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Boundary value problems involving upper and lower solutions in reverse order[☆]

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ABSTRACT

In this paper, we discuss the existence of extreme solutions of the boundary value problem for a class of first-order functional equations with a nonlinear boundary condition. In the presence of a lower solution α and an upper solution β with $\beta \leq \alpha$, we establish existence results of extreme solutions by using the method of upper and lower solutions and a monotone iterative technique.

1. Introduction

We are concerned with the existence of a solution of the boundary value for functional differential equation

$$\begin{cases} u'(t) = f(t, u(t), u(\theta(t))), & t \in J, \\ g(u(0)) = u(T), \end{cases} \tag{1.1}$$

where $J = [0, T], f \in C(J \times R^2, R), \theta \in C(J, J)$. By a solution of (1.1), we mean a function $x \in C^1(J, R)$ that satisfies (1.1).

Monotone iterative techniques coupled with the method of upper and lower solutions have been widely used in the study of value boundary problems for nonlinear differential equations in recent years, see [5,8,9,11,12,14] and the reference therein. In 2003, Nieto and Lopez [13] considered the existence of extreme solutions of special case of (1.1)

$$\begin{cases} u'(t) = h(t, u(t), u(\theta(t))), & t \in J, \\ u(0) = u(T), \end{cases} \tag{1.2}$$

where $h \in C(J \times R^2, R)$ and $\theta \in C(J, R), 0 \leq \theta(t) \leq t$. They introduced a new concept of lower and upper solutions and obtained an existence result of extreme solutions in the presence of a lower solution α and an upper solution β with the classical condition $\alpha \leq \beta$ on J. Also, a similar method has already succeeded in being employed in nonlinear impulsive integro-differential equations and impulsive functional differential equations [4,6,10].

In all the quoted papers, the usual order for the lower and upper solutions is considered. Recently, many researchers have considered existence results for the non-ordered case [1–3,7]. To the best of the author's knowledge, no scholar has

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considered boundary value problems for functional differential equations with nonlinear boundary conditions under the assumption of existing upper and lower solutions in the reverse order.

In this paper, we discuss the existence of extreme solutions of (1.1) under the condition of lower and upper solutions with reverse order. The paper is organized as follows. In Section 2, we establish several comparison principles. In Section 3, we consider a linear problem associated with Eq. (1.1). In Section 4, by using the method of upper and lower solutions and the monotone iterative technique, we obtain the existence of extreme solutions of Eq. (1.1).

2. Preliminaries and comparison principle

In the following, we denote

$$c(t) = 1 - \sin\frac{\pi t}{2T}, \qquad E = C^1(J, R).$$

Throughout this paper, we always assume that the following condition is satisfied:

(H) The constants $M > 0, N \geq 0, g \in C^1(R, R), g(0) \leq 0$ and there exists a $r > 0$ such that

$$r \leq g'(t) < \left(1 + N\int_0^T e^{M(\theta(t)-t)}dt\right)e^{MT}, \tag{2.1}$$

$$(M+N)T \leq \frac{r}{1+r}. \tag{2.2}$$

Remark. When (2.1) is satisfied, the reverse of g exists since the function g is strictly increasing.

Definition 2.1. Functions $\alpha, \beta \in E$ are called lower solution and upper solutions of Eq. (1.1) if

$$\alpha'(t) \leq f(t, \alpha(t)), \alpha(\theta(t)) + a_\alpha(t), \quad t \in J,$$
$$\beta'(t) \geq f(t, \beta(t)), \beta(\theta(t)) - b_\beta(t), \quad t \in J,$$

where

$$a_\alpha(t) = \begin{cases} 0, & g(\alpha(0)) \leq \alpha(T), \\ (c'(t) - Mc(t) - Nc(\theta(t)))(\alpha(0) - G(\alpha(T))), & g(\alpha(0)) \geq \alpha(T), \end{cases}$$

$$b_\beta(t) = \begin{cases} 0, & g(\beta(0)) \geq \beta(T), \\ (c'(t) - Mc(t) - Nc(\theta(t)))(G(\beta(T)) - \beta(0)), & g(\beta(0)) \leq \beta(T), \end{cases}$$

here G denotes the reverse of g.

We present the main results of this section.

Theorem 2.1. *Assume that $u \in E$ satisfies*

$$\begin{cases} u'(t) \geq Mu(t) + Nu(\theta(t)), & t \in J, \\ g(u(0)) \geq u(T), \end{cases} \tag{2.3}$$

then $u(t) \leq 0$ for $t \in J$.

Proof. Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$. We consider the following two possible cases.

Case 1. $u(t) \geq 0$ for all $t \in J$. Let $x(t) = e^{-Mt}u(t)$, then $x(t) \geq 0$ for any $t \in J$ and $x(t)$ is nondecreasing on J,

$$\begin{cases} x'(t) \geq Ne^{M(\theta(t)-t)}x(\theta(t)) \geq Ne^{M(\theta(t)-t)}x(0), & t \in J, \\ g(x(0)) \geq e^{MT}x(T). \end{cases} \tag{2.4}$$

Integrating the above inequality from 0 to t, we have

$$x(t) \geq \left(1 + N\int_0^t e^{M(\theta(s)-s)}ds\right)x(0).$$

Hence,

$$\left(1 + N\int_0^T e^{M(\theta(t)-t)}dt\right)e^{MT}x(0) \leq g(x(0)) = g'(\xi)x(0) + g(0) \leq g'(\xi)x(0),$$

where ξ is between $x(0)$ and 0. The condition (2.1) implies that $x(0) = 0$. Since $0 \leq x(T) \leq g(x(0)) \leq 0$ and $x(t)$ is nondecreasing on J, $x(t) \equiv 0$ on J. Thus $u(t) \equiv 0$.

Case 2. There exist $t_1, t_2 \in J$ such that $u(t_1) > 0$ and $u(t_2) < 0$. Put $u(t_0) = \min_{t\in J} u(t) = -\lambda$, then $\lambda > 0$ and

$$u'(t) \geq -\lambda(M+N), \quad t \in J.$$

We claim that $u(0) \le -\lambda + \lambda(M+N)T$. In fact, if $t_0 = 0$, then $u(0) \le -\lambda + \lambda(M+N)T$, if $t_0 > 0$, then there exist a t_* such that $u(0) = u(t_0) - u'(t_*)t_0 \le -\lambda + \lambda(M+N)T$. On the other hand, there is $t^* \in [t_1, T]$ such that

$$u(T) = u(t_1) + u'(t^*)(T - t^*) > -\lambda(M+N)T.$$

Hence,

$$-\lambda + \lambda(M+N)T > G(-\lambda(M+N)T) = G(0) - G'(\rho)\lambda(M+N)T,$$

where $-\lambda(M+N)T \le \rho \le 0$. Noting that $G(0) \ge 0$ and $0 < G' \le r^{-1}$, we have

$$r[-\lambda + \lambda(M+N)T] > -\lambda(M+N)T,$$

that is, $(r+1)(M+N)T > r$, a contradiction. The proof is complete. □

Corollary 2.1. *Assume that $u \in E$ satisfies*

$$\begin{cases} u'(t) \ge Mu(t) + Nu(\theta(t)) - (c'(t) - Mc(t) - Nc(\theta(t)))(G(u(T)) - u(0)), & t \in J, \\ g(u(0)) < u(T), \end{cases}$$

then $u(t) \le 0$ for $t \in J$.

Proof. Put

$$y(t) = u(t) + c(t)(G(u(T)) - u(0)), \quad t \in J,$$

then $y(t) \ge u(t)$ for all $t \in J$ and $y(0) = G(u(T))$, $y(T) = u(T)$,

$$y'(t) - My(t) - Ny(\theta(t)) = u'(t) - Mu(t) - Ny(\theta(t)) + [c'(t) - Mc(t) - Nc(\theta(t))](G(u(T)) - u(0)) \ge 0.$$

By Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$. This ends the proof. □

3. Linear problem

Consider a boundary value problem for the linear differential equation

$$\begin{cases} u'(t) - Mu(t) - Nu(\theta(t)) = \delta(t), & t \in J, \\ g(u(0)) = u(T), \end{cases} \tag{3.1}$$

where $\delta \in C(J)$.

Theorem 3.1. *Assume that there exist $\alpha, \beta \in E$ which are lower and upper solutions of (3.1) and $\beta \le \alpha$ on J. Then there exists one unique solution u to problem (3.1) and $\beta \le u \le \alpha$ on J.*

Proof. First, we show that the solution of Eq. (3.1) is unique. Let u_1, u_2 be the solutions of (3.1) and set $v = u_1 - u_2$. Then,

$$\begin{cases} v'(t) - Mv(t) - Nv(\theta(t)) = 0, & t \in J, \\ v(T) = g(u_1(0)) - g(u_2(0)) = g'(\tau_1)v(0), \end{cases}$$

where τ_1 is between $u_1(0)$ and $u_2(0)$. Noting that $r \le g'(\tau_1)$, by Theorem 2.1, we have that $v \le 0$ for $t \in J$, that is, $u_1 \le u_2$ on J. Similarly, one can obtain $u_2 \le u_1$ on J. Hence $u_1 = u_2$.

Next, we prove that if u is a solution of Eq. (3.1), then $\beta \le u \le \alpha$. Let $m = u - \alpha$.

If $g(\alpha(0)) \le \alpha(T)$, then $a_\alpha(t) = 0$ on J. So we have

$$\begin{cases} m'(t) - Mm(t) - Nm(\theta(t)) \ge 0, & t \in J, \\ m(T) \le g(u(0)) - g(\alpha(0)) = g'(\tau_2)m(0), \end{cases}$$

where τ_2 is between $u(0)$ and $\alpha(0)$. By Theorem 2.1, we have that $m = u - \alpha \le 0$ on J.

If $g(\alpha(0)) \ge \alpha(T)$, then $a_\alpha(t) = (c'(t) - Mc(t) - Nc(\theta(t)))(\alpha(0) - G(\alpha(T)))$. Thus,

$$\begin{aligned} m'(t) - Mm(t) - Nm(\theta(t)) &= u'(t) - Mu(t) - Nu(\theta(t)) - \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \\ &\ge -(c'(t) - Mc(t) - Nc(\theta(t)))(\alpha(0) - G(\alpha(T))) \\ &= -(c'(t) - Mc(t) - Nc(\theta(t)))(\alpha(0) - G(\alpha(T)) - u(0) + G(u(T))) \\ &= -(c'(t) - Mc(t) - Nc(\theta(t)))(G(u(T)) - G(\alpha(T)) - m(0)) \\ &= -(c'(t) - Mc(t) - Nc(\theta(t)))\left(\frac{1}{g'(\tau_3)}m(T) - m(0)\right), \end{aligned}$$

where τ_3 is between $u(T)$ and $\alpha(T)$. Noting that

$$m(0) \leq G(u(T)) - G(\alpha(T)) = \frac{1}{g'(\tau_3)} m(T),$$

by Corollary 2.1, we have that $u \leq \alpha$ on J. Analogously, $u \geq \beta$ on J.

Finally, we show that Eq. (3.1) has a solution by some steps as follows.

Step 1. Let

$$p(t) = \begin{cases} \alpha(t), & g(\alpha(0)) \leq \alpha(T), \\ \alpha(t) - c(t)(\alpha(0) - G(\alpha(T))), & g(\alpha(0)) \geq \alpha(T), \end{cases}$$

$$q(t) = \begin{cases} \beta(t), & g(\beta(0)) \geq \beta(T), \\ \beta(t) + c(t)(G(\beta(T)) - \beta(0)), & g(\beta(0)) \leq \beta(T). \end{cases}$$

We shall show that p, q are the lower and upper solutions of (3.1) respectively, and

$$\beta \leq q \leq p \leq \alpha \quad \text{for } t \in J. \tag{3.2}$$

Obviously, $g(p(0)) \leq p(T)$, $g(q(0)) \geq q(T)$,

$$p'(t) - Mp(t) - Np(\theta(t)) \leq \delta(t), \quad t \in J, \tag{3.3}$$

$$q'(t) - Mq(t) - Nq(\theta(t)) \geq \delta(t), \quad t \in J. \tag{3.4}$$

It is easy to see that $\beta \leq p$ and $q \leq \alpha$ on J. Let $m = q - p$, then $m(T) \leq g(q(0)) - g(p(0)) = g'(\eta)m(0)$ and

$$m'(t) - Mm(t) - Nm(\theta(t)) \geq 0, \quad t \in J, \tag{3.5}$$

where η is between $p(0)$ and $q(0)$. By Theorem 2.1, we have that $q \leq p$ on J. Thus (3.2) holds.

Step 2. We consider the equation

$$\begin{cases} u'(t) - Mu(t) - Nu(\theta(t)) = \delta(t), & t \in J, \\ u(T) = \lambda, \end{cases} \tag{3.6}$$

where $\lambda \in R$. We show that Eq. (3.6) has a unique solution $u(t, \lambda)$ and $u(t, \lambda)$ is continuous in λ.

Define a mapping $A : C(J) \to C(J)$ by

$$Au(t) = \lambda - \int_t^T [\delta(s) + Mu(s) + Nu(\theta(s))]\mathrm{d}s.$$

Clearly, the fixed point of A in $C(J)$ is a solution of (3.6) and $C(J)$ with norm $\|u\| = \sup_{t\in J} |u(t)|$ is a Banach space. For any $x, y \in C(J)$, we have

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| &\leq \int_0^T [M|y(s) - x(s)| + N|x(\theta(s)) - y(\theta(t))|]\mathrm{d}s \\ &\leq (M + N)T\|x - y\|. \end{aligned}$$

The condition (2.2) implies that $A : C(J) \to C(J)$ is a contraction mapping. Thus there exists a $u \in C(J)$ such that $Au = u$. The Eq. (3.6) has a unique solution.

Let $u(t, \lambda_1)$, $u(t, \lambda_2)$ be the solution of

$$\begin{cases} u'(t) - Mu(t) - Nu(\theta(t)) = \delta(t), & t \in J, \\ u(T) = \lambda_i, \quad i = 1, 2, \end{cases} \tag{3.7}$$

then

$$u(t, \lambda_i) = \lambda_i - \int_t^T [\delta(s) - Mu(s, \lambda_i) - Nu(\theta(s), \lambda_i)]\mathrm{d}s, \quad i = 1, 2,$$

$$\max_{t\in J} |u(t, \lambda_1) - u(t, \lambda_2)| \leq \frac{1}{1 - (M + N)T} |\lambda_1 - \lambda_2|.$$

Step 3. We show that

$$q(0) \leq u(0, \lambda) \leq p(0) \tag{3.8}$$

for any $\lambda \in [g(q(0)), g(p(0))]$, where $u(t, \lambda)$ is the unique solution of (3.6).

Let $m(t) = u(t,\lambda) - p(t)$. Suppose that $u(0,\lambda) > p(0)$, then $m(0) = u(0,\lambda) - p(0) > 0$, $m(T) = u(T,\lambda) - p(T) \le u(T,\lambda) - g(p(0)) \le 0$ and

$$m'(t) - Mm(t) - Nm(\theta(t)) \ge 0.$$

By Theorem 2.1, we obtain that $m(t) \le 0$ for all $t \in J$, which is a contradiction.

Let $n(t) = q(t) - u(t,\lambda)$. Suppose that $u(0,\lambda) < q(0)$, then $n(0) > 0$, $n(T) = q(T) - u(T,\lambda) \le g(q(0)) - u(T,\lambda) \le 0$ and

$$n'(t) - Mn(t) - Nn(\theta(t)) \ge 0.$$

By Theorem 2.1, we obtain that $n(t) \le 0$ for all $t \in J$, which is a contradiction.

Step 4. Let $P(\lambda) = g(u(0,\lambda)) - \lambda$, where $u(t,\lambda)$ is the unique solution of (3.6). From step 3, we have

$$P(g(q(0)))P(g(p(0))) \le 0.$$

Since P is continuous in λ, it follows that there exists a $\lambda_0 \in [g(q(0)), g(p(0))]$ such that $g(u(0,\lambda_0)) = \lambda_0$. Obviously, $u(t,\lambda_0)$ is the unique solution of (3.1). This ends the proof. $\square$

4. Main result

Our main result is the following theorem.

Theorem 4.1. *Assume that the following conditions are satisfied*

(i) *α, β are lower and upper solutions for the boundary value problem (1.1) respectively, $\beta(t) \le \alpha(t)$ on J.*

(ii) *The constants $M > 0$, $N \ge 0$ in the Definition 2.1 satisfy*

$$f(t,x,y) - f(t,\bar{x},\bar{y},) \le M(x-\bar{x}) + N(y-\bar{y}),$$

for $\beta(t) \le \bar{x} \le x \le \alpha(t)$, $\beta(\theta(t)) \le \bar{y} \le y \le \alpha(\theta(t))$, $t \in J$.

Then, there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that

$$\lim_{n\to\infty} \alpha_n(t) = e(t), \qquad \lim_{n\to\infty} \beta_n(t) = r(t)$$

uniformly on J, and e, r are the maximal and the minimal solutions of (1.1) on $[\beta,\alpha] = \{u \in C^1(J,R) : \beta(t) \le x(t) \le \alpha(t)\}$.

Proof. For any $\gamma \in [\beta,\alpha]$, we consider the equation

$$\begin{cases} u'(t) - Mu(t) - Nu(\theta(t)) = f(t,\gamma(t),\gamma(\theta(t))) - M\gamma(t) - N\gamma(\theta(t)), & t \in J, \\ g(u(0)) = u(T). \end{cases} \tag{4.1}$$

Since α, β are lower and upper solutions of (1.1), by (ii), we have that

$$\begin{aligned} \alpha'(t) - M\alpha(t) - N\alpha(\theta(t)) &\le f(t,\alpha(t),\alpha(\theta(t))) - M\alpha(t) - N\alpha(\theta(t)) + a_\alpha(t) \\ &\le f(t,\gamma(t),\gamma(\theta(t))) - M\gamma(t) - N\gamma(\theta(t)) + a_\alpha(t) \\ \beta'(t) - M\beta(t) - N\beta(\theta(t)) &\ge f(t,\beta(t),\beta(\theta(t))) - M\beta(t) - N\beta(\theta(t)) - b_\beta(t) \\ &\ge f(t,\gamma(t),\gamma(\theta(t))) - M\gamma(t) - N\gamma(\theta(t)) - b_\beta(t). \end{aligned}$$

Therefore, α, β are lower and upper solutions of (4.1), by Theorem 3.1, the Eq. (4.1) has a unique solution $u \in E$. We define an operator T by $u = T\gamma$, then T is an operator from $[\beta,\alpha]$ to $[\beta,\alpha]$. Hence, $\alpha \ge T\alpha$, $T\beta \ge \beta$.

Next, we show that $T\mu_1 \le T\mu_2$ if $\beta \le \mu_1 \le \mu_2 \le \alpha$.

Let $m = T\mu_1 - T\mu_2$, then by (ii) and (4.1), we have

$$\begin{aligned} m'(t) - Mm(t) - Nm(\theta(t)) &= f(t,\mu_1(t),\mu_1(\theta(t))) - f(t,\mu_2(t),\mu_2 u(\theta(t))) \\ &\quad - M(\mu_1(t) - \mu_2(t)) - N(\mu_1(\theta(t)) - \mu_2(\theta(t))) \\ &\ge 0 \end{aligned}$$

and

$$m(T) = \mu_1(T) - \mu_2(T) = g(\mu_1(0)) - g(\mu_2(0)) = g'(\varsigma)m(0),$$

where ς is between $\mu_1(0)$ and $\mu_2(0)$. By Theorem 2.1, $m(t) \le 0$, which implies $T\mu_1 \le T\mu_2$. Hence, T is nondecreasing in $[\beta,\alpha]$.

Define the sequences $\{\alpha_n\}_{n\in N}, \{\beta_n\}_{n\in N}$ such that $\alpha_n = T\alpha_{n-1}$, $\beta_n = T\beta_{n-1}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ for $n \in N = \{1, 2, 3, \ldots\}$, then

$$\beta_1 \le \beta_2 \le \cdots \le \beta_n \le \alpha_n \le \cdots \le \alpha_2 \le \alpha_1$$

on $t \in J$, and each $\alpha_n, \beta_n \in E$ satisfies

$$\begin{cases} \alpha_n'(t) - M\alpha_n(t) - N\alpha_n(\theta(t)) = f(t, \alpha_{n-1}(t), \alpha_{n-1}(\theta(t))) - M\alpha_{n-1}(t) - N\alpha_{n-1}(\theta(t)), \quad t \in J, \\ g(\alpha_n(0)) = \alpha_n(T) \end{cases}$$

and

$$\begin{cases} \beta_n'(t) - M\beta_n(t) - N\beta_n(\theta(t)) = f(t, \beta_{n-1}(t), \beta_{n-1}(\theta(t))) - M\beta_{n-1}(t) - N\beta_{n-1}(\theta(t)), \quad t \in J, \\ g(\beta_n(0)) = \beta_n(T). \end{cases}$$

Therefore there exist e, r such that

$$\lim_{n\to\infty} \alpha_n(t) = e(t), \qquad \lim_{n\to\infty} \beta_n(t) = r(t)$$

uniformly on J. Clearly, e, r are solutions of (1.1).

Finally, we prove that if $x \in [\beta, \alpha]$ is any solution of (1.1), then $r(t) \le x(t) \le e(t)$ on J. Since $\beta(t) \le x(t) \le \alpha(t)$ and $x = Tx$, we easily obtain by the fact that T is nondecreasing in $[\beta, \alpha]$ that

$$\beta_1(t) \le x(t) \le \alpha_1(t), \quad t \in J.$$

Hence, we can conclude that

$$\beta_n(t) \le x(t) \le \alpha_n(t) \quad \text{for all } n.$$

Passing to the limit as $n \to \infty$, we obtain that $r(t) \le x(t) \le e(t),\ t \in J$. This ends the proof. □

Example 1. Consider the equation

$$\begin{cases} x' = \dfrac{1 + x^2(t)}{10} + \dfrac{2}{5}\sin x(0.1t), \quad 0 \le t \le 0.25, \\ \dfrac{1}{2}x(0) - \dfrac{1}{4}\sin x(0) = x(0.25). \end{cases} \tag{4.2}$$

Clearly, $\alpha = 0$ and $\beta = -1$ are lower and upper solutions of (4.2) respectively. Let $M = 1/5$, $N = 2/5$, then the condition of Theorem 4.1 is satisfied. Hence, (4.2) has maximal and minimal solutions on $[-1, 0]$.

Example 2. Consider the equation

$$\begin{cases} x' = \dfrac{3x(1-t) - x(t)}{20}, \quad 0 \le t \le 1, \\ x(0) - p = x(1), \end{cases} \tag{4.3}$$

where $p > 0$ is a constant.

Let $M = 0.01$ and $N = 0.15$. Clearly, $\alpha = 0$ is a lower solution of (4.3). Put $\beta = -20p$. Noting that $\beta(0) - p < \beta(1)$, we have that $G(\beta(1)) - \beta(0) = p$,

$$\begin{aligned} \beta' &= 0 > \frac{3\beta(1-t) - \beta(t)}{20} - (c'(t) - Mc(t) - Nc(\theta(t)))(G(\beta(T)) - \beta(0)) \\ &= \left(\frac{\pi}{2}\cos\frac{\pi t}{2} + Mc(t) + Nc(\theta(t)) - 2\right)p. \end{aligned}$$

So $\beta = -20p$ is an upper solution of (4.3). By Theorem 4.1, (4.3) has maximal and minimal solutions on $[-20p, 0]$.

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