using the discrete maximum principle for the difference operator $L_{h,\varepsilon}$ and we can deduce that

$$|u(x_j, \varepsilon) - U_j| \leqslant Ch_j^{p-1}\max\{\varepsilon, h_j\}\left(1 + \exp\left(\frac{-1}{\varepsilon}\int_0^{x_j} b(t)\,\mathrm{d}t\right)\right) + \exp\left(-\frac{1}{\varepsilon}\int_{x_j}^1 b(t)\,\mathrm{d}t\right). \tag{4.6}$$

5. Numerical examples

In this section we present numerical solutions of several problems, using the finite difference methods described in previous sections. The theoretical order p of ε-uniform convergence depends on both the problem and the chosen difference method. The computations show that in all cases the numerical order of ε-uniform convergence is in good agreement with the theoretical predictions.

We determine the numerical order of uniform convergence in the following way. If we know the exact solution u_ε, we denote by E_N the maximum nodal error where

$$E_N = \max_j |u_\varepsilon(x_j) - U_j|, \quad j = 0, \dots, N.$$

We then define the numerical order $p^\star$, using two successive values E_N and E_{2N}, by

$$p^\star = (\log(E_N) - \log(E_{2N}))/\log 2. \tag{5.1}$$

On the other hand, if we do now know the exact solution, we define the numerical order $p^\star$ by

$$p^\star = (\log(E_N^\star) - \log(E_{2N}^\star))/\log 2, \tag{5.2}$$

where

$$E_N^\star = \max_j |u_\varepsilon^\star(x_j) - U_j|, \quad j = 0, \dots, N$$

and, for each fixed ε, $u_\varepsilon^\star$ is the numerical solution on the finest available mesh.

In all of the examples we begin the integration with an initial mesh. We then modify this mesh, using the algorithm of equidistribution of arclength with piecewise linear interpolation of the initial discrete solution [4, 23]. Similar results may be obtained using the equidistribution of other quantities involving derivatives of the solution.

In the tables we show the maximum nodal error for each mesh and the numerical order of ε-uniform convergence for different values of the parameter p. The computational cost increases with p and so, in practice, we should use the method for small p only.

In Problem 5.3 we do not know the exact solution and, for a non uniform mesh, linear interpolation is used to determine the approximate error $E_N^\star$. This is not satisfactory, however, because the accuracy of this linear interpolation is less than the accuracy of the method, and overall accuracy of the method is thus reduced unnecessarily. For this reason we give the numerical results for this problem only on uniform meshes. The finest available mesh in this case has 160 points in the integration interval.

Problem 5.1 (Herceg [12]).

$$-\varepsilon^2 u''(x) + u(x) = -\cos^2(\pi x) - 2(\varepsilon\pi)^2 \cos(2\pi x),$$

$$u(0) = 0, \qquad u(1) = 0.$$

The exact solution is

$$u_\varepsilon(x) = (\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon))/(1 + \exp(-1/\varepsilon)) - \cos^2(\pi x).$$

Problem 5.2 (Clavero [4]).

$$-\varepsilon^2 u''(x) + u(x) = -72\varepsilon^2 x^7 + x^9,$$
$$u(-1) = 0, \qquad u(1) = 0.$$

The exact solution is

$$u_\varepsilon(x) = (\exp(-(1+x)/\varepsilon) - \exp(-(1-x)/\varepsilon))/(1-\exp(-2/\varepsilon)) + x^9.$$

Problem 5.3.

$$-\varepsilon^2 u''(x) + \frac{1}{1+x^2} u(x) = x^2 + 1,$$
$$u(0) = 1, \qquad u(1) = 0.$$

If we compare the results in Tables 1–3 with the numerical results given in [12] we see that the value of the maximum error in all cases is less using our method, and that we do not need to use

Table 1
Maximum nodal error E_N and numerical order p^* for Problem 5.1

$p = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	Average
$\varepsilon^2 = 2^{-4}$	0.630E − 10	0.254E − 12	0.247E − 14	0.185E − 14	5.017
		7.950	6.687	0.415	
$\varepsilon^2 = 2^{-8}$	0.550E − 7	0.818E − 9	0.666E − 11	0.233E − 13	7.056
		6.071	6.940	8.156	
$\varepsilon^2 = 2^{-12}$	0.129E − 4	0.497E − 6	0.639E − 8	0.323E − 9	5.096
		4.705	6.280	4.305	
$\varepsilon^2 = 2^{-16}$	0.358E − 5	0.140E − 5	0.434E − 6	0.476E − 7	2.077
		1.351	1.692	3.189	
$\varepsilon^2 = 2^{-20}$	0.188E − 4	0.101E − 6	0.399E − 7	0.138E − 7	3.470
		7.540	1.339	1.532	

Table 2
Maximum nodal error E_N and numerical order p^* for Problem 5.2

$p = 3$	$N = 25$	$N = 50$	$N = 100$	$N = 200$	Average
$\varepsilon^2 = 10^{-1}$	0.881E − 10	0.32E − 12	0.251E − 14	0.111E − 14	5.442
		8.103	6.992	1.1712	
$\varepsilon^2 = 10^{-2}$	0.120E − 9	0.396E − 12	0.199E − 14	0.108E − 16	7.801
		8.25	7.63	7.525	
$\varepsilon^2 = 10^{-3}$	0.701E − 6	0.559E − 8	0.704E − 10	0.144E − 12	7.404
		6.969	6.311	8.933	
$\varepsilon^2 = 10^{-4}$	0.407E − 4	0.168E − 5	0.232E − 7	0.165E − 9	5.969
		4.597	6.176	7.135	
$\varepsilon^2 = 10^{-5}$	0.102E − 5	0.246E − 5	0.397E − 6	0.140E − 8	3.171
		2.056	2.632	4.825	

Table 3
Maximum nodal error E_N and numerical order p^* for Problem 5.3

$p = 3$	$N = 10$	$N = 20$	$N = 40$	$N = 80$	Average
$\varepsilon^2 = 10^{-1}$	0.412E − 11	0.723E − 13 5.832	0.115E − 14 5.975	0.177E − 16 6.016	5.941
$\varepsilon^2 = 10^{-2}$	0.227E − 9	0.103E − 11 7.776	0.531E − 14 7.611	0.341E − 16 7.086	7.491
$\varepsilon^2 = 10^{-3}$	0.158E − 7	0.103E − 9 7.269	0.467E − 12 7.785	0.188E − 14 7.949	7.667
$\varepsilon^2 = 10^{-4}$	0.115E − 6	0.322E − 8 5.167	0.315E − 10 6.675	0.170E − 12 7.526	6.456
$\varepsilon^2 = 10^{-5}$	0.207E − 7	0.744E − 8 1.475	0.404E − 9 4.201	0.941E − 11 5.425	3.700

a special mesh to obtain good results in the boundary layers. Better results, for the maximum error and for the numerical order of uniform convergence, have been obtained using more points in the integration mesh, in particular for $N = 25, 50, 100, 200$.

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