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Existence results for time scale boundary value problem

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Abstract

In this paper, we establish some existence results for positive solutions to a class of singular boundary value problem on time scale by using the Krasnosel'skii fixed point theorem. Two examples are presented as applications. The conditions we used in this paper are different from those in [D.R. Anderson, Eigenvalue intervals for a two-point boundary value problem on a measure chain, J. Comput. Appl. Math. 141 (2002) 57–64; C.J. Chyan, J. Henderson, Eigenvalues problems for nonlinear differential equations on a measure chain, J. Math. Anal. Appl. 245 (2000) 547–559; L.H. Erbe, A. Peterson Positive solutions for nonlinear differential equation on a measure chain, Math. Comput. Modelling 32 (2000) 571–585; L.H. Erbe, H.Y. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (1994) 743–748; J. Henderson, H.Y. Wang, Positive solutions for nonlinear eigenvalue problems, J. Math. Anal. Appl. 208 (1997) 252–259; C.H. Hong, C.C. Yeh, Positive solutions for eigenvalue problems on a measure chain, Nonlinear Anal. 51 (2002) 499–507; W.C. Lian; W.F. Wong; C.C. Yeh, On the existence of positive solutions of nonlinear differential equations, Proc. Amer. Math. Soc. 124 (1996) 1117–1126; J. Liang, T.J. Xiao, Z.C. Hao, Positive solutions of singular differential equations on measure chain, Comput. Math. Appl. 49 (2005) 651–663].

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1. Introduction

In this paper, we consider the existence of positive solutions of the following boundary value problem (BVP in short) on a time scale (see appendix for this terminology.)

$$\begin{cases} x^{\triangle\triangle}(t) + m(t)f(t, x(\sigma(t))) = 0, \quad t \in [a, b], \\ \alpha x(a) - \beta x^{\triangle}(a) = 0, \\ \gamma x(\sigma(b)) + \delta x^{\triangle}(\sigma(b)) = 0, \end{cases} \tag{1.1}$$

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and

$$\begin{cases} x^{\triangle\triangle}(t) + m(t)f(t, x(\sigma(t)), x^{\triangle}(\sigma(t))) = 0, \quad t \in [a, b], \\ \alpha x(a) - \beta x^{\triangle}(a) = 0, \\ \gamma x(\sigma(b)) + \delta x^{\triangle}(\sigma(b)) = 0, \end{cases} \tag{1.2}$$

where f is continuous; $m(\cdot) : (a, \sigma(b)) \to [0, \infty)$ is rd-continuous (see Appendix, Definition 5) and may be singular at $t = a$ and/or $t = \sigma(b)$; $\alpha, \beta, \gamma, \delta \geqslant 0$ such that

$$d := \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(b) - a) > 0 \quad \text{and} \quad \delta \geqslant \gamma[\sigma^2(b) - \sigma(b)]. \tag{1.3}$$

Throughout this paper, we assume that $a < b$.

To make this work reasonable self-contained we have included the basic definitions from the theory of time scale in Appendix.

Recently, much interest has developed regarding the existence of positive solutions to problem

$$\begin{cases} x^{\triangle\triangle}(t) + \lambda m(t)f(t, x(\sigma(t))) = 0, \quad t \in [a, b], \\ \alpha x(a) - \beta x^{\triangle}(a) = 0, \\ \gamma x(\sigma(b)) + \delta x^{\triangle}(\sigma(b)) = 0, \end{cases} \tag{1.4}$$

the eigenvalue problem of (1.1). For example, Anderson [1], Chyan and Henderson [6], Erbe and Peterson [8] and Hong and Yeh [14] considered BVP (1.4) on time scale when $m(\cdot) \in C[a, \sigma(b)]$. Erbe and Wang [10], Henderson and Wang [11], Lian et al. [17] studied BVP (1.4) when $\mathbb{T} = \mathbb{R}$, $m(\cdot) \in C[a, \sigma(b)]$ and $\lambda = 1$. Liang et al.[18] studied relating singular BVP on time scale

$$\begin{cases} [\rho(t)x^{\triangle}(t)]^{\triangle} + \lambda m(t)f(t, x(\sigma(t))) = 0, \quad t \in [a, b], \\ \alpha x(a) - \beta x^{\triangle}(a) = 0, \\ \gamma x(\sigma(b)) + \delta x^{\triangle}(\sigma(b)) = 0. \end{cases}$$

Problems in [1,6,8,10,11,14,17] are all nonsingular. Furthermore, limit conditions on f are all required in papers [1,6,8,10,11,14,17,18] (see Remark 2.4).

Stimulated by these works, in this paper, we investigate the singular BVP (1.1) and BVP (1.2), which is more general because of the term $x^{\Delta}(t)$. We obtain some criteria of the existence for positive solutions of BVP (1.1) and BVP (1.2) via the Krasnosel'skii fixed point theorem. Growth conditions, not limit conditions, on f are required in this paper. In view of different aspect, main results of papers [1,6,8,10,11,14,17,18] do not apply to Examples 2.3 and 3.2 because of different conditions on f required in this paper (see Remarks 2.4 and 3.3).

This paper is organized as follows. In Section 2, we obtain the existence theorem (see Theorem 2.2) of positive solution of BVP (1.1) and Example 2.3 is given to illustrate it. Section 3 is devoted to BVP (1.2) and Theorem 3.1 is obtained, which application can be seen from Example 3.2. For the convenience of the readers, we conclude this paper with Appendix which should serve as a time scale primer for those unfamiliar with the area.

2. Existence of positive solution of BVP (1.1)

Let us first present some properties of $G(t, s)$, the *Green* function of the following BVP:

$$\begin{cases} x^{\triangle\triangle}(t) = 0, \quad t \in [a, b], \\ \alpha x(a) - \beta x^{\triangle}(a) = 0, \quad \gamma x(\sigma(b)) + \delta x^{\triangle}(\sigma(b)) = 0. \end{cases}$$

We know form Erbe and Peterson [7] that

$$G(t, s) = \frac{1}{d} \begin{cases} u(t)v(\sigma(s)), & t \leqslant s, \\ u(\sigma(s))v(t), & \sigma(s) \leqslant t, \end{cases} \tag{2.1}$$

where d is given by (1.3) and

$$u(t)=\alpha(t-a)+\beta, \quad v(t)=\gamma(\sigma(b)-t)+\delta.$$

Obviously, the partial derivative of $G(t,s)$ with respect to t is given by

$$G_t(t,s)=\frac{1}{d}\begin{cases}\alpha v(\sigma(s)), & t\leqslant s,\\ -\gamma u(\sigma(s)), & \sigma(s)\leqslant t.\end{cases} \tag{2.2}$$

(1.3), together with (2.1), implies that

$$0\leqslant G(t,s)\leqslant G(\sigma(s),s), \quad (t,s)\in[a,\sigma^2(b)]\times[a,b]. \tag{2.3}$$

We also need the following fixed point theorem.

Lemma 2.1. (Krasnosel'skii[16]) *Let X be a Banach space, and let P be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let $A: P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ be a completely continuous operator such that, either*

(i) $\|Ax\|\leqslant\|x\|, \ x\in P\cap\partial\Omega_1, \quad \|Ax\|\geqslant\|x\|, \ x\in P\cap\partial\Omega_2,$

or

(ii) $\|Ax\|\geqslant\|x\|, \ x\in P\cap\partial\Omega_1, \quad \|Ax\|\leqslant\|x\|, \ x\in P\cap\partial\Omega_2.$

Then A has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Now, we give the existence theorem of positive solution of BVP (1.1).

Theorem 2.2. *Assume that* (H_1) $f(\cdot,\cdot):[a,\sigma(b)]\times[0,\infty)\to[0,\infty)$ *is continuous and*

$$f(t,x)\not\equiv 0, \quad (t,x)\in\mathbb{S}_1\times[0,\infty), \tag{2.4}$$

$$f(t,x)\leqslant p(t)+q(t)x, \quad (t,x)\in[a,\sigma(b)]\times[0,\infty), \tag{2.5}$$

where $\mathbb{S}_1$ is any closed subinterval of $(a,\sigma(b))$, $p(\cdot)$ and $q(\cdot):[a,\sigma(b)]\to[0,\infty)$ are continuous functions.

(H_2) $m(\cdot):(a,\sigma(b))\to[0,\infty)$ *is rd-continuous and may be singular at $t=a$ and/or $t=\sigma(b)$ and also satisfies*

$$m(t)\not\equiv 0 \text{ on any closed subinterval of } (a,\sigma(b)). \tag{2.6}$$

(H_3) *The integrals*

$$\int_a^{\sigma(b)} G(\sigma(s),s)m(s)\triangle s$$

and

$$M_1:=\int_a^{\sigma(b)} m(s)p(s)\triangle s$$

and

$$M_2:=\int_a^{\sigma(b)} m(s)q(s)\triangle s$$

are convergent and

$$M_2 < \frac{d}{u(\sigma(b))v(\sigma(a))}.$$

(H$_4$) *There exist*

$$C > 0, \quad C \neq \frac{M_1 u(\sigma(b))v(\sigma(a))}{d - M_2 u(\sigma(b))v(\sigma(a))},$$

$$\xi \text{ and } \omega \in \left[\frac{\sigma(b)+3a}{4}, \frac{3\sigma(b)+a}{4}\right], \quad \xi < \omega,$$

$$\gamma\sigma(\omega) < \gamma\sigma(b) + \delta, \tag{2.7}$$

and

$$t_0 \in [a, \sigma^2(b)],$$

such that

$$f(t,x) \geqslant C\left[\int_\xi^\omega G(t_0,s)m(s)\triangle s\right]^{-1}, \quad (t,x) \in [\xi,\omega] \times [m_1 C, C], \tag{2.8}$$

where

$$m_1 = \min\left\{\frac{u(\sigma(b)+3a/4)}{u(\sigma(b))}, \frac{v(3\sigma(b)+a/4)}{v(\sigma(a))}, \min_{s\in[a,b]} \frac{G(\sigma(\omega),s)}{G(\sigma(s),s)}\right\}. \tag{2.9}$$

Then, *BVP* (1.1) *has at least one positive solution*.

Proof. Let

$$E := \mathbb{C}_{\mathrm{rd}}[a, \sigma^2(b)],$$

equipped with the norm

$$\|x\| := \max_{t\in[a,\sigma^2(b)]} |x(t)|.$$

In the Banach space$(E, \|\cdot\|)$ we define

$$P := \left\{x \in E;\ x \geqslant 0 \text{ on } [a,\sigma^2(b)] \text{ and } \min_{t\in[\xi,\sigma(\omega)]} x(t) \geqslant m_1\|x\|\right\}, \tag{2.10}$$

where m_1 is given by (2.9) and it also satisfies $m_1 > 0$ with the aid of (2.7). It is clear that P is a cone in E. By (2.3) and assumptions (H$_1$) and (H$_3$), we may define an operator A on P by

$$Ax(t) := \int_a^{\sigma(b)} G(t,s)m(s)f(s,x(\sigma(s)))\triangle s, \quad x \in P, \quad t \in [a,\sigma^2(b)]. \tag{2.11}$$

From (2.11) and the fact that $G(t,s)$ is *Green* function, we know that

Fixed points of operator A are also solutions of the BVP (1.1). (2.12)

We know from [18, Lemma 2.3] that

$$A : P \to P \text{ is completely continuous.} \tag{2.13}$$

Now, let us prove the existence of positive solutions of BVP (1.1). Write

$$R_1 := \frac{M_1 u(\sigma(b))v(\sigma(a))}{d - M_2 u(\sigma(b))v(\sigma(a))}, \quad R_2 := C.$$

$$\Omega_1 := \{x \in E : \|x\| < R_1\}, \quad \Omega_2 := \{x \in E : \|x\| < R_2\}.$$

Without loss of generality, we may assume that $R_1 < R_2$.

Firstly, for any $x \in P \cap \partial\Omega_1$, $t \in [a, \sigma^2(b)]$, we have from (H_1), (H_3) and (2.5),

$$\begin{aligned} Ax(t) &\leqslant \int_a^{\sigma(b)} G(\sigma(s), s)m(s)[p(s) + q(s)x(\sigma(s))]\Delta s \\ &\leqslant d^{-1}u(\sigma(b))v(\sigma(a))\left[\int_a^{\sigma(b)} m(s)p(s)\Delta s + \int_a^{\sigma(b)} m(s)q(s)x(\sigma(s))\Delta s\right] \\ &= d^{-1}u(\sigma(b))v(\sigma(a))[M_1 + M_2 R_1] \\ &= R_1. \end{aligned}$$

Hence,

$$\|Ax\| \leqslant \|x\|, \quad \forall x \in P \cap \partial\Omega_1. \tag{2.14}$$

Secondly, if $x \in P \cap \partial\Omega_2$, then $\min_{t\in[\xi,\sigma(\omega)]} x(t) \geqslant m_1 R_2$. So

$$m_1 R_2 \leqslant x(\sigma(s)) \leqslant R_2, \quad s \in [\xi, \omega], \quad x \in P \cap \partial\Omega_2.$$

Thus, for any $x \in P \cap \partial\Omega_2$, we know from (2.4), (2.6) and (2.8) that

$$\begin{aligned} Ax(t_0) &\geqslant \int_\xi^\omega G(t_0, s)m(s)f(s, x(\sigma(s)))\Delta s \\ &\geqslant \int_\xi^\omega G(t_0, s)m(s)C\left[\int_\xi^\omega G(t_0, s)m(s)\Delta s\right]^{-1} \Delta s \\ &= C. \end{aligned}$$

Then

$$\|Ax\| \geqslant \|x\|, \quad \forall x \in P \cap \partial\Omega_2. \tag{2.15}$$

(2.13)–(2.15) and Lemma 2.1 imply that the operator A has one fixed point $x \in \overline{\Omega_2}\backslash\Omega_1$. (2.12) tells us that x is a positive solution of BVP (1.1). □

Let us give an example to illustrate Theorem 2.2.

Example 2.3. Consider BVP

$$\begin{cases} x'' + \dfrac{1}{t(1-t)} f(t, x) = 0, & t \in (0, 1), \\ x(0) = x(1) = 0, \end{cases} \tag{2.16}$$

where

$$f(t, x) = e^{-x}\left[\frac{11e\sqrt{3}}{\ln 3} t\sqrt{1-t} + \frac{(1-t)\sqrt{t}}{4} x\right].$$

Clearly, (2.16) is a singular BVP on [0, 1]. We choose

$$p(t)=\frac{11e\sqrt{3}}{\ln 3}t\sqrt{1-t},\quad q(t)=\frac{(1-t)\sqrt{t}}{4},\quad m(t)=\frac{1}{(1-t)t},$$

$$\alpha=\gamma=1,\quad \beta=\delta=0.$$

Then we compute

$$\int_a^{\sigma(b)} G(\sigma(s),s)m(s)\Delta s=1,\quad M_1=\frac{22e\sqrt{3}}{\ln 3},\quad M_2=\frac{1}{2},$$

$$C=1,\quad \xi=\frac{1}{4},\quad \omega=\frac{3}{4},\quad t_0=\frac{1}{4},\quad m_1=\frac{1}{4}.$$

For any $(t,x)\in[\frac{1}{4},\frac{3}{4}]\times[\frac{1}{4},1]$, we know

$$\begin{aligned} f(t,x)\int_\xi^\omega G(t_0,s)m(s)\Delta s &= e^{-x}\left[\frac{11e\sqrt{3}}{\ln 3}t\sqrt{1-t}+\frac{(1-t)\sqrt{t}}{4}x\right]\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{t}{s}ds \\ &> t\ln 3e^{-x}\frac{11e\sqrt{3}}{\ln 3}t\sqrt{1-t} \\ &\geqslant 11t^2\sqrt{1-t} \\ &\geqslant \frac{33}{32}>C. \end{aligned}$$

Hence conditions of Theorem 2.2 hold all. Then applying Theorem 2.2 we know that BVP (2.16) has at least one positive solution.

Remark 2.4. As usual we write

$$\max f_\infty := \lim_{u\to\infty}\max_{t\in[a,\sigma(b)]}\frac{f(t,u)}{u},\quad \min f_\infty := \lim_{u\to\infty}\min_{t\in[a,\sigma(b)]}\frac{f(t,u)}{u},$$

$$\max f_0 := \lim_{u\to 0^+}\max_{t\in[a,\sigma(b)]}\frac{f(t,u)}{u},\quad \min f_0 := \lim_{u\to 0^+}\min_{t\in[a,\sigma(b)]}\frac{f(t,u)}{u}.$$

Function f in [8,10] is assumed to be superlinear ($\max f_0=0$ and $\min f_\infty=\infty$) or sublinear ($\max f_\infty=0$ and $\min f_0=\infty$). It is required in papers [1,6,11,14,17] that limits $\max f_\infty$, $\min f_\infty$ and $\min f_0$ and $\max f_0$ all exist and are positive. The following condition:

$$0\leqslant f^0<L,\quad l<f_\infty\leqslant\infty\quad or\quad 0\leqslant f^\infty<L,\quad l<f_0\leqslant\infty,$$

is required in [18], where L and l are given. In this paper, we can see from Example 2.3 that

$$\max f_0=\infty,\quad \min f_0=\max f_\infty=\min f_\infty=0.$$

So the function f in Example 2.3 does not satisfy either superlinear (sublinear) conditions, or positive. Consequently, in view of different aspect, we can say that main results of papers [1,6,8,10,11,14,17,18] do not apply to Examples 2.3.

3. Existence of positive solution of BVP (1.2)

Main result of this section is as follows.

Theorem 3.1. *Assume* (H_2) *in Theorem* 2.2 *holds. Moreover, we assume also*

(H_5) $f(\cdot,\cdot,\cdot):[a,\sigma(b)]\times[0,\infty)\times R\to[0,\infty)$ *is continuous and*

$$f(t,x,y)\not\equiv 0,\ (t,x,y)\in\mathbb{S}_2\times[0,\infty)\times(-\infty,\infty), \tag{3.1}$$

$$f(t,x,y)\leqslant p^*(t)+q^*(t)(x+|y|),\quad (t,x,y)\in[a,\sigma(b)]\times[0,\infty)\times R, \tag{3.2}$$

where $\mathbb{S}_2$ *is any closed subinterval of* $(a,\sigma(b))$, $p^*(\cdot)$ *and* $q^*(\cdot):[a,\sigma(b)]\to[0,\infty)$ *are continuous functions satisfying the following condition* (H_6).

(H_6) *The integrals*

$$\int_a^{\sigma(b)} G(\sigma(s),s)m(s)\triangle s$$

and

$$M_1^*:=\int_a^{\sigma(b)} m(s)p(s)\triangle s$$

and

$$M_2^*:=\int_a^{\sigma(b)} m(s)q(s)\triangle s$$

are convergent and

$$M_2^*<\frac{1}{M^{**}}, \tag{3.3}$$

where

$$M^{**}:=d^{-1}[u(\sigma(b))v(\sigma(a))+\max\{\alpha v(\sigma(a)),\gamma u(\sigma(b))\}].$$

(H_7) *There exist*

$$C^*>0,\quad C^*\neq\frac{M_1^*M^{**}}{1-M^{**}M_2^*},$$
$$\xi^* \text{ and } \omega^*\in\left[\frac{\sigma(b)+3a}{4},\frac{3\sigma(b)+a}{4}\right],\quad \xi^*<\omega^*,\quad \gamma\sigma(\omega^*)<\gamma\sigma(b)+\delta,$$

and

$$t_0^*\in[a,\sigma^2(b)],$$

such that

$$f(t,x,y)\geqslant\frac{m_2C^*}{\int_{\xi^*}^{\omega^*}[m_2G(t_0^*,s)+G_t(t_0^*,s)]m(s)\triangle s}, \tag{3.4}$$

for all

$$(t,x,y)\in[\xi^*,\omega^*]\times\mathbb{D},$$

where

$$\mathbb{D} := \{(x, y) \in [0, \infty) \times R : m_2 x + y \geqslant m_3 m_4 C^* \text{ and } x + |y| \leqslant C^*\}, \tag{3.5}$$

$$m_2 := \max\left\{1, \frac{\gamma + 1}{\gamma(\sigma(b) - \sigma(\omega^*)) + \delta}\right\}, \quad m_3 := \min\left\{\frac{m_2 v(\sigma(\omega^*)) - \gamma}{v(\sigma(\xi^*)) + \gamma}, \frac{m_2 u(\xi^*) + \alpha}{u(\sigma(b)) + \alpha}\right\},$$

and m_4 is chosen as

$$\min\left\{\frac{v(\sigma^2(\omega^*)) + \gamma}{v(a) + \gamma}, \frac{u(\sigma(a))[v(\sigma^2(\omega^*)) + \gamma]}{v(\sigma(a))[u(\sigma^2(b)) + \alpha]}, \frac{v(\sigma^2(b))[u(\sigma(\xi^*)) + \alpha]}{u(\sigma^2(b))[v(a) + \gamma]}, \frac{u(\sigma(\xi^*)) + \alpha}{u(\sigma^2(b)) + \alpha}\right\}.$$

Then, *BVP* (1.2) *has at least one positive solution.*

Proof. We first write

$$E^* := \{x \in E : x^{\Delta}(t) \in C_{\mathrm{rd}}[a, \sigma^2(b)]\},$$

with the norm

$$\|x\| := \max_{t \in [a, \sigma^2(b)]} \{|x(t)| + |x^{\Delta}(t)|\}.$$

Obviously, $(E^*, \|\cdot\|)$ is a Banach space. Our aim is to construct a special cone (see P^* given by (3.10)) in the Banach space $(E^*, \|\cdot\|)$, in which we may find a positive solution of BVP (1.2). However, the success of this thing reckons on more profound properties of the Green function $G(t, s)$ given by (2.1).

Denote by $G_t(s+0, s)$ ($G_t(s-0, s)$) the right-hand side derivative (left-hand side derivative) of (2.1) at (s, s). We can see that, for any $s \in [a, \sigma(b)]$,

$$\frac{m_2 G(t, s) + G_t(t, s)}{G(\sigma(s), s) + |G_t(s+0, s)|} = \frac{m_2 v(t) - \gamma}{v(\sigma(s)) + \gamma} \geqslant m_3, \quad t \in [\xi^*, \sigma(\omega^*)], \quad s < t, \tag{3.6}$$

$$\frac{m_2 G(t, s) + G_t(t, s)}{G(\sigma(s), s) + |G_t(s-0, s)|} = \frac{m_2 u(t) + \alpha}{u(\sigma(s)) + \alpha} \geqslant m_3, \quad t \in [\xi^*, \sigma(\omega^*)], \quad s \geqslant t. \tag{3.7}$$

Furthermore, we know that, for any fixed $t \in [\xi^*, \sigma(\omega^*)]$,

$$\begin{aligned}
&\frac{G(\sigma(s), s) + |G_t(s+0, s)|}{G(\tau, s) + |G_t(\tau, s)|}, \quad s \in [a, t], \quad \tau \in [a, \sigma^2(b)], \\
&= \begin{cases} \dfrac{v(\sigma(s)) + \gamma}{v(\tau) + \gamma}, & s < \tau, \\[2mm] \dfrac{u(\sigma(s))[v(\sigma(s)) + \gamma]}{v(\sigma(s))[u(\tau) + \alpha]}, & s \geqslant \tau, \end{cases} \\
&\geqslant \begin{cases} \dfrac{v(\sigma^2(\omega^*)) + \gamma}{v(a) + \gamma}, & s < \tau, \\[2mm] \dfrac{u(\sigma(a))[v(\sigma^2(\omega^*)) + \gamma]}{v(\sigma(a))[u(\sigma^2(b)) + \alpha]}, & s \geqslant \tau, \end{cases}
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{G(\sigma(s),s)+|G_t(s-0,s)|}{G(\tau,s)+|G_t(\tau,s)|}, \quad s\in[t,\sigma(b)],\ \ \tau\in[a,\sigma^2(b)],\\
&\quad=\begin{cases}\dfrac{v(\sigma(s))[u(\sigma(s))+\alpha]}{u(\sigma(s))[v(\tau)+\gamma]}, & s<\tau,\\[2mm] \dfrac{u(\sigma(s))+\alpha}{u(\tau)+\alpha}, & s\geqslant\tau,\end{cases}\\
&\quad\geqslant\begin{cases}\dfrac{v(\sigma^2(b))[u(\sigma(\xi^*))+\alpha]}{u(\sigma^2(b))[v(a)+\gamma]}, & s<\tau,\\[2mm] \dfrac{u(\sigma(\xi^*))+\alpha}{u(\sigma^2(b))+\alpha}, & s\geqslant\tau.\end{cases}
\end{aligned}
$$

So we get, for any fixed $t\in[\xi^*,\sigma(\omega^*)]$,

$$\frac{G(\sigma(s),s)+|G_t(s+0,s)|}{G(\tau,s)+|G_t(\tau,s)|}\geqslant m_4, \quad s\in[a,t],\ \ \tau\in[a,\sigma^2(b)], \tag{3.8}$$

and

$$\frac{G(\sigma(s),s)+|G_t(s-0,s)|}{G(\tau,s)+|G_t(\tau,s)|}\geqslant m_4, \quad s\in[t,\sigma(b)],\ \ \tau\in[a,\sigma^2(b)]. \tag{3.9}$$

Now, let us define a cone P^* in $(E^*,\|\cdot\|)$ such that

$$P^*:=\left\{x\in E^*; x(t)\geqslant 0, t\in[a,\sigma^2(b)];\ \ \min_{t\in[\xi^*,\sigma(\omega^*)]}[m_2x(t)+x^{\Delta}(t)]\geqslant m_3m_4\|x\|\right\}. \tag{3.10}$$

Define an operator B as follows:

$$Bx(t):=\int_a^{\sigma(b)} G(t,s)m(s)f(s,x(\sigma(s)),\quad x^{\Delta}(\sigma(s)))\Delta s,\ \ x\in P^*,\ \ t\in[a,\sigma^2(b)].$$

We will show $B(P^*)\subset P^*$. It is easy to see that B maps P^* into E^* and $Bx(t)\geqslant 0$ for all $x\in P^*$ and $t\in[a,\sigma^2(b)]$. Moreover, we have, by (3.6)–(3.9) , for any $t\in[\xi^*,\sigma(\omega^*)]$, $\tau\in[a,\sigma^2(b)]$, $x\in P^*$,

$$
\begin{aligned}
&m_2Bx(t)+(Bx(t))^{\Delta}\\
&\quad=\int_a^{\sigma(b)}\{m_2G(t,s)+G_t(t,s)\}m(s)f(s,x(\sigma(s)),x^{\Delta}(\sigma(s)))\Delta s\\
&\quad\geqslant m_3\int_a^t\{G(\sigma(s),s)+|G_t(s+0,s)|\}m(s)f(s,x(\sigma(s)),x^{\Delta}(\sigma(s)))\Delta s\\
&\qquad+m_3\int_t^{\sigma(b)}\{G(\sigma(s),s)+|G_t(s-0,s)|\}m(s)f(s,x(\sigma(s)),x^{\Delta}(\sigma(s)))\Delta s\\
&\quad\geqslant m_3\int_a^t m_4\{G(\tau,s)+|G_t(\tau,s)|\}m(s)f(s,x(\sigma(s)),x^{\Delta}(\sigma(s)))\Delta s\\
&\qquad+m_3\int_t^{\sigma(b)} m_4\{G(\tau,s)+|G_t(\tau,s)|\}m(s)f(s,x(\sigma(s)),x^{\Delta}(\sigma(s)))\Delta s\\
&\quad=m_3m_4\{|Bx(\tau)|+|(Bx(\tau))^{\Delta}|\}.
\end{aligned}
$$

Thus,

$$\min_{t\in[\xi^*,\ \sigma(\omega^*)]} [m_2 Bx(t) + (Bx(t))^\Delta] \geqslant m_3 m_4 \|Bx\|.$$

This implies $B(P^*) \subset P^*$. We know from [18, Lemma 2.3] that

$$B : P^* \to P^* \textit{is completely continuous}, \tag{3.11}$$

hold also.

At last, let us consider the existence of positive solution of BVP (1.2). Set

$$R_1^* := \frac{M_1^* M^{**}}{1 - M^{**} M_2^*}, \quad R_2^* := C^*,$$

$$\Omega_1^* := \{x \in E^* : \|x\| < R_1^*\}, \quad \Omega_2^* := \{x \in E^* : \|x\| < R_2^*\}.$$

Without loss of generality, we may assume that $R_1^* < R_2^*$. For any $x \in \partial\Omega_1^* \cap P^*$, $t \in [a, \sigma^2(b)]$, we see from (2.1)–(2.2) and (3.1)–(3.3) that

$$\begin{aligned} |Bx(t)| + |(Bx(t))^\Delta| &= \int_a^{\sigma(b)} [G(t,s) + |G_t(t,s)|]\, m(s) f(s, x(\sigma(s)), x^\Delta(\sigma(s)))\Delta s \\ &\leqslant M^{**} \int_a^{\sigma(b)} m(s) \left\{ p(s) + q(s) \left[x(\sigma(s)) + |x^\Delta(\sigma(s))| \right] \right\} \Delta s \\ &\leqslant M^{**}(M_1^* + M_2^* R_1^*) = R_1^*. \end{aligned}$$

Hence,

$$\|Bx\| \leqslant \|x\|, \quad \forall x \in P^* \cap \partial\Omega_1^*. \tag{3.12}$$

Furthermore, if $x \in P^* \cap \partial\Omega_2^*$, then for any $t \in [\xi^*, \omega^*]$, we get

$$m_3 m_4 C^* \leqslant m_2 x(\sigma(t)) + x^\Delta(\sigma(t)),$$

$$x(\sigma(t)) + |x^\Delta(\sigma(t))| \leqslant C^*.$$

Thus, we have from (H_2) and (H_7) that for any $x \in P^* \cap \partial\Omega_2^*$,

$$\begin{aligned} & m_2\{|Bx(t_0)| + |(Bx(t_0))^\Delta|\} \\ & \quad \geqslant \int_a^{\sigma(b)} [m_2 G(t_0^*, s) + |G_t(t_0^*, s)|] m(s) f(s, x(\sigma(s)), x^\Delta(\sigma(s)))\Delta s \\ & \quad \geqslant \int_{\xi^*}^{\omega^*} [m_2 G(t_0^*, s) + |G_t(t_0^*, s)|] m(s) f(s, x(\sigma(s)), x^\Delta(\sigma(s)))\Delta s \\ & \quad \geqslant \int_{\xi^*}^{\omega^*} \frac{m_2 C^* [m_2 G(t_0^*, s) + |G_t(t_0^*, s)|] m(s)}{\int_{\xi^*}^{\omega^*} [m_2 G(t_0^*, s) + |G_t(t_0^*, s)|] m(s)\Delta s} \Delta s \\ & \quad = m_2 C^*. \end{aligned}$$

Consequently,

$$\|Bx\| \geqslant \|x\|, \quad \forall x \in P^* \cap \partial\Omega_2^*. \tag{3.13}$$

(3.11)–(3.13) and Lemma 2.1 imply that the operator B has one fixed point $x^* \in \overline{\Omega_2^*} \backslash \Omega_1^*$. Furthermore, (2.12) implies also that x^* is a positive solution of BVP (1.2). □

We end this Section by the following example.

Example 3.2. Consider the singular BVP

$$\begin{cases} x'' + \dfrac{1}{t(1-t)} f(t, x, x') = 0, \quad t \in (0, 1), \\ x(0) = x(1) = 0, \end{cases} \tag{3.14}$$

where

$$f(t, x, x') = \mathrm{e}^{-x} \left[\frac{11\mathrm{e}\sqrt{3}}{\ln 3} t\sqrt{1-t} + \frac{(1-t)\sqrt{t}}{8}(x + |\sin x'|) \right].$$

We may set

$$p^*(t) = \frac{11\mathrm{e}\sqrt{3}}{\ln 3} t\sqrt{1-t}, \quad q^*(t) = \frac{(1-t)\sqrt{t}}{8}, \quad m(t) = \frac{1}{(1-t)t}.$$

Then

$$f(t, x, y) \leqslant p^*(t) + q^*(t)(x + |y|), \quad (t, x, y) \in [a, \sigma(b)] \times [0, \infty) \times R.$$

We compute

$$\alpha = \gamma = 1, \quad \beta = \delta = 0, \quad C^* = 1, \quad \xi^* = \frac{1}{4}, \quad \omega^* = \frac{3}{4}, \quad t_0^* = \frac{1}{4},$$

$$\int_a^{\sigma(b)} G(\sigma(s), s) m(s) \mathrm{d}s = 1, \quad M_1^* = \frac{22\mathrm{e}\sqrt{3}}{\ln 3}, \quad M_2^* = \frac{1}{4}, \quad M^* = 3.$$

It is easy to testify that all conditions of Theorem 3.1 hold. Therefore, Theorem 3.1 implies the existence of positive solutions of BVP (3.14).

Remark 3.3. Main results in [1,6,8,10,11,14,17,18] do not apply to Examples 3.3 because functions *f* in those papers do not contain the term $x^{\Delta}(t)$.

Now let us end this paper by the following Appendix.

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Appendix

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$, the set of real numbers, with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. An alternative terminology for time scale is measure chain.

The theory of time scale was introduced and developed by Aulbach and Hilger [2] to unify continuous and discrete analysis. Now, there have been many publications (see [4,5,12,15]) relating difference equations with differential equations. A result for a dynamic equation contains simultaneously a corresponding result for a differential equation, one for a difference equation, as well as results for other dynamic equations in arbitrary time scales. Time scales theory present us with the tools necessary to understand and explain the mathematical structure underpinning the theories of discrete and continuous dynamical systems and allows us to connect them. That is certainly the goal with this work. In fact, the potential impact of dynamic equations on time scales in applications is showcased in a recent article in *New Scientist* magazine (see [19]). The following definitions on time scales can be found in papers [2–4,9,12,15].

Definition 4. Define the interval in $\mathbb{T}$

$$[a, b] := \{t \in \mathbb{T} \text{ such that } a \leqslant t \leqslant b\}.$$

Open intervals and half-open intervals etc. are defined accordingly.

Definition 5. A time scale may or may not be connected, so we define the forward jump operator and backward jump operator σ, ρ by

$$\sigma(t) := \inf\{\tau > t; \tau \in \mathbb{T}\} \in \mathbb{T}, \quad \rho(t) := \sup\{\tau < t; \tau \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$ with $t < \sup \mathbb{T}$.

In this definition, we put $\inf \phi = \sup \mathbb{T}$ (*i.e.*, $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \phi = \inf \mathbb{T}$ (*i.e.*, $\rho(m) = m$ if $\mathbb{T}$ has a minimum m), where ϕ denotes the empty set. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say that t is left scattered. Points that are right scattered and left scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense are called dense. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k := \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k := \mathbb{T}$.

Definition 6. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^k$. Then we define $x^\triangle(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\|[x(\sigma(t)) - x(s)] - x^\triangle(t)[\sigma(t) - s]\| \leqslant \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^\triangle(t)$ the delta (or Hilger) derivative of $x(t)$ at $t \in \mathbb{T}$. The derivative can also be defined in terms of a limit as follows:

$$x^\triangle(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{x(\sigma(s)) - x(t)}{\sigma(s) - t} = \lim_{s \to t, \sigma(t) \neq s} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s}.$$

The second derivative of $x(t)$ is defined by $x^{\triangle\triangle}(t) = (x^\triangle)^\triangle(t)$.

It is obvious that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}^k$ and t is right scattered, then

$$x^\triangle(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, the set of integers, then it follows that $x^\triangle(t) = \triangle x(t) := x(t+1) - x(t)$. In particular, if $\mathbb{T} = \mathbb{R}$, we find $\sigma(s) = s$ for all $s \in \mathbb{T}$. Thus $x^\triangle(t)$ reduces to the usual derivative

$$x'(t) = \lim_{s \to t} \frac{x(s) - x(t)}{s - t}.$$

Definition 7. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\triangle(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. In this case we define the integral of f by $\int_a^t f(\tau)\triangle\tau = F(t) - F(a)$.

Further property of this integral can be seen in Bohner and Guseinov [3].

Definition 8. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Remark. Hilger [13, p.2688, line 8] shows that, for the usual time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, rd-continuity coincides with continuity. So main results of this paper are new also when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.

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