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## Spectral problems with mixed Dirichlet–Neumann boundary conditions: Isospectrality and beyond<sup>☆</sup>

Dmitry Jakobson<sup>a</sup>, Michael Levitin<sup>b,\*</sup>, Nikolai Nadirashvili<sup>c</sup>, Iosif Polterovich<sup>d</sup>

<sup>a</sup>*Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montreal, Que., Canada H3A 2K6*

<sup>b</sup>*Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK*

<sup>c</sup>*CNRS Laboratoire d'Analyse, Topologie, Probabilités UMR 6632, Centre de Mathématiques et Informatique, Université de Provence, 39 rue F. Joliot-Curie, 13453 Marseille cedex 13, France*

<sup>d</sup>*Dépt. de mathématiques et de statistique, Université de Montréal, CP 6128 succ Centre-Ville, Montreal, Canada, QC H3C 3J7*

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### Abstract

Consider a bounded domain with the Dirichlet condition on a part of the boundary and the Neumann condition on its complement. Does the spectrum of the Laplacian determine uniquely which condition is imposed on which part? We present some results, conjectures and problems related to this variation on the isospectral theme.

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\* Corresponding author.

*E-mail addresses:* [jakobson@math.mcgill.ca](mailto:jakobson@math.mcgill.ca) (D. Jakobson), [m.levitin@ma.hw.ac.uk](mailto:m.levitin@ma.hw.ac.uk) (M. Levitin), [nicholas@cmi.univ-mrs.fr](mailto:nicholas@cmi.univ-mrs.fr) (N. Nadirashvili), [iossif@dms.umontreal.ca](mailto:iossif@dms.umontreal.ca) (I. Polterovich).

## 1. Introduction

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain, its piecewise smooth boundary being decomposed as  $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$ , where  $\partial_1\Omega, \partial_2\Omega$  are finite unions of open segments of  $\partial\Omega$  and  $\partial_1\Omega \cap \partial_2\Omega = \emptyset$ . Suppose that there are no isometries of  $\mathbb{R}^2$  exchanging  $\partial_1\Omega$  and  $\partial_2\Omega$ . We call such a decomposition of the boundary *nontrivial*. Consider a Laplace operator on  $\Omega$  and assume that on one part of the boundary we have the Dirichlet condition and on the other part the Neumann condition. (Such a boundary value problem is sometimes called a *Zaremba problem* [24].) Does the spectrum of the Laplacian determine uniquely which condition is imposed on which part?

We recall the classical question of Mark Kac, “*Can one hear the shape of a drum?*” [17] related to the (Dirichlet) Laplacian on the plane, which still remains open for smooth (as well as for convex) domains. For arbitrary planar domains it was answered negatively in [14] using an algebraic construction of [22]; see also reviews and extensions [6,4,5] and references therein. Some related numerics can be found in [10].

We may reformulate our question in a similar way. Consider two identical drums with drumheads which are partially attached to them. The drumhead of the first drum is attached exactly where the drumhead of the second drum is free and vice versa. Can one distinguish between the two drums by hearing them?

Similarly to the question of Kac, the answer to our question is in general negative. In this note we construct a family of domains, each of them having a nontrivial isospectral (with respect to exchanging the Neumann and Dirichlet boundary conditions) boundary decomposition. We say that such domains *admit Dirichlet–Neumann isospectrality*.

Our main example is a half-disk which is considered in Sections 2.1–2.4. In Sections 3.1 and 3.2 we construct generalisations of the main example, in particular to nonplanar domains. Section 3.3 provides a simple necessary condition for a boundary decomposition to be Dirichlet–Neumann isospectral. In Section 3.4 we discuss if there exist domains not admitting Dirichlet–Neumann isospectrality and conjecture that a disk should be one of them. Some numerical evidence in favour of this conjecture is also presented.

Our motivation for the study of Dirichlet–Neumann isospectrality, rather surprisingly, comes from a seemingly unrelated problem of obtaining a sharp upper bound for the first eigenvalue on a surface of genus two. We discuss it, as well as some other relevant eigenvalue inequalities, in Sections 4.1 and 4.2.

## 2. Domains isospectral with respect to the Dirichlet–Neumann swap

### 2.1. Main example

Our principal example is constructed using the half-disk. Let  $\Omega := \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$  be an upper half of a disc centred at the point  $O$  (here and further on we shall often identify the real plane  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$  and shall use the complex variable  $z = x + iy$  instead of real coordinates  $(x, y)$  on the plane). Consider the following boundary decomposition:

$$\partial_1\Omega = \{\operatorname{Re} z \in (-1, 0), \operatorname{Im} z = 0\} \cup \{|z| = 1, |\arg z - \pi/2| < \pi/4\}, \quad (2.1.1)$$

$$\partial_2\Omega = \{\operatorname{Re} z \in (0, 1), \operatorname{Im} z = 0\} \cup \{|z| = 1, \pi/4 < |\arg z - \pi/2| < \pi/2\}. \quad (2.1.2)$$

Obviously, such a decomposition is nontrivial.

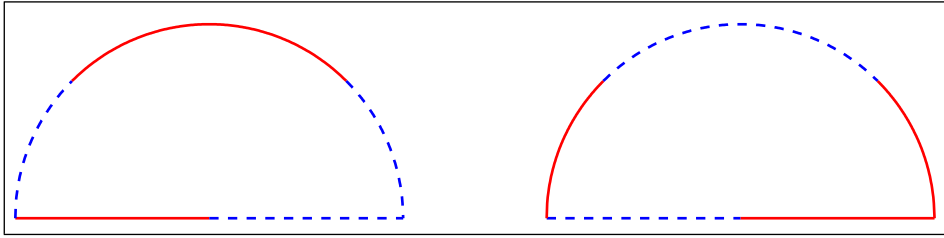


Fig. 1. Problems I and II on the half-disk. Here and further on, a red solid line denotes the Dirichlet boundary conditions and a blue dashed line the Neumann ones.

Consider the following boundary value spectral problems on  $\Omega$ :

**Problem I.**

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u|_{\partial_1 \Omega} = 0, \quad \left. \frac{\partial u}{\partial n} \right|_{\partial_2 \Omega} = 0,$$

and

**Problem II.**

$$-\Delta v = \lambda v \quad \text{in } \Omega, \quad v|_{\partial_2 \Omega} = 0, \quad \left. \frac{\partial v}{\partial n} \right|_{\partial_1 \Omega} = 0$$

(see Fig. 1). Here after  $\partial/\partial n$  is the normal derivative and  $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$  is the Laplace operator.

Let  $\sigma_I$  denote the spectrum of Problem I and  $\sigma_{II}$  the spectrum of Problem II. Both spectra are discrete and positive. Our main claim is the following

**Theorem 2.1.3.** *With account of multiplicities,  $\sigma_I \equiv \sigma_{II}$ .*

We give three different proofs of Theorem 2.1.3, each of them, in our opinion, instructive in its own right, and generalise this theorem later on for a certain class of examples.

## 2.2. Proof of Theorem 2.1.3 by transplantation

This proof uses the transplantation trick similar to [2,6]. Let  $u(z) = u(r, \phi)$  be an eigenfunction of Problem I corresponding to an eigenvalue  $\lambda$ ; here  $(r, \phi)$  denote the usual polar coordinates. Let us introduce a mapping  $T : u \mapsto v$ , where

$$\begin{aligned} v(z) &= (Tu)(z) \\ &:= \frac{1}{\sqrt{2}} \begin{cases} u(r, \frac{\pi}{2} - \phi) - u(r, \frac{\pi}{2} + \phi) & \text{if } \phi = \arg z \in (0, \frac{\pi}{2}], \\ u(r, \frac{3\pi}{2} - \phi) + u(r, \phi - \frac{\pi}{2}) & \text{if } \phi = \arg z \in [\frac{\pi}{2}, \pi). \end{cases} \end{aligned}$$

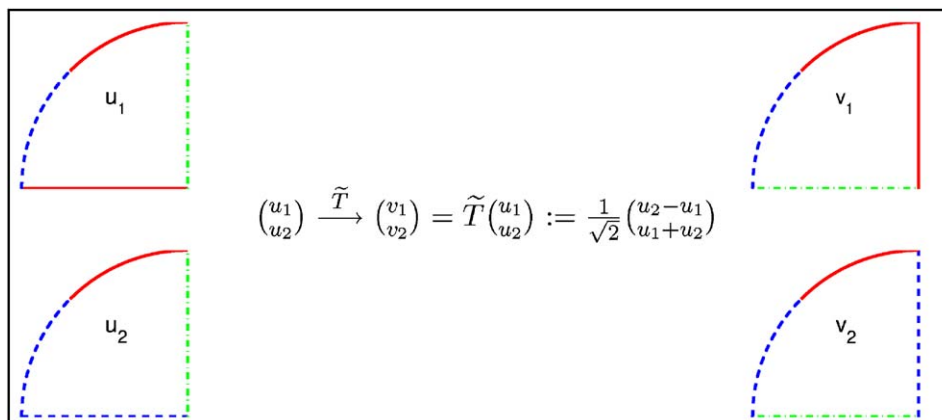


Fig. 2. Problems I and II on the quarter-disk and the map  $\tilde{T}$ . Here and further on, a green dash-dotted line denotes the matching conditions.

Then it is easily checked that  $v(z)$  is an eigenfunction of Problem II: it satisfies the equation and the boundary conditions as well as the matching conditions for the trace of the function and the trace of the normal derivative on the central symmetry line  $r \in (0, 1)$ ,  $\phi = \pi/2$ .

Similarly, if  $v(z)$  is an eigenfunction of Problem II, in order to construct an eigenfunction  $u(z)$  of Problem I we use an inverse mapping  $T^{-1}$  (one may check that  $T^8 = \text{Id}$  and hence  $T^{-1} = T^7$ ):

$$u(z) = (T^{-1}v)(z) \\ := \frac{1}{\sqrt{2}} \begin{cases} v(r, \frac{\pi}{2} - \phi) + v(r, \frac{\pi}{2} + \phi) & \text{if } \phi = \arg z \in (0, \frac{\pi}{2}], \\ v(r, \frac{3\pi}{2} - \phi) - v(r, \phi - \frac{\pi}{2}) & \text{if } \phi = \arg z \in [\frac{\pi}{2}, \pi). \end{cases}$$

This proves that the sets  $\sigma_I$  and  $\sigma_{II}$  coincide. The equality of multiplicities for each eigenvalue follows immediately from the linearity of the map  $T$ .  $\square$

It is easy to visualise the mapping  $T$  in the following way. Cutting a half-disk along the symmetry line, we can rewrite Problem I as a system of two boundary value problems with respect to functions  $u_1, u_2$  on a quarter-disk  $\mathcal{Y} := \{z \in \mathbb{C} : |z| < 1, \text{Im } z > 0, \text{Re } z < 0\}$ ;  $u_1, u_2$  should satisfy the matching conditions ( $u_1 = u_2, \partial u_1 / \partial n = -\partial u_2 / \partial n$ ) on the line  $\arg z = \pi/2$ , see Fig. 2. We call this equivalent statement Problem I. The map  $\tilde{T}$  shown in Fig. 2 maps the eigenfunction  $(u_1, u_2)$  of Problem I into an eigenfunction  $(v_1, v_2)$  of Problem II, which is in turn equivalent to Problem II (it is obtained by rotating a half-disk in Problem II clockwise by  $\pi/2$  and then cutting along the symmetry line).

The possibility of transplanting the eigenfunctions indicates that our problem on a half-disk has a “hidden” symmetry. An attempt to unveil it is presented in the next section.

### 2.3. Proof of Theorem 2.1.3 using a branched double covering of the disk

Consider an auxiliary spectral problem for the Laplacian on the branched double covering  $\mathbb{D}$  of the unit disk  $D$ , with alternating Dirichlet and Neumann boundary conditions on quarter-circle arcs, see Fig. 3. Let  $(r, \phi)$  and  $(r, \theta)$  denote the polar coordinates on  $\mathbb{D}$  and  $D$ , respectively, with  $r \in (0, 1]$ ,

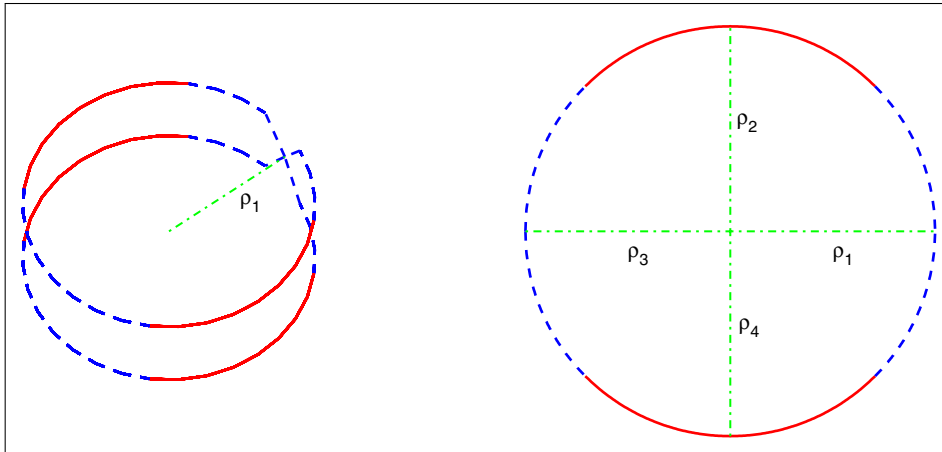


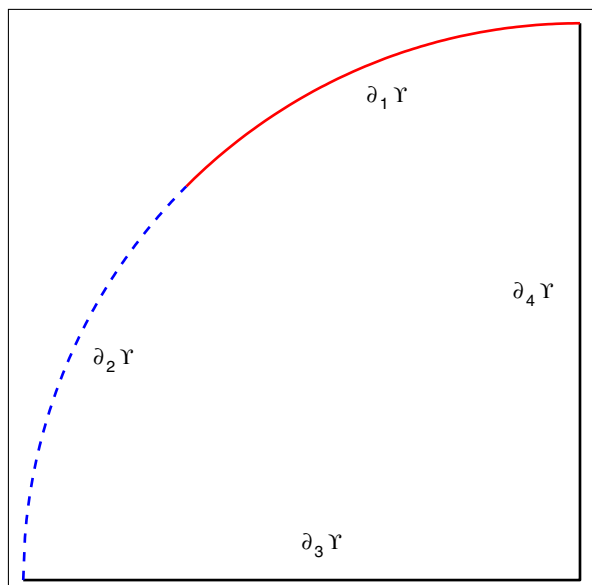
Fig. 3. The problem on the double covering  $\mathbb{D}$  (left) of the disk  $D$  (right), and the radii on  $D$  which lift to the lines of symmetry for this problem on  $\mathbb{D}$ .

$\phi \in [0, 4\pi)$ ,  $\theta = \phi \pmod{2\pi} \in [0, 2\pi)$ . The metric on  $\mathbb{D} \setminus \{O\}$  is a pull-back of the Euclidean metric from  $D$ , where  $O = (0, 0)$  is the branch point of the covering. Though  $\mathbb{D}$  has a conical singularity at  $O$ , eigenvalues and eigenfunctions on  $\mathbb{D}$  are well defined using the variational formulation. Consider three symmetries of  $\mathbb{D}$ :  $U : (r, \phi) \rightarrow (r, 4\pi - \phi)$ ,  $T : (r, \phi) \rightarrow (r, (\phi + 2\pi) \pmod{4\pi})$  and  $V : (r, \phi) \rightarrow (r, (2\pi - \phi) \pmod{4\pi})$ . These symmetries are involutions, they commute with each other, and satisfy  $V = U \circ T$ . Symmetries  $U$  and  $V$  are axial symmetries, and  $T$  is an intertwining of sheets of  $\mathbb{D}$ . By the spectral theorem we find a basis of eigenfunctions that are either even or odd with respect to  $T$ ,  $U$  and  $V$ . Consider a space  $E_-$  of eigenfunctions on  $\mathbb{D}$  that are *odd* with respect to  $T$  and the corresponding spectrum  $\sigma_-(\mathbb{D})$ . We have  $E_- = E_-^{+, -} \cup E_-^{-, +}$ , where  $E_-^{+, -}$  is a subspace of eigenfunctions that are even with respect to  $U$  and odd with respect to  $V$ , and  $E_-^{-, +}$  is a subspace of eigenfunctions that are odd with respect to  $U$  and even with respect to  $V$ . Denote  $F_U = \{\phi = 0\} \cup \{\phi = 2\pi\}$  and  $F_V = \{\phi = \pi\} \cup \{\phi = 3\pi\}$  the fixed point sets of  $U$  and  $V$ . Any  $f \in E_-^{+, -}$  (respectively,  $f \in E_-^{-, +}$ ) satisfies Neumann (respectively, Dirichlet) condition on  $F_U$  and Dirichlet (respectively, Neumann) condition on  $F_V$ .

Choose a coordinate system on  $\mathbb{D}$  in such a way that  $\theta=0$  corresponds to the radius  $\rho_1 \subset D$  in Fig. 3. For any eigenfunction  $f \in E_-^{+, -}$  consider its restriction on the “upper” part  $\tilde{\mathbb{D}} = \{(r, \phi) \mid 0 < r < 1, 0 \leq \phi < 2\pi\}$ . Then  $f|_{\tilde{\mathbb{D}}}$  projects to an eigenfunction of our boundary problem on a disk  $D$  with a cut along a diameter  $\rho_1 \cup \rho_3$ : on  $\rho_1$  it satisfies the Neumann condition and on  $\rho_3$  it satisfies the Dirichlet condition. Similarly, any eigenfunction  $f \in E_-^{-, +}$  can be projected from  $\tilde{\mathbb{D}}$  to an eigenfunction of our boundary problem on a disk  $D$  with the same cut, but now it satisfies Dirichlet condition on  $\rho_1$  and Neumann condition on  $\rho_3$ . In either case, we obtain an eigenfunction of Problem I. Hence,  $\sigma_-(\mathbb{D})$  equals  $\sigma_{II}$  with doubled multiplicities.

Now, let us choose the coordinate system differently so that  $\theta=0$  corresponds to the radius  $\rho_2$ . Arguing in exactly the same way as above we obtain that  $\sigma_-(\mathbb{D})$  equals  $\sigma_{II}$  with doubled multiplicities. Therefore,  $\sigma_I = \sigma_{II}$  with account of multiplicities which completes the proof of the theorem.  $\square$

**Remark.** The construction of a “common” covering for Problems I and II described above can be viewed as an application of Sunada’s approach [22] to mixed Dirichlet–Neumann problems.

Fig. 4. The quarter-disk  $\Upsilon$ .

#### 2.4. Proof of Theorem 2.1.3 by Dirichlet–Neumann-type mappings

For those who prefer operator theory to geometric constructions we give yet another proof of the main theorem. Consider the following auxiliary problem. Let  $\Upsilon$  be a quarter-disk introduced in Section 2.2. Denote  $\partial_1 \Upsilon = \{|z| = 1, \arg z \in (\pi/2, 3\pi/4)\}$ ,  $\partial_2 \Upsilon = \{|z| = 1, \arg z \in (3\pi/4, \pi)\}$ ,  $\partial_3 \Upsilon = \{\operatorname{Re} z \in (-1, 0), \operatorname{Im} z = 0\}$ ,  $\partial_4 \Upsilon = \{\operatorname{Re} z = 0, \operatorname{Im} z \in (0, 1)\}$ , so that  $\partial \Upsilon = \partial_1 \Upsilon \cup \partial_2 \Upsilon \cup \partial_3 \Upsilon \cup \partial_4 \Upsilon$ , see Fig. 4.

Let, for a given  $\lambda \in \mathbb{R}$ ,  $w(z)$  satisfy the equation

$$-\Delta w = \lambda w \quad \text{in } \Upsilon \quad (2.4.1)$$

and the boundary conditions

$$w|_{\partial_1 \Upsilon} = 0, \quad \frac{\partial w}{\partial n} \Big|_{\partial_2 \Upsilon} = 0 \quad (2.4.2)$$

(we do not impose at the moment any boundary conditions on  $w$  on  $\partial_{3,4} \Upsilon$ ). Denote  $\xi = w|_{\partial_4 \Upsilon}$ ,  $\eta = \partial w / \partial n|_{\partial_4 \Upsilon}$ ,  $p = w|_{\partial_3 \Upsilon}$ ,  $q = \partial w / \partial n|_{\partial_3 \Upsilon}$ . Consider four linear operators which depend on  $\lambda$  as a parameter:

$$(\mathcal{D}\mathcal{D})_\lambda : \xi \mapsto p, \quad \text{subject to } q = 0, \quad (2.4.3)$$

$$(\mathcal{D}\mathcal{N})_\lambda : \xi \mapsto q, \quad \text{subject to } p = 0, \quad (2.4.4)$$

$$(\mathcal{N}\mathcal{D})_\lambda : \eta \mapsto p, \quad \text{subject to } q = 0, \quad (2.4.5)$$

$$(\mathcal{N}\mathcal{N})_\lambda : \eta \mapsto q, \quad \text{subject to } p = 0. \quad (2.4.6)$$

These operators acting on the radius  $\partial_4 \Upsilon$  are well defined as long as  $\lambda$  does not belong to the spectra of any of the four homogeneous boundary value problems (2.4.1), (2.4.2) with Dirichlet or Neumann boundary

conditions imposed on  $\partial_4\mathcal{Y}$  and  $\partial_3\mathcal{Y}$  (cf. [13]). Consider an operator  $\mathcal{C}_\lambda := (\mathcal{D}\mathcal{D})_\lambda^{-1}(\mathcal{N}\mathcal{D})_\lambda(\mathcal{N}\mathcal{N})_\lambda^{-1}(\mathcal{D}\mathcal{N})_\lambda$ . Theorem 2.1.3 then follows from

**Proposition 2.4.7.**

$$\lambda \in \sigma_I \iff \mu = -1 \text{ is an eigenvalue of } \mathcal{C}_\lambda \iff \lambda \in \sigma_{II}.$$

To prove Proposition 2.4.7, let us now return to our original Problem I, or, more precisely, to its equivalent formulation Problem  $\tilde{I}$  described at the end of Section 2.2 and illustrated on the left-hand side of Fig. 2. Assume that an eigenvalue  $\lambda$  of Problem  $\tilde{I}$  does not belong to the spectra of any of the four homogeneous boundary value problems (2.4.1), (2.4.2) with Dirichlet or Neumann boundary conditions imposed on  $\partial_3\mathcal{Y}$  and  $\partial_4\mathcal{Y}$  (an easy argument by contradiction shows that it is indeed the case). Then, with the account of boundary conditions on  $u_1|_{\partial_3\mathcal{Y}}$  and  $\partial u_2/\partial n|_{\partial_3\mathcal{Y}}$ , the matching conditions on  $\partial_4\mathcal{Y}$  can be written as

$$u_1|_{\partial_4\mathcal{Y}} = u_2|_{\partial_4\mathcal{Y}} = (\mathcal{D}\mathcal{D})_\lambda(u_1|_{\partial_3\mathcal{Y}}) = (\mathcal{N}\mathcal{D})_\lambda\left(\frac{\partial u_1}{\partial n}\Big|_{\partial_3\mathcal{Y}}\right),$$

$$\frac{\partial u_1}{\partial n}\Big|_{\partial_4\mathcal{Y}} = -\frac{\partial u_2}{\partial n}\Big|_{\partial_4\mathcal{Y}} = (\mathcal{D}\mathcal{N})_\lambda(u_1|_{\partial_3\mathcal{Y}}) = -(\mathcal{N}\mathcal{N})_\lambda\left(\frac{\partial u_1}{\partial n}\Big|_{\partial_3\mathcal{Y}}\right),$$

which implies

$$(\mathcal{D}\mathcal{D})_\lambda^{-1}(\mathcal{N}\mathcal{D})_\lambda(\mathcal{N}\mathcal{N})_\lambda^{-1}(\mathcal{D}\mathcal{N})_\lambda(u_1|_{\partial_4\mathcal{Y}}) = -u_1|_{\partial_4\mathcal{Y}}.$$

Thus,  $\lambda \in \sigma_I$  iff  $\mu = -1$  is an eigenvalue of the operator  $\mathcal{C}_\lambda$ .

Similarly, consider Problem  $\tilde{II}$ , which is equivalent to the original Problem II, and is illustrated on the right-hand side of Fig. 2. With the account of matching conditions on  $\partial_3\mathcal{Y}$ , the boundary conditions on  $v_1|_{\partial_4\mathcal{Y}}$  and  $\partial v_2/\partial n|_{\partial_4\mathcal{Y}}$  can be written as

$$0 = v_1|_{\partial_4\mathcal{Y}} = (\mathcal{D}\mathcal{D})_\lambda^{-1}(v_1|_{\partial_3\mathcal{Y}}) + (\mathcal{D}\mathcal{N})_\lambda^{-1}\left(\frac{\partial v_1}{\partial n}\Big|_{\partial_3\mathcal{Y}}\right),$$

$$0 = \frac{\partial v_2}{\partial n}\Big|_{\partial_4\mathcal{Y}} = (\mathcal{N}\mathcal{D})_\lambda^{-1}(v_1|_{\partial_3\mathcal{Y}}) - (\mathcal{N}\mathcal{N})_\lambda^{-1}\left(\frac{\partial v_1}{\partial n}\Big|_{\partial_3\mathcal{Y}}\right),$$

which again implies that  $\lambda \in \sigma_{II}$  iff  $\mu = -1$  is an eigenvalue of the operator  $\mathcal{C}_\lambda$ , with a corresponding eigenfunction  $(\mathcal{D}\mathcal{D})_\lambda^{-1}(v_1|_{\partial_3\mathcal{Y}})$ .

Thus, the spectra of Problems  $\tilde{I}$  and  $\tilde{II}$ , and therefore of Problems I and II, coincide.

### 3. Extensions, generalisations, open questions

#### 3.1. From half-disks to quarter-spheres

Consider two quarter-spheres with the boundary conditions as shown in Fig. 5. To prove that they are isospectral one can use the same trick as shown in Fig. 2. In general, and analogous argument

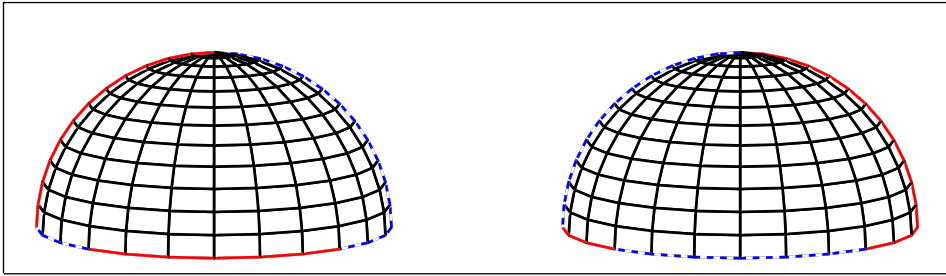


Fig. 5. Dirichlet–Neumann isospectral problems on quarter-spheres. The upper semicircles are divided into two equal parts, and the lower semicircles are divided in proportion 1:2:1.

works for half-disks endowed with an arbitrary radial metric  $ds^2 = f(|z|) dz d\bar{z}$  (note that the matching conditions on Fig. 5 are imposed along the radii), quarter-spheres being a special case for a metric  $ds^2 = 4 dz d\bar{z} / (1 + |z|^2)^2$ . The example in Fig. 5 was in fact the first nontrivial isospectral boundary decomposition that we observed, and it motivated our study, see Section 4.1.

### 3.2. Domains built from sectorial blocks

The example of Section 2.1 can be also generalised to a class of domains constructed by gluing together four copies of a sectorial block, i.e. a domain bounded by the sides of an acute angle and an arbitrary continuous curve (without self-intersections) inside it. Namely, let  $0 < \alpha < \pi/2$ , and choose any points  $z_1, z_2 \neq 0$  such that  $\arg z_1 = 0$  and  $\arg z_2 = \alpha$ . Now, let  $\Gamma_1$  be a piecewise smooth non-self-intersecting curve with end-points  $z_1, z_2$  which lie in the sector  $\{0 < \arg z < \alpha\}$ , and let  $K_1$  denote an open set bounded by the radii  $[0, z_1]$ ,  $[0, z_2]$ , and the curve  $\Gamma_1$ .

Let now  $\mathcal{S}_\beta : (r, \phi) \mapsto (r, 2\beta - \phi)$  be a map which sends a point into its mirror image with respect to the axis  $\{\arg z = \beta\}$ . Let

$$\Gamma_2 := \mathcal{S}_\alpha \Gamma_1, \quad \Gamma_3 := \mathcal{S}_{2\alpha} \Gamma_2, \quad \Gamma_4 := \mathcal{S}_{2\alpha} \Gamma_1 \quad (3.2.1)$$

and

$$K_2 := \mathcal{S}_\alpha K_1, \quad K_3 := \mathcal{S}_{2\alpha} K_2, \quad K_4 := \mathcal{S}_{2\alpha} K_1$$

and let  $K$  be the interior of  $\overline{K_1 \cup K_2 \cup K_3 \cup K_4}$ . The domain  $K$  is bounded by the radii  $[0, z_1]$ ,  $[0, \mathcal{S}_{2\alpha} z_2]$  and the curve  $\overline{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}$ . We additionally assume that  $\partial K$  satisfies the internal cone condition (i.e. there are no outward pointing cusps, see e.g. [11, chapter V.4]), thus ensuring that the spectra of all the boundary problems considered below are discrete, or some other less restrictive smoothness condition guaranteeing the discreteness of the spectrum of the Neumann Laplacian on  $K$ , see e.g. [20] for a recent discussion.

We construct a family of pairwise Dirichlet–Neumann isospectral boundary value problems on  $K$  in the following way. Suppose  $\Gamma_1$  is decomposed into a union of two nonintersecting sets  $\Gamma_{1,1}$  and  $\Gamma_{1,2}$  (one of which may be empty). We define the sets  $\Gamma_{j,m}$ ,  $j = 1, 2, 3, 4$ ,  $m = 1, 2$  similarly to 3.2.1. We now set

$$\partial_1 K := \Gamma_{1,1} \cup \Gamma_{2,2} \cup \Gamma_{3,2} \cup \Gamma_{4,1} \cup [0, \mathcal{S}_{2\alpha} z_2]$$



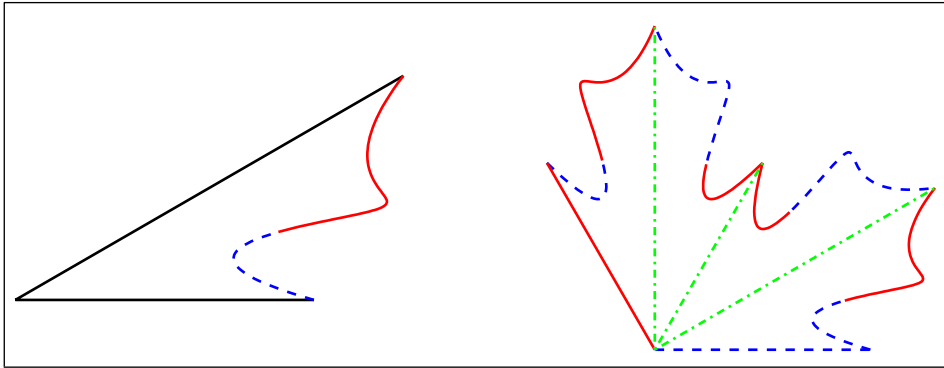


Fig. 6. A sectorial block  $K_1$  and the resulting domain  $K$ . The spectral problem on  $K$  with boundary conditions as shown is Dirichlet–Neumann isospectral.

and

$$\partial_2 K := [0, z_1] \cup \Gamma_{1,2} \cup \Gamma_{2,1} \cup \Gamma_{3,1} \cup \Gamma_{4,2}$$

(see Fig. 6).

The following result generalises Theorem 2.1.3.

**Theorem 3.2.2.** *With the above notation, the problem*

$$-\Delta u = \lambda u \quad \text{in } K, \quad u|_{\partial_1 K} = 0, \quad \left. \frac{\partial u}{\partial n} \right|_{\partial_2 K} = 0$$

*is isospectral with respect to exchanging the Dirichlet and Neumann boundary conditions.*

The first proof of Theorem 2.1.3 (see Section 2.2) is straightforwardly adapted for Theorem 3.2.2. Note that to obtain Theorem 2.1.3 we just set  $z_1 = 1$ ,  $z_2 = e^{i\pi/4}$ ,  $\Gamma_1 = \Gamma_{1,2} = \{e^{it}, t \in (0, \pi/4)\}$ ,  $\Gamma_{1,2} = \emptyset$  in Theorem 3.2.2.

Other simple examples are illustrated in Fig. 7.

**Remark.** All our examples of domains admitting Dirichlet–Neumann isospectrality are constructed using essentially the same principle. Are there other examples of such domains? For instance, all our domains have one axis of symmetry. Do there exist nonsymmetric domains that admit Dirichlet–Neumann isospectrality? In general, can one characterise in geometric terms domains admitting Dirichlet–Neumann isospectrality?

### 3.3. A necessary condition for Dirichlet–Neumann isospectrality

After presenting various examples of Dirichlet–Neumann isospectrality it would be natural to ask about restrictions. Intuitively, isospectral decompositions should occur rarely. A simple necessary condition for a boundary decomposition to be isospectral is given by

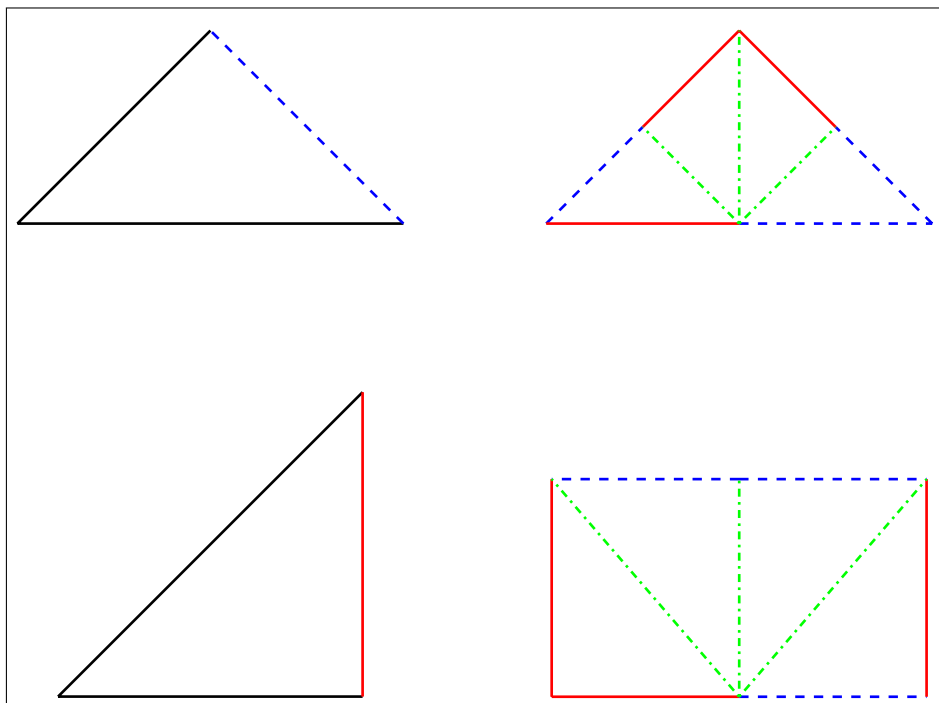


Fig. 7. Two more examples built using sectorial blocks. In the first example,  $z_1 = 1$ ,  $z_2 = 1/2 + i/2$ ,  $\Gamma_1 = \Gamma_{1,2} = [z_1, z_2]$ ,  $\Gamma_{1,1} = \emptyset$ ; the resulting set  $K$  is a triangle. In the second example,  $z_1 = 1$ ,  $z_2 = 1 + i$ ,  $\Gamma_1 = \Gamma_{1,1} = [z_1, z_2]$ ,  $\Gamma_{1,2} = \emptyset$ ; the resulting set  $K$  is a  $2 \times 1$  rectangle.

**Proposition 3.3.1.** *If a boundary decomposition  $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$  of a bounded planar domain is isospectral with respect to the Dirichlet–Neumann swap then the total lengths of the parts are equal:  $|\partial_1\Omega| = |\partial_2\Omega|$ .*

**Proof.** We use the asymptotics of the heat trace for a domain with mixed boundary conditions, which has recently attracted a lot of attention (see [3,21,1,9] and references therein). The first heat invariant  $a_1$  is equal (up to a multiplicative constant) to  $|\partial_N\Omega| - |\partial_D\Omega|$ ,  $\partial_N\Omega$  and  $\partial_D\Omega$  being Neumann and Dirichlet parts of the boundary, respectively. This immediately implies the proposition.  $\square$

**Remark.** Note that the second heat invariant  $a_2$  does not change under the Dirichlet–Neumann swap, see e.g. [9, formulae (4) and (5)], and therefore does not produce an additional necessary condition of the Dirichlet–Neumann isospectrality.

### 3.4. Are there domains not admitting Dirichlet–Neumann isospectrality?

Though in general the question of Kac has a negative answer, there exist domains that are determined by their Dirichlet spectrum (see [25]), for example, a disk. It would be natural to ask if there are domains not admitting nontrivial Dirichlet–Neumann isospectral decompositions of their boundaries.

**Conjecture 3.4.1.** *A disk does not admit Dirichlet–Neumann isospectrality.*

We conducted a simple numerical experiment providing some evidence for this conjecture, by considering boundary decompositions of a unit disk such that  $\partial_1\Omega$  and  $\partial_2\Omega$  are unions of two segments each,  $|\partial_1\Omega| = |\partial_2\Omega| = \pi$  by Proposition 3.3.1. Each partition is parametrised by a pair  $(\alpha, \beta)$ , where  $\alpha, \pi - \alpha$  are lengths of segments in  $\partial_1\Omega$ , and  $\beta, \pi - \beta$  are lengths of segments in  $\partial_2\Omega$ . For every pair  $(k\pi/24, n\pi/24)$ ,  $0 < k \leq n < 12$  we compute numerically using FEMLAB [12] the  $L^2$ -norm  $v(k, n)$  of a vector  $(\lambda_1^I - \lambda_1^{II}, \lambda_2^I - \lambda_2^{II}, \lambda_3^I - \lambda_3^{II})$ . Here  $\lambda_i^I$  are the eigenvalues of the mixed problem with Dirichlet conditions on  $\partial_1\Omega$  and Neumann conditions on  $\partial_2\Omega$ , and  $\lambda_i^{II}$  are the eigenvalues of the problem with the conditions swapped. We observe that for trivial decompositions ( $n = k$ ) the norm  $v(k, n)$  is by at least an order of magnitude smaller than for any nontrivial decomposition. For example, in a trivially isospectral case  $v(12, 12) = 0.0012$ , and in a nonisospectral case  $v(11, 12) = 0.0725$  (this value is in fact the minimal one achieved among all nontrivial combinations).

#### 4. Dirichlet–Neumann isospectrality and eigenvalue inequalities

##### 4.1. Genus 2: where did Dirichlet–Neumann isospectrality come from

In this section we briefly describe our motivation to study Dirichlet–Neumann isospectrality. It comes, quite unexpectedly, from a problem to obtain a sharp upper bound for the first positive eigenvalue  $\lambda_1$  of the Laplacian on a surface of genus 2. It is known [23,19] that on a surface  $M$  of genus  $p$

$$\lambda_1 \text{Area}(M) \leq 8\pi \left\lceil \frac{p+3}{2} \right\rceil, \quad (4.1.1)$$

where  $\lceil \cdot \rceil$  denotes the integer part. On a surface  $\mathcal{P}$  of genus 2 this implies

$$\lambda_1 \text{Area}(\mathcal{P}) \leq 16\pi. \quad (4.1.2)$$

In general (4.1.1) is not sharp, for example for  $p = 1$  [19]. In [16] we work towards proving the following

**Conjecture 4.1.3.** *There exists a metric on a surface of genus 2 that attains the upper bound in (4.1.2).*

The candidate for the extremal metric is a singular metric of constant curvature  $+1$  that is lifted from a sphere  $\mathbb{S}^2$ . The surface  $\mathcal{P}$  here is viewed as a branched double covering over a sphere with 6 branching points. The branching points are chosen to be the intersections of  $\mathbb{S}^2$  with the coordinate axes in  $\mathbb{R}^3$ . The punctured sphere has an octahedral symmetry group, and the corresponding hyperelliptic cover corresponds to *Bolza's surface*  $w^2 = z^5 - z$  (known also as the *Burnside curve*), and has a symmetry group with 96 elements (a central extension by  $\mathbb{Z}_2$  of an octahedral group), the largest possible symmetry group for surfaces of genus 2, see e.g. [15,18].

Note that  $\text{Area}(\mathcal{P}) = 2\text{Area}(\mathbb{S}^2) = 8\pi$  and, therefore it suffices to show that

$$\lambda_1(\mathcal{P}) = \lambda_1(\mathbb{S}^2) = 2. \quad (4.1.4)$$

It remains to be proved that there exists a first eigenfunction on  $\mathcal{P}$  that projects to  $\mathbb{S}^2$ , i.e. which is even with respect to the hyperelliptic involution  $\tau$  intertwining the sheets of the double cover. We conjecture (see Conjecture 4.2.2) that a first eigenfunction on  $\mathcal{P}$  cannot be odd with respect to  $\tau$ . The symmetry group of  $\mathcal{P}$  contains many commuting involutions, and this allows us to exploit the ideas of Section 2.3.

Consider an odd eigenfunction (with respect to  $\tau$ ) and symmetrise it with respect to those involutions. On their fixed point sets we get either Dirichlet or Neumann conditions. Applying the projection  $\mathcal{P} \rightarrow \mathbb{S}^2$  we obtain an eigenfunction of a spectral problem on a sphere *with cuts* along certain arcs of great circles, where Dirichlet or Neumann conditions are imposed. In particular, in this way we obtain mixed boundary problems shown in Fig. 5. These problems are isospectral, since the spectrum of each problem coincides with the odd (with respect to  $\tau$ ) part of the spectrum of  $\mathcal{P}$ .

All the details of this argument will appear in [16].

#### 4.2. Bounds on the first eigenvalue of mixed boundary problems

Dirichlet–Neumann isospectrality can be viewed as a special case of the following question. Consider a mixed Dirichlet–Neumann problem on a domain with a boundary of length  $l$ , where the Dirichlet and the Neumann conditions are specified on parts of the boundary of total length  $l/2$  each. For a given domain, how does the geometry of the boundary decomposition affect the spectrum? We discuss this question in relation to the first eigenvalue  $\lambda_1$ .

It is natural to ask how large and how small can  $\lambda_1$  be. Extremal boundary decompositions for the first eigenvalue of a mixed Dirichlet–Neumann problem are studied in [8]. In particular, it is proved that  $\lambda_1$  can get arbitrarily close to the first eigenvalue of the pure Dirichlet problem (which is hence a sharp upper bound for  $\lambda_1$ ): it is achieved in the limit as Dirichlet and Neumann conditions get uniformly distributed on the boundary. It is also shown that a decomposition minimising  $\lambda_1$  always exists for bounded Lipschitz domains. However, an explicit minimiser is found only for a disk, where Dirichlet and Neumann conditions have to be imposed on half-circles [8].

The problem of comparing the first eigenvalues for different boundary decompositions seems to be rather transcendental in general. Below we present a result, communicated to us by Brian Davies and Leonid Parnovski, that applies to a special case of axisymmetric/centrally symmetric decompositions.

Let  $\Xi_a$  be a simply connected planar domain, which is symmetric with respect to an axis  $d$ . We consider a mixed boundary value spectral problem for the Laplacian on  $\Xi_a$  with some combination of Dirichlet and Neumann boundary conditions on  $\partial\Xi_a$  which is also symmetric with respect to  $d$ . Denote the first eigenvalue of this problem by  $\lambda_1(\Xi_a)$ .

Let  $\Xi_{1,2}$  denote two halves of  $\Xi_a$  lying on either side of  $d$ , and let  $\tilde{\Xi}_2$  be an image of  $\Xi_1$  under the central symmetry with respect to the midpoint  $O$  of the interval  $d_{\Xi} := \Xi_a \cap d$ . Consider a centrally symmetric domain  $\Xi_c$  which is the interior of  $\overline{\Xi_1 \cup \tilde{\Xi}_2}$ , and the spectral mixed boundary value problem on  $\Xi_c$  with boundary conditions on  $\partial\tilde{\Xi}_2$  centrally symmetric to the ones on  $\partial\Xi_1$ , see Fig. 8. Denote the first eigenvalue of this problem by  $\lambda_1(\Xi_c)$ .

**Theorem 4.2.1** (Davies and Parnovski [7]).  $\lambda_1(\Xi_c) \geq \lambda_1(\Xi_a)$ .

**Proof.** Consider an auxiliary boundary value problem for the Laplacian on  $\Xi_1$  obtained by keeping the given boundary conditions on  $\partial\Xi_1 \setminus d_{\Xi}$  and imposing the Neumann condition on  $d_{\Xi}$ . Denote the first eigenvalue of this auxiliary problem by  $\lambda_1(\Xi_1)$ . By the variational principle and the Dirichlet–Neumann bracketing argument,  $\lambda_1(\Xi_c) \geq \lambda_1(\Xi_1)$ . On the other hand, as the first eigenfunction of the symmetric problem (corresponding to the eigenvalue  $\lambda_1(\Xi_a)$ ) should be symmetric with respect to  $d$  and therefore should satisfy the Neumann condition on  $d_{\Xi}$ , we have  $\lambda_1(\Xi_a) = \lambda_1(\Xi_1)$ , thus implying the result. Note that the equality  $\lambda_1(\Xi_c) = \lambda_1(\Xi_1)$  (and therefore the equality  $\lambda_1(\Xi_c) = \lambda_1(\Xi_a)$ ) can be attained if and only

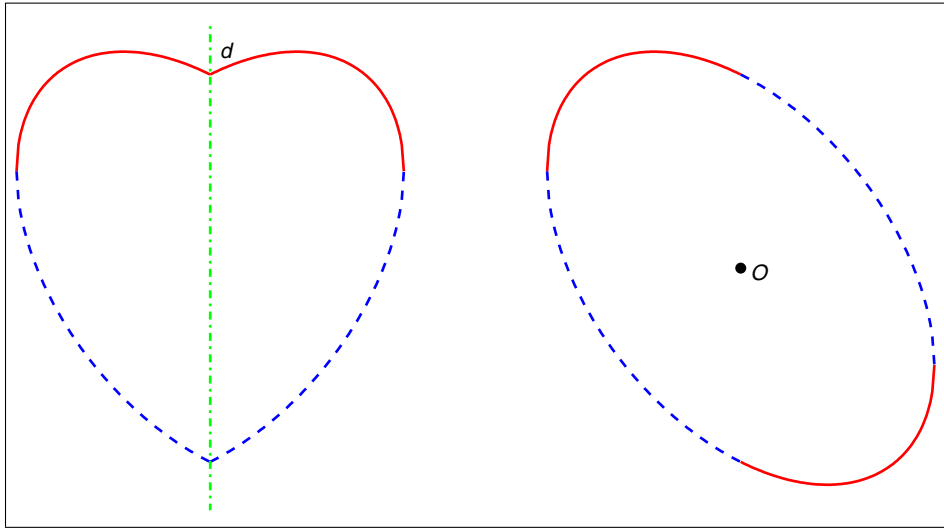


Fig. 8. Axisymmetric domain  $\Xi_a$  and centrally symmetric domain  $\Xi_c$ .

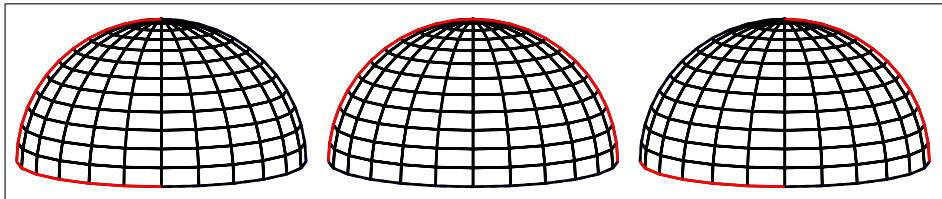


Fig. 9. Axisymmetric (domains  $Q_{a,1}$ , left, and  $Q_{a,2}$ , centre) and centrally symmetric (domain  $Q_c$ , right) positioning of Dirichlet and Neumann boundary conditions on the quarter-sphere. The first eigenvalue of the Laplacian is larger in the centrally symmetric case:  $\lambda_1(Q_c) > \lambda_1(Q_{a,j})$ ,  $j = 1, 2$ .

if  $\Xi_1$  has an additional line of symmetry  $d_1$  perpendicular to  $d$  and passing through the midpoint  $O$  of  $d_\Xi$ , with the boundary conditions being imposed on  $\partial\Xi_1$  symmetrically with respect to  $d_1$ .  $\square$

Theorem 4.2.1 can be used for obtaining estimates of eigenvalues of boundary value problems on domains with two lines of symmetry. For example, the boundary of a quarter-sphere has a natural decomposition into two halves of great circles. If we impose the mixed Dirichlet–Neumann boundary conditions on the halves of these great circles as shown in Fig. 9, we immediately obtain that the first eigenvalue in any of the axisymmetric cases is smaller than the first eigenvalue in the centrally symmetric case.

We would like to conclude with another inequality on the first eigenvalue for quarter-spheres that we need to check in order to complete the proof of sharpness of (4.1.2) in [16]. Let  $Q$  be any of the two isospectral problems on a quarter-sphere shown in Fig. 5, and recall that  $Q_{a,2}$  is the problem shown in the middle of Fig. 9 (with the Dirichlet condition imposed on one half of the big circle and the Neumann condition on another).

**Conjecture 4.2.2.**  $\lambda_1(Q) > \lambda_1(Q_{a,2})$ .

One can immediately check that  $\lambda_1(Q_{a,2}) = \lambda_1(\mathbb{S}^2) = 2$ . An affirmative solution of Conjecture 4.2.2 excludes the possibility that the first eigenfunction on  $\mathcal{P}$  is odd with respect to the intertwining of sheets (see Section 4.1), and hence we have

**Theorem 4.2.3** (Jakobson et al. [16]). *Conjecture 4.2.2 implies Conjecture 4.1.3.*

Using FEMLAB [12] one can verify Conjecture 4.2.2 numerically:  $\lambda_1(Q) \approx 2.28 > 2$ . Our current project is to find a rigorous (possibly, computer-assisted) proof of this conjecture.

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## References

- [1] I.G. Avramidi, Heat kernel asymptotics of Zaremba boundary value problem, *Math. Phys. Anal. Geom.* 7 (2004) 9–46.
- [2] P. Berard, Transplantation et isospectralité I, *Math. Ann.* 292 (1992) 547–559.
- [3] T. Branson, P. Gilkey, K. Kirsten, D. Vassilevich, Heat kernel asymptotics with mixed boundary conditions, *Nuclear Phys. B* 563 (1999) 603–626.
- [4] R. Brooks, Constructing isospectral manifolds, *Amer. Math. Monthly* 95 (1988) 823–839.
- [5] P. Buser, Isospectral Riemannian surfaces, *Ann. Inst. Fourier* 36 (1986) 167–192.
- [6] P. Buser, J. Conway, P. Doyle, K.-D. Semmler, Some planar isospectral domains, *IMRN* 9 (1994) 391–400.
- [7] E.B. Davies, L. Parnovski, 2000, private communication.
- [8] J. Denzler, Windows of given area with minimal heat diffusion, *Trans. AMS* 351 (1999) 569–580.
- [9] J.S. Dowker, The hybrid spectral problem and the Robin boundary conditions, *arXiv:math.SP/0409442* (2004).
- [10] T. Driscoll, H.P.W. Gottlieb, Isospectral shapes with Neumann and alternating boundary conditions, *Phys. Rev. E* 68 (2003) 016702-1–016702-6.
- [11] D.E. Edmunds, W.D. Evans, *Spectral theory and differential operators*, Clarendon Press, Oxford, 1987.
- [12] FEMLAB User's Guide, Comsol AB, Stockholm, 2004.
- [13] L. Friedlander, Some inequalities between Dirichlet and Neumann eigenvalues, *Arch. Rational Mech. Anal.* 116 (1991) 153–160.
- [14] C. Gordon, D. Webb, S. Wolpert, Isospectral plane domains and surfaces via Riemannian orbifolds, *Invent. Math.* 110 (1992) 1–22.
- [15] J. Igusa, Arithmetic variety of moduli for genus two, *Ann. Math.* 72 (1960) 612–649.
- [16] D. Jakobson, M. Levitin, N. Nadirashvili, N. Nigam, I. Polterovich, A sharp upper bound for the first eigenvalue on a surface of genus 2, in preparation.
- [17] M. Kac, Can one hear the shape of a drum?, *American Math. Monthly* 73 (1966) 1–23.
- [18] H. Karcher, M. Weber, The geometry of Klein's Riemann surface. The eightfold way, *MSRI Publ.* 35 (1999) 9–49, Cambridge Univ. Press.
- [19] N. Nadirashvili, Berger's isoperimetric problem and minimal immersions of surfaces, *GAFA* 6 (1996) 877–897.
- [20] Yu. Netrusov, Yu. Safarov, Weyl asymptotic formula for the Laplacian on domains with rough boundaries, *Comm. Math. Phys.* 253 (2005) 481–509.

- [21] R.T. Seeley, Trace expansions for the Zaremba problem, *Comm. Partial Differential Equations* 28 (2003) 601–616.
- [22] T. Sunada, Riemannian coverings and isospectral manifolds, *Ann. Math.* 121 (1985) 169–186.
- [23] P. Yang, S.-T. Yau, Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 7 (1) (1980) 55–63.
- [24] S. Zaremba, Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann, *J. Math. Pures Appl.* 6 (1927) 127–163.
- [25] S. Zelditch, Spectral determination of analytic axi-symmetric plane domains, *GAFA* 10 (2000) 628–677.