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Stability of linear multistep methods for nonlinear neutral delay differential equations in Banach space ☆

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ABSTRACT

This paper is devoted to investigating the nonlinear stability properties of linear multistep methods for the solution to neutral delay differential equations in Banach space. Two approaches to numerically treating the "neutral term" are considered, which allow us to prove several results on numerical stability of linear multistep methods. These results provide some criteria for choosing the step size such that the numerical method is stable. Some examples of application and a numerical experiment, which further confirms the main results, are given.

1. Introduction

Neutral delay differential equations (NDDEs) have found applications in many areas of science (see, e.g., [1–3]). A multitude of papers have been devoted to the linear stability of numerical methods in the last few decades (see, e.g., [4–10]). Recently, the nonlinear stability of theoretical solutions and numerical solutions to nonlinear NDDEs with different forms in Hilbert space has been studied by several authors. In 2000, using a one-sided Lipschitz condition and some classical Lipschitz conditions, Bellen, Guglielmi and Zennaro [11] discussed the contractivity and asymptotic stability of Runge–Kutta methods for nonlinear NDDEs:

$$\begin{cases} y'(t) = f(t, y(t), G(t, y(t-\tau(t)), y'(t-\tau(t)))), & t \geq t_0, \\ y(t) = \phi(t), & t \leq t_0. \end{cases} \tag{1.1}$$

Following this paper, the nonlinear stability of numerical methods for NDDEs of the "Hale' form" [12,13] and for general NDDEs [14–19] has been examined.

On the other hand, in order to surmount the restriction of the inner product norm, Nevanlinna and Liniger [20] first, in 1979, considered an ODE test problem and studied the nonlinear stability of one-leg methods applied to ODEs in Banach space. In 1983, Vanselow [21] researched the stability of linear multistep methods for classes $K1$, $K2\lambda^*$ and $K3\mu$ in Banach space. In 1987, Shoufu Li [22,23] introduced the test problem class $K(\alpha, \lambda^*)$ of stiff ODEs in Banach space and obtained a series of stability results for numerical methods, including linear multistep methods and explicit and diagonal implicit Runge–Kutta methods, for the test problem class $K(\alpha, \lambda^*)$ $(\alpha \leq 0)$. Recently, Wen et al. [24,25] have extended these results

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to the test problem class $D(\alpha, \beta, \lambda^*)$ of stiff delay differential equations (DDEs) and obtained some stability results for numerical methods ($\alpha \le 0$). It is natural to study the stability of the theoretical and numerical solutions to nonlinear NDDEs in Banach space. The nonlinear stability properties of θ-methods for NDDEs with constant delays and explicit and diagonal implicit Runge–Kutta methods for NDDEs with general variable delay in Banach space are studied in the papers [26,27], respectively. In this paper, we consider the stability properties of a class of linear multistep methods for nonlinear NDDEs with general variable delay.

We follow the approach designed by Shoufu Li for ODEs. In particular, Lemma 4.1 has its analogue in [28], with major changes in the proof. The main obstacle to our development is the numerical treatment of the "neutral term", resulting in direct evaluation and interpolation approximation for the "neutral term". This makes many of the arguments essentially different compared with the case of ODEs and DDEs. As a reward, we obtain some novel numerical results on stiff ODEs with $\alpha > 0$ and stiff DDEs with $\alpha > 0$.

The main results are proved in Section 4. These results provide some criteria for choosing the step size. The preceding sections give some basic concepts, including the problem classes $D(\alpha, \beta, \gamma, L, \lambda^*)$, $D(\alpha, \beta, \gamma, \sigma, \lambda^*)$ introduced in Section 2 and the variable coefficient linear multistep methods considered in Section 3. In Section 5, we give some examples, which describe how the main results are applied to practical problems, and numerical experiments, which further verify the main results.

2. Test problems

Let X be a real Banach space with the norm $\|\cdot\|$, D be a infinite subset of X, $T > 0$ be a constant.

2.1. ODE and DDE test problems

Definition 2.1 (*[28]*). Let α, λ^* be real constants, and $\lambda^* \le 0$, $1 + \alpha\lambda^* > 0$. The class of all problems

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [0, T], \\ y(0) = y_0, \end{cases} \tag{2.1}$$

with continuous mapping $f : [0, T] \times D \to X$ satisfying the following condition:

$$[1 - \alpha(\lambda - \lambda^*)]G(\lambda^*) \le G(\lambda), \quad \forall \lambda \ge 0, y_1, y_2 \in D, t \in [0, T], \tag{2.2}$$

is denoted by $D(\alpha, \lambda^*)$, where

$$G(\lambda) := G_{y_1,y_2,t,f}(\lambda) = \|y_1 - y_2 - \lambda[f(t, y_1) - f(t, y_2)]\|, \quad \lambda \in R.$$

The function $G(\lambda)$ appears in the investigation into the logarithmic norm of $\partial f/\partial y$ and plays an important role in the stability analysis of solutions to ODEs in Banach space. The numerical counterpart also plays a key role in numerical stability analysis for ODEs in Banach space (cf. [29]).

We note that the constants α and λ^* depend on the norm of the Banach space and our sufficient conditions for stability depend on these constants. Consequently, our sufficient conditions for stability depend on the norm of the Banach space. However, we know that the stability of a system does not depend on the norm of the Banach space. Therefore, for a stable system, with our sufficient conditions we do not confirm the stability of the system under a norm but can do under another norm. Alternatively, one can choose an appropriate norm for analyzing the stability of a system. This is an important reason for studying the stability of the system in Banach space. We also observe that the stability result obtained with our sufficient conditions based on the constants α and λ^* may be different under two equivalent norms in a finite dimensional space.

Definition 2.2 (*[24]*). Let α, β, λ^* be real constants, and $\lambda^* \le 0$, $1 + \alpha\lambda^* > 0$. The class of all problems

$$\begin{cases} y'(t) = f(t, y(t), y(t - \tau(t))), & t \in [0, T], \\ y(t) = \phi(t), & t \in [t_{-1}, 0], \end{cases} \tag{2.3}$$

where $t_{-1} = \inf_{t \in [0,T]}\{t - \tau(t)\}$, with continuous mapping $f : [0, T] \times D \times D \to X$ satisfying the following conditions:

$$[1 - \alpha(\lambda - \lambda^*)]G(\lambda^*) \le G(\lambda), \quad \forall \lambda \ge 0, y_1, y_2, u \in D, t \in [0, T], \tag{2.4}$$

$$\|f(t, y, u_1) - f(t, y, u_2)\| \le \beta\|u_1 - u_2\|, \quad \forall y, u_1, u_2 \in D, t \in [0, T], \tag{2.5}$$

is denoted by $D(\alpha, \beta, \lambda^*)$, where

$$G(\lambda) := G_{y_1,y_2,u,t,f}(\lambda) = \|y_1 - y_2 - \lambda[f(t, y_1, u) - f(t, y_2, u)]\|, \quad \lambda \in R.$$

As stated in the Introduction, the stability properties of numerical methods applied to the problem classes $D(\alpha, \lambda^*)$ and $D(\alpha, \beta, \lambda^*)$ with $\alpha \le 0$ have been investigated by several authors (see, e.g., [22,23,28,24,25]). However, for many practical problems, we have $\alpha > 0$.

2.2. Examples of stiff ODEs and stiff DDEs ($\alpha > 0$)

Example 2.1. Consider the classical stiff ODEs [30]

$$\begin{cases} y_1' = -0.04y_1 + 10^4 y_2 y_3, & y_1(0) = 1, \\ y_2' = 0.04y_1 - 10^4 y_2 y_3 - 3\cdot 10^7 y_2^2, & y_2(0) = 0, \\ y_3' = 3\cdot 10^7 y_2^2, & y_3(0) = 0. \end{cases} \tag{2.6}$$

For the problem, when we consider the 1-*norm*, we have $\alpha = 2\cdot 10^4|y_2|$. "We observe that the solution y_2 rapidly reaches a quasi-stationary position in the vicinity of $y_2' = 0$, which in the beginning ($y_1 = 1, y_3 = 0$) is at $0.04 \approx 3\cdot 10^7 y_2^2$, hence $y_2 \approx 3.65\cdot 10^{-5}$, and then very slowly goes back to zero again" (from [30], pp. 3–4). Therefore, we have $\alpha > 0$.

From this example, we know that there exist some practical problems with $\alpha > 0$. The fact that the constants α and λ^* depend on the norm of the Banach space is further verified.

Example 2.2. As an example of stiff DDEs with $\alpha > 0$, consider

$$\begin{cases} y_1'(t) = -2y_1(t) + (m + 3\mathrm{e}^{-t})y_2(t) + y_2^2(t) - 0.125y_1(t-1), \\ y_2'(t) = -(2+m)y_2(t) - y_2^2(t) - 0.125y_2(t-1), \end{cases} \tag{2.7}$$

where $t \geq 0$, and $m \gg 0$. From the second equation of (2.7) we observe that the solution y_2 rapidly decays and then from the first equation of (2.7) we observe that the solution y_1 very slowly goes back to zero after reaching a peak. Moreover, the classical Lipschitz constant of the *Jacobi* matrix

$$J = \frac{\partial f}{\partial y} = \begin{pmatrix} -2 & m + 3\mathrm{e}^{-t} + 2y_2(t) \\ 0 & -(2 + m + 2y_2(t)) \end{pmatrix}$$

is very large. Thus, we think that this problem is stiff. On the other hand, the logarithmic norm induced by standard inner product of the *Jacobi* matrix J is

$$\mu_2(J) = \frac{1}{2}[\sqrt{(m + 2y_2(t))^2 + (m + 3\mathrm{e}^{-t} + 2y_2(t))^2} - m - 2y_2(t)] - 2 \gg 0.$$

But in the 1-*norm*, we have $\alpha = \mu_1(J) = 1$.

This example also illustrates the necessity of investigation into the stability of the system in Banach space.

2.3. NDDE test problems

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau(t)), y'(t-\tau(t))), & t \in [0, T], \\ y(t) = \phi(t), & t \in [t_{-1}, 0], \end{cases} \tag{2.8}$$

where $t_{-1} = \inf_{t\in[0,T]}\{t - \tau(t)\}$, $f : [0, T] \times D \times D \times D \to X$ is a given continuous mapping, $\tau(\cdot)$ is continuous, and $\phi(\cdot)$ is differentiable on its domain of definition. Conditions will be imposed later upon f, and the existence of a unique solution of (2.8) will be assumed. We also assume that

$\mathcal{H}1$ $\quad \tau_0 \geq 0$ with $\tau_0 := \inf_{t\in[0,T]}\{\tau(t)\}$.

For any given $y_1, y_2, u, v \in D$, $t \in [0, T]$, a nonnegative function $G(\lambda)$ can be defined from the mapping f:

$$G(\lambda) = G_{y_1,y_2,u,v,t,f}(\lambda) = \|y_1 - y_2 - \lambda[f(t, y_1, u, v) - f(t, y_2, u, v)]\|, \quad \lambda \in R. \tag{2.9}$$

Following [28,24], we give the following definition.

Definition 2.3. Let $\alpha, \beta, \gamma, \sigma, L, \lambda^*$ be real constants, and $\lambda^* \leq 0$, $1 + \alpha\lambda^* > 0$. The class of all problems (2.8) with f satisfying the following conditions:

$$[1 - \alpha(\lambda - \lambda^*)]G(\lambda^*) \leq G(\lambda), \tag{2.10}$$

$$\|f(t, y_1, u_1, v_1) - f(t, y_2, u_2, v_2)\| \leq L\|y_1 - y_2\| + \beta\|u_1 - u_2\| + \gamma\|v_1 - v_2\|, \tag{2.11}$$

for any $\lambda \geq 0$, $y_1, y_2, u, u_1, u_2, v, v_1, v_2 \in D$, $t \in [0, T]$, is denoted by $D(\alpha, \beta, \gamma, L, \lambda^*)$. The class of all problems (2.8) with f satisfying the conditions (2.10) and

$$\|f(t, y, u_1, v_1) - f(t, y, u_2, v_2)\| \leq \beta\|u_1 - u_2\| + \gamma\|v_1 - v_2\|, \tag{2.12}$$

$$\|H(t, y, u_1, v, w) - H(t, y, u_2, v, w)\| \leq \sigma\|u_1 - u_2\|, \tag{2.13}$$

for any $y, u_1, u_2, v_1, v_2, w \in D$, $t \in [0, T]$, is denoted by $D(\alpha, \beta, \gamma, \sigma, \lambda^*)$, where

$$H(t, y, u, v, w) := f(t, y, u, f(t-\tau(t), u, v, w)).$$

Remark 2.1. When $\gamma = 0$, the problem classes mentioned in Definition 2.3 have been used as the test problem classes $D(\alpha, \beta, \lambda^*)$ in [24,25] with respect to the nonlinear stability of numerical methods for stiff DDEs in Banach space.

Remark 2.2. The class $K(\alpha, \lambda^*)$ for stiff ODEs introduced in [28] can be viewed as the class $D(\alpha, 0, 0, L, \lambda^*)$ for NDDEs.

Remark 2.3. It should be pointed out that in view of the assumption $\mathscr{H}1$, the problem classes mentioned in Definition 2.3 contain various delay problems, including constant delay and proportional delay.

Below, we collect several results, which are slightly modified to coincide with our terminology, that are required in what follows.

Proposition 2.1 (*[28]*). *Suppose system (2.8) belongs to the class $D(\alpha, \beta, \gamma, L, \lambda^*)$ (or $D(\alpha, \beta, \gamma, \sigma, \lambda^*)$), and real constants λ_1, λ_2 satisfy $\lambda^* \le \lambda_1 \le \lambda_2$, $\alpha\lambda_2 < 1 + \alpha\lambda^*$ and $\lambda_2 \ge 0$. Then*

$$G(\lambda_1) \le \frac{1-\alpha(\lambda_1-\lambda^*)}{1-\alpha(\lambda_2-\lambda^*)} G(\lambda_2), \quad \forall y_1, y_2, u, v \in D, t \in [0, T].$$

Proposition 2.2 (*[28]*). *Let X be a (real or complex) Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Then when $\lambda^* \le 0$ and $1 + \alpha\lambda^* > 0$, condition (2.10) is equivalent to the following condition:*

$$\begin{aligned} Re\langle y_1 - y_2, f(t, y_1, u, v) - f(t, y_2, u, v)\rangle &\le \frac{\alpha(2+\alpha\lambda^*)}{2(1+\alpha\lambda^*)^2}\|y_1 - y_2\|^2 \\ &+ \frac{\lambda^*}{2}\|f(t, y_1, u, v) - f(t, y_2, u, v)\|^2, \quad \forall \lambda \ge 0, y_1, y_2, u, v \in D, t \in [0, T]. \end{aligned}$$

In order to discuss the stability of the theoretical solution and the numerical solution to the nonlinear NDDEs (2.8), we introduce the perturbed problem

$$\begin{cases} z'(t) = f(t, z(t), z(t-\tau(t)), z'(t-\tau(t))), & t \in [0, T], \\ z(t) = \psi(t), & t \in [t_{-1}, 0], \end{cases} \tag{2.14}$$

and assume that the problem (2.14) has a unique true solution $z(t)$. Let

$$C_L = \max\left\{\alpha + \frac{\beta+\gamma L}{1-\gamma}, \alpha+\beta+\gamma\right\}, \qquad C_\sigma = \max\left\{\alpha + \frac{\sigma}{1-\gamma}, \alpha+\beta+\gamma\right\},$$

$$d = \begin{cases} \max\left\{1, \dfrac{\gamma}{|\alpha+\beta|}\right\}, & \alpha+\beta \ne 0, \\ 1, & \alpha+\beta = 0, \end{cases}$$

$$M_1 = \max_{t_{-1}\le s\le 0}\|\phi(s) - \psi(s)\|, \qquad M_2 = \max_{t_{-1}\le s\le 0}\|\phi'(s) - \psi'(s)\|.$$

Then the following stability results of the analytical solution to the problem (2.8) have been given in [31].

Proposition 2.3. *If the problem (2.8) belongs to $D(\alpha, \beta, \gamma, L, 0)$, then:*

(i) *when $C_L > 0$, we have*

$$\|y(t) - z(t)\| \le d\exp(C_L(t-t_0))\max\{M_1, M_2\}, \quad \forall t \in [0, T]; \tag{2.15}$$

(ii) *when $C_L \le 0$, we have*

$$\|y(t) - z(t)\| \le \max\{M_1, M_2\}, \quad \forall t \in [0, T]. \tag{2.16}$$

Remarks 2.4. Even if the functions f and ϕ are sufficiently smooth, the solution is continuous with a discontinuous derivative if the initial function ϕ does not satisfy the necessary and sufficient sewing condition (see, for example, [32])

$$\phi'(0) = f(0, y(0), y(-\tau(0)), y'(-\tau(0))).$$

Remarks 2.5. In general, the solution $y(t)$ of problem (2.8) is continuous with a discontinuous derivative. But the solution $y(t)$ to NDDEs with proportional delay, that is, $t - \tau(t) = qt$, $q \in (0, 1)$, and to NDDEs with $\tau(t) = 0$, that is, implicit ODEs, is sufficiently differentiable if the functions f and ϕ are sufficiently smooth.

Proposition 2.4. *If the problem (2.8) belongs to $D(\alpha, \beta, \gamma, \sigma, 0)$, then:*

(i) *when $C_\sigma > 0$, we have*

$$\|y(t) - z(t)\| \le d \exp(C_\sigma(t - t_0)) \max\{M_1, M_2\}, \quad \forall t \in [0, T];$$

(ii) *when $C_\sigma \le 0$, we have (2.16).*

2.4. Examples of NDDEs

Example 2.3. Slightly modifying the equation

$$\begin{cases} y'(t) = a(t)(y(t) - g(t)) + g'(t), & t \in [0, T], \\ y(0) = g(0), \end{cases} \tag{2.17}$$

which is used as a stiff ODE test (see, e.g., [30,33]), we can obtain NDDEs

$$\begin{cases} y'(t) = a(t)[y(t) - g(t)] + g'(t) + b(t)[y(t-\tau) - g(t-\tau)] + c(t)[y'(t-\tau) - g'(t-\tau)], & t \in [0, T] \\ y(t) = g(t), & t \le 0, \end{cases} \tag{2.18}$$

where $\tau > 0$ is a real constant, where $g : R \to R$ is a given smooth function, $a, b, c : R \to R$ are given and continuous, and $a(t) \le 0$ whenever $t \in [0, T]$. It is easily obtained by simple calculation that

$$\alpha = \sup_{t \in [0,T]} a(t), \qquad L = \sup_{t \in [0,T]} |a(t)|, \qquad \beta = \sup_{t \in [0,T]} |b(t)|, \qquad \gamma = \sup_{\in [0,T]} |c(t)|, \qquad \sigma = \sup_{t \in [0,T]} |b(t) + a(t)c(t)|.$$

Thus Eq. (2.18) belongs to classes $D(\alpha, \beta, \gamma, L, 0)$ and $D(\alpha, \beta, \gamma, \sigma, 0)$. Take $a(t) = -50$, $b(t) = 0.5$, $c(t) = -0.4$. Then problem (2.18) satisfies the assumptions of Propositions 2.3 and 2.4 and $\beta + \gamma L = \sigma = 20.5$. If we take $a(t) = -50$, $b(t) = 0.5$, $c(t) = 0.5$, we obtain (2.16) from Proposition 2.4 even though problem (2.18) does not satisfy $\frac{\beta + \gamma L}{-\alpha} + \gamma \le 1$. In particular, if we take $a(t) = -50$, $b(t) = 5$, $c(t) = 0.1$, we have $\sigma = 0$.

Example 2.4. As a specific example, consider the nonlinear problem

$$y'(t) = -ay(t) + \frac{by'(t - \tau(t))}{1 + [y'(t - \tau(t))]^n}, \tag{2.19}$$

where $y(t)$ is a real-valued scalar function, $a > 0$ and b are real parameters and n is an even positive integer. It is easy to verify that it belongs to class $D(\alpha, \beta, \gamma, L, 0)$ with $\alpha = -a$, $\beta = 0$, $\gamma = |b|$, $L = a$. And when $|b| \le \frac{1}{2}$, we have (2.16).

3. Variable coefficient linear multistep methods

A variable coefficient linear multistep method

$$\sum_{i=0}^{k} \alpha_i [y_{n+i} - h\beta_i f(t_{n+i}, y_{n+i})] = 0, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

for addressing ODEs (see [28]) can generally lead to a variable coefficient linear multistep method

$$\sum_{i=0}^{k} \alpha_i [y_{n+i} - h\beta_i f(t_{n+i}, y_{n+i}, y^h(\eta(t_{n+i})), \bar{y}^h(\eta(t_{n+i})))] = 0, \quad n = 0, 1, \ldots, \tag{3.2}$$

for solving problem (2.8), where $y_n \in D$ is an approximation to the exact solution $y(t_n)$,$\eta(t) = t - \tau(t)$, $t_n = nh$ $(n = 0, 1, \ldots)$ are net points, $h > 0$ is the fixed integration step size, α_i, β_i are real functions of h, and $\alpha_k > 0$, $\sum_{i=0}^{k} \alpha_i = 0$, $\beta_k \ge 0$ for any given $h > 0$, $y^h(t)$ and $\bar{y}^h(t)$ are approximations to the exact solution $y(t)$ and the derivative of $y(t)$ on the interval $[t_{-1}, t_{n+k}]$, respectively. $y_0 = y^h(0) = \phi(0)$, $y^h(t) = \phi(t)$ and $\bar{y}^h(t) = \phi'(t)$ for $t \in [t_{-1}, 0]$. In this paper, we consider an appropriate interpolation operator $\Pi^h : C[t_{-1}, 0] \times D^{n+k} \to C[t_{-1}, t_{n+k}]$ which satisfies a canonical condition (see [34,35])

$$\begin{aligned} &\max_{t_{-1} \le t \le t_{n+k}} \|\Pi^h(t; \phi, y_1, y_2, \ldots, y_{n+k}) - \Pi^h(t; \psi, z_1, z_2, \ldots, z_{n+k})\| \\ &\le c_\pi \max\left\{ \max_{1 \le i \le n+k} \|y_i - z_i\|, M_1 \right\}, \quad \forall \phi, \psi \in C[t_{-1}, 0], y_i, z_i \in D, i = 1, 2, \ldots, n+k, \end{aligned} \tag{3.3}$$

for approximation $y^h(t)$. Here the constant c_π is independent of n, k, h and L.

For approximation $\bar{y}^h(t)$, we consider two schemes: one is based on the direct evaluation

$$\bar{y}^h(t) = f(t, y^h(t), y^h(\eta(t)), \bar{y}^h(\eta(t))), \quad t \in [0, t_{n+k}]; \tag{3.4}$$

the other is based on the interpolation operator $\bar{\Pi}^h$

$$\bar{y}^h(t) = \bar{\Pi}^h(t, \phi', \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{n+k}), \quad t \in [0, t_{n+k}], \tag{3.5}$$

where $\bar{y}_i$ ($i = 1, 2, \dots, n+k$) denote $\bar{y}^h(t_i)$ which is computed using the formula

$$\bar{y}_i = f(t_i, y_i, y^h(\eta(t_i)), \bar{y}^h(\eta(t_i))), \quad i = 1, 2, \dots, n+k.$$

Like for the interpolation operator Π^h, we always assume that the interpolation operator $\bar{\Pi}^h : C[t_{-1}, 0] \times D^{n+k} \to C[t_{-1}, t_{n+k}]$ also satisfies a canonical condition

$$\max_{t_{-1} \le t \le t_{n+k}} \left\| \bar{\Pi}^h(t; \phi', \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{n+k}) - \bar{\Pi}^h(t; \psi', \bar{z}_1, \bar{z}_2, \dots, \bar{z}_{n+k}) \right\|$$
$$\le c_{\bar{\pi}} \max\left\{ \max_{1 \le i \le n+k} \|\bar{y}_i - \bar{z}_i\|, M_2 \right\}, \quad \forall \phi', \psi' \in C[t_{-1}, 0], \bar{y}_i, \bar{z}_i \in D, i = 1, 2, \dots, n+k. \tag{3.6}$$

The constant $c_{\bar{\pi}}$ is also independent of n, k, h and L.

As we have just noted in Remarks 2.4, for an arbitrarily smooth functional f and initial function ϕ the solution $y(t)$ to (2.8) may fail to be continuously differentiable if the sewing condition is not fulfilled. This leads to numerically solving NDDEs being more difficult than numerically solving DDEs since the solution to DDEs has the property of solution smoothing for increasing t (see, for example, [36]). The problem of how to treat the derivative discontinuities in numerically solving NDDEs has attracted the attention of some researchers. Some approaches to handling discontinuities have been proposed, for example, discontinuity tracking [37,38], discontinuity detection [39,4,40–42], perturbing the initial function [43] and using dissipative approximations [44]. Since the main purpose of this paper is to present some stability results and not to discuss the treatment of discontinuities, we will assume that one of the approaches mentioned above has been used.

Remark 3.1. We note that if we consider the long time behavior of the numerical solution, the method (3.2) with (3.4) does not seem to be applicable to the general problem (2.8) for a general mesh in practice because it requires one to trace back the recursion until the initial interval is reached. Thus, in practice, we generally use the method (3.2) with (3.5) to solve the problem (2.8). For the NDDEs (2.8) with a constant delay $\tau(t) = \tau$, however, if a suitable constrained mesh, that is, $\tau = mh$ with positive integer m, is used, the method (3.2) with (3.4) is identical to the method (3.2) with (3.5).

For simplicity, let $\omega_n = y_n - z_n$, $\bar{\omega}_n = \bar{y}_n - \bar{z}_n$, $\omega^h(t) = y^h(t) - z^h(t)$, $\bar{\omega}^h(t) = \bar{y}^h(t) - \bar{z}^h(t)$, $F(t) = f(t, z^h(t), y^h(\eta(t)), \bar{y}^h(\eta(t))) - f(t, z^h(t), z^h(\eta(t)), \bar{z}^h(\eta(t)))$ and

$$G_n(\lambda) = \|y_n - z_n - \lambda[f(t_n, y_n, y^h(\eta(t_n)), \bar{y}^h(\eta(t_n))) - f(t_n, z_n, y^h(\eta(t_n)), \bar{y}^h(\eta(t_n)))]\|.$$

4. Stability analysis for $D(\alpha, \beta, \gamma, L, \lambda^*)$

For any given method (3.2) and step size h, let

$$I_0 = \{0, 1, \dots, k-1\}, \qquad I_1 = \{i \in I_0 | \alpha_i \neq 0\}, \qquad I_+ = \{i \in I_0 | \alpha_i > 0\}$$

and

$$A = A(h) = \frac{\alpha_k}{\sum_{i \in I_1} |\alpha_i|} = \frac{\alpha_k}{\alpha_k + 2\sum_{i \in I_+} \alpha_i}, \qquad B = B(h) = \beta_k + \sum_{i \in I_1} \frac{|\alpha_i|}{\alpha_k} |\beta_i|.$$

4.1. Numerical methods based on direct evaluation

We first analyze the stability of numerical methods based on direct evaluation and give the following theorems.

Lemma 4.1. *Apply the method (3.2) with interpolation Π^h and direct evaluation (3.4) to problem (2.8) belonging to class $D(\alpha, \beta, \gamma, L, \lambda^*)$ with $\gamma < 1$, and assume that the set*

$$H_{\lambda^*} = \{h \in R | h > 0; \lambda^* \le h\beta_i \le h\beta_k, \forall i \in I_1; \alpha h \beta_k < 1 + \alpha\lambda^*\}$$

is nonempty. Then as $h \in H_{\lambda^*}$, for any $\mu \in [\mu_0, \beta_k]$, for any $n \ge 0$, we have

$$\frac{1 - \alpha(h\mu - \lambda^*)}{1 + \alpha\lambda^*} \|\omega_{n+k}\| \le G_{n+k}(h\mu) \le C_h \max_{i \in I_0} G_{n+i}(h\mu)$$
$$+ \frac{\beta + \gamma L}{1 - \gamma} h c_\pi B c_\mu \max\left\{ \max_{1 \le i \le n+k} \|\omega_i\|, M_1 \right\} + \frac{hBc_\mu}{1 - \gamma}(\beta M_1 + \gamma M_2), \tag{4.1}$$

where $\mu_0 = \max\{0, \max_{i\in I_1} \beta_i\}$, $c_\mu = \frac{1-\alpha(h\mu-\lambda^*)}{1-\alpha(h\beta_k-\lambda^*)}$ *and*

$$C_h = \frac{1-\alpha(h\max\limits_{i\in I_1}\beta_i - \lambda^*)}{A[1-\alpha(h\beta_k-\lambda^*)]}, \quad \alpha \le 0; \qquad C_h = \frac{1-\alpha(h\min\limits_{i\in I_1}\beta_i - \lambda^*)}{A[1-\alpha(h\beta_k-\lambda^*)]}, \quad \alpha > 0. \tag{4.2}$$

Proof. It follows from (3.2) that

$$\alpha_k G_{n+k}(h\beta_k) \le \sum_{i\in I_1} |\alpha_i| G_{n+i}(h\beta_i) + h\beta_k\alpha_k \|F(t_{n+k})\| + h\sum_{i\in I_1}|\alpha_i||\beta_i|\|F(t_{n+i})\|. \tag{4.3}$$

On the other hand, for any $\mu \in [\mu_0, \beta_k]$, since $\lambda^* \le 0 \le h\mu \le h\beta_k$, from Proposition 2.1, one gets

$$\frac{1-\alpha(h\mu-\lambda^*)}{1+\alpha\lambda^*} G_{n+k}(0) \le G_{n+k}(h\mu) \le \frac{1-\alpha(h\mu-\lambda^*)}{1-\alpha(h\beta_k-\lambda^*)} G_{n+k}(h\beta_k). \tag{4.4}$$

For any $i \in I_1$, $\lambda^* \le h\beta_i \le h\mu$ and $h\mu \ge 0$, from Proposition 2.1, one also gets

$$G_{n+i}(h\beta_i) \le \frac{1-\alpha(h\beta_i-\lambda^*)}{1-\alpha(h\mu-\lambda^*)} G_{n+i}(h\mu). \tag{4.5}$$

A combination of (4.3), (4.4), (4.5) and (3.3) leads directly to

$$\begin{aligned}
&\frac{1-\alpha(h\mu-\lambda^*)}{1+\alpha\lambda^*}\|\omega_{n+k}\| \le G_{n+k}(h\mu)\\
&\quad\le \frac{1-\alpha(h\mu-\lambda^*)}{1-\alpha(h\beta_k-\lambda^*)}\sum_{i\in I_1}\frac{|\alpha_i|}{\alpha_k}\cdot\frac{1-\alpha(h\beta_i-\lambda^*)}{1-\alpha(h\mu-\lambda^*)} G_{n+i}(h\mu)\\
&\qquad + hc_\mu\beta_k\left[(\beta+\gamma L)\|\omega^h(\eta(t_{n+k}))\| + \gamma\|F(\eta(t_{n+k}))\|\right]\\
&\qquad + hc_\mu\sum_{i\in I_1}\frac{|\alpha_i|}{\alpha_k}|\beta_i|\left[(\beta+\gamma L)\|\omega^h(\eta(t_{n+i}))\| + \gamma\|F(\eta(t_{n+i}))\|\right]\\
&\quad\le C_h\max_{i\in I_0} G_{n+i}(h\mu) + hc_\mu\beta_k\left[\frac{(\beta+\gamma L)c_\pi}{1-\gamma}\max\left\{\max_{1\le i\le n+k}\|\omega_i\|, M_1\right\} + \frac{1}{1-\gamma}(\beta M_1+\gamma M_2)\right]\\
&\qquad + hc_\mu\sum_{i\in I_1}\frac{|\alpha_i|}{\alpha_k}|\beta_i|\left[\frac{(\beta+\gamma L)c_\pi}{1-\gamma}\max\left\{\max_{1\le i\le n+k}\|\omega_i\|, M_1\right\} + \frac{1}{1-\gamma}(\beta M_1+\gamma M_2)\right],
\end{aligned}$$

which implies inequality (4.1). □

Corollary 4.2. *Apply the method (3.1) to (2.1) belonging to class $D(\alpha, \lambda^*)$, and assume that the set H_{λ^*} is nonempty. Then as $h \in H_{\lambda^*}$, for any $\mu \in [\mu_0, \beta_k]$, for any $n \ge 0$, we have*

$$\frac{1-\alpha(h\mu-\lambda^*)}{1+\alpha\lambda^*}\|\omega_{n+k}\| \le G_{n+k}(h\mu) \le C_h\max_{i\in I_0} G_{n+i}(h\mu). \tag{4.6}$$

Corollary 4.3. *Apply the method (3.2) with interpolation Π^h to problem (2.3) belonging to class $D(\alpha, \beta, \lambda^*)$, and assume that the set H_{λ^*} is nonempty. Then as $h \in H_{\lambda^*}$, for any $\mu \in [\mu_0, \beta_k]$, for any $n \ge 0$, we have*

$$\frac{1-\alpha(h\mu-\lambda^*)}{1+\alpha\lambda^*}\|\omega_{n+k}\| \le G_{n+k}(h\mu) \le C_h\max_{i\in I_0} G_{n+i}(h\mu) + \beta hc_\pi Bc_\mu\max\left\{\max_{1\le i\le n+k}\|\omega_i\|, M_1\right\}. \tag{4.7}$$

Theorem 4.4. *Apply the method (3.2) with interpolation Π^h and direct evaluation (3.4) to problem (2.8) belonging to class $D(\alpha, \beta, \gamma, L, \lambda^*)$ with $\alpha \le 0$, $\gamma < 1$, and assume that the set*

$$\begin{aligned}
H_{\alpha,\lambda^*} = \Big\{ & h\in R | h > 0;\ \lambda^* \le h\beta_i \le h\beta_k,\ \forall i\in I_1;\ -\alpha h(A\beta_k - \max_{i\in I_1}\beta_i)\\
& \ge (1+\alpha\lambda^*)(1-A);\ \frac{h(\beta+\gamma L)}{1-\gamma} \le \frac{\zeta}{Bc_\pi},\ \exists\zeta\in(0,1)\Big\}
\end{aligned}$$

is nonempty. Then there exists a constant $c > 0$, which only depends on the constants $\beta, \gamma, \gamma L$ and the method, for any $h \in H_{\alpha,\lambda^*}$, $\mu \in [\mu_0, \beta_k]$ and $n \geq k$, such that the following inequality holds:

$$\|y_n - z_n\| \leq \exp(c(t_n - t_{k-1})) \max\left\{\max_{0\leq i\leq k-1} G_i(h\mu), M_1, M_2\right\}. \tag{4.8}$$

Proof. Since $\alpha \leq 0$ and $\mu \leq \beta_k$, from the proof of Lemma 4.1, we have

$$\|\omega_{n+k}\| \leq \frac{1+\alpha\lambda^*}{1-\alpha(h\mu-\lambda^*)} G_{n+k}(h\mu) \leq G_{n+k}(h\mu) \tag{4.9}$$

and

$$\begin{aligned} G_{n+k}(h\mu) \leq\; & C_h \max_{i\in I_0} G_{n+i}(h\mu) + \frac{h(\beta+\gamma L)c_\pi B}{1-\gamma}\|\omega_{n+k}\| \\ & + \frac{hB}{1-\gamma}\left[c_\pi(\beta+\gamma L)+\beta+\gamma\right] \max\left\{\max_{1\leq i\leq n+k-1}\|\omega_i\|, M_1, M_2\right\}. \end{aligned} \tag{4.10}$$

Substituting (4.10) with (4.9) and taking note of $0 < C_h \leq 1$, we further have

$$\begin{aligned} \left[1 - \frac{h(\beta+\gamma L)c_\pi B}{1-\gamma}\right] G_{n+k}(h\mu) &\leq C_h \max_{i\in I_0} G_{n+i}(h\mu) + \frac{hB}{1-\gamma}\left[c_\pi(\beta+\gamma L)+\beta+\gamma\right] \max\left\{\max_{1\leq i\leq n+k-1}\|\omega_i\|, M_1, M_2\right\} \\ &\leq \left[1 + \frac{hB}{1-\gamma}(c_\pi(\beta+\gamma L)+\beta+\gamma)\right] X_n, \end{aligned}$$

where

$$X_n := \max\left\{\max_{0\leq i\leq n} G_i(h\mu), M_1, M_2\right\}, \quad n \geq k.$$

Since $\frac{h(\beta+\gamma L)c_\pi B}{1-\gamma} \leq \zeta < 1$, we obtain

$$\begin{aligned} G_{n+k}(h\mu) &\leq \frac{1 + \frac{hB}{1-\gamma}\left[c_\pi(\beta+\gamma L)+\beta+\gamma\right]}{1 - \frac{h(\beta+\gamma L)c_\pi B}{1-\gamma}} X_{n+k-1} \\ &\leq (1+ch)X_{n+k-1}, \end{aligned}$$

where $c = \frac{2c_\pi B(\beta+\gamma L)+B(\beta+\gamma)}{(1-\gamma)(1-\zeta)}$. Therefore, we have

$$\begin{aligned} X_{n+k} &\leq (1+ch)X_{n+k-1} \leq (1+ch)^{n+1}X_{k-1} \\ &\leq \exp(c(t_{n+k}-t_{k-1})) \max\left\{\max_{0\leq i\leq k-1} G_i(h\mu), M_1, M_2\right\}. \end{aligned}$$

This completes the proof of the theorem. □

Closely related to the foregoing Theorem 4.4 is the following result.

Theorem 4.5. *Under the assumptions of Theorem 4.4, if $\beta + \gamma L \neq 0$, then there exists a constant $c_2 > 0$, which only depends on the constants $\beta, \gamma, \gamma L$ and the method, for any $h \in H_{\alpha,\lambda^*}$, $\mu \in [\mu_0, \beta_k]$ and $n \geq k$, such that the following inequality holds:*

$$\begin{aligned} \|y_n - z_n\| \leq\; & \exp(c_2(t_n - t_{k-1})) \max\left\{\max_{0\leq i\leq k-1} G_i(h\mu), M_1\right\} \\ & + \frac{1}{2(\beta+\gamma L)c_\pi}\left[\exp(c_2(t_n - t_{k-1})) - 1\right](\beta M_1 + \gamma M_2). \end{aligned} \tag{4.11}$$

Proof. Like in the proof of Theorem 4.4, we have

$$\begin{aligned} G_{n+k}(h\mu) \leq\; & C_h \max_{i\in I_0} G_{n+i}(h\mu) + \frac{h(\beta+\gamma L)c_\pi B}{1-\gamma}\|\omega_{n+k}\| \\ & + \frac{hBc_\pi(\beta+\gamma L)}{1-\gamma} \max\left\{\max_{1\leq i\leq n+k-1}\|\omega_i\|, M_1\right\} + \frac{hB}{1-\gamma}(\beta M_1 + \gamma M_2), \end{aligned} \tag{4.12}$$

and

$$G_{n+k}(h\mu) \le (1+c_2h)\left\{\max_{1\le i\le n+k-1}\|\omega_i\|, M_1\right\} + C_Bh(\beta M_1+\gamma M_2),$$

where $c_2 = \frac{2(\beta+\gamma L)c_\pi B}{(1-\gamma)(1-\zeta)}$ and $C_B = \frac{B}{(1-\gamma)(1-\zeta)}$. Hence, we further get

$$G_{n+k}(h\mu) \le (1+c_2h)^{n+1}\max\left\{\max_{1\le i\le k-1}\|\omega_i\|, M_1\right\} + \frac{1}{2(\beta+\gamma L)c_\pi}\left[(1+c_2h)^{n+1}-1\right](\beta M_1+\gamma M_2).$$

The desired inequality (4.11) follows from above inequality. □

The following results are related to the problem class $D(\alpha,\beta,\gamma,L,\lambda^*)$ with $\alpha>0$, $\gamma<1$.

Theorem 4.6. *Apply the method (3.2) with interpolation Π^h and direct evaluation (3.4) to problem (2.8) belonging to class $D(\alpha,\beta,\gamma,L,\lambda^*)$ with $\alpha>0$, $\gamma<1$, and assume that the set*

$$\tilde{H}_{\alpha,\lambda^*} = \left\{h\in R|h>0;\ \lambda^*\le h\beta_i\le h\beta_k, \forall i\in I_1;\ \alpha h\beta_k\le\xi<1+\alpha\lambda^*;\ \frac{h(\beta+\gamma L)}{1-\gamma}\le\frac{\zeta^*}{Bc_\pi c_\alpha}, \exists\zeta^*\in(0,1)\right\}$$

is nonempty, where $c_\alpha = \frac{1+\alpha\lambda^*}{1+\alpha\lambda^*-\xi}$. Then there exists a constant $c^*>0$, which only depends on the constants β, γ, λ^*, γL and the method, for any $h\in\tilde{H}_{\alpha,\lambda^*}$ and $n\ge k$, such that the following inequality holds:

$$\|y_n-z_n\| \le c_\alpha\exp(c^*(t_n-t_{k-1}))\max\left\{\max_{0\le i\le k-1}G_i(h\beta_k), M_1, M_2\right\}. \tag{4.13}$$

Proof. Like for (4.4) and (4.5), one gets

$$\|\omega_{n+k}\| \le \frac{1+\alpha\lambda^*}{1-\alpha(h\beta_k-\lambda^*)}G_{n+k}(h\beta_k) \tag{4.14}$$

and

$$G_{n+i}(h\beta_i) \le \frac{1-\alpha(h\beta_i-\lambda^*)}{1-\alpha(h\beta_k-\lambda^*)}G_{n+i}(h\beta_k),\quad n=0,1,2,\ldots. \tag{4.15}$$

Therefore, it follows from (4.3) that

$$\begin{aligned} llG_{n+k}(h\beta_k) &\le C_h\max_{i\in I_0}G_{n+i}(h\beta_k) + \frac{h(\beta+\gamma L)c_\pi Bc_\alpha}{1-\gamma}G_{n+k}(h\beta_k) \\ &\quad + \frac{hB\left[c_\pi c_\alpha(\beta+\gamma L)+\beta+\gamma\right]}{1-\gamma}\max\left\{\max_{1\le i\le n+k-1}G_i(h\beta_k), M_1, M_2\right\}. \end{aligned}$$

For any $n\ge k$, define

$$\tilde{X}_n = \max\left\{\max_{0\le i\le n}G_i(h\beta_k), M_1, M_2\right\}.$$

By analogy to the proof of Theorem 4.4, we have

$$\begin{aligned} G_{n+k}(h\beta_k) &\le \frac{C_h+\frac{hB}{1-\gamma}\left[c_\pi c_\alpha(\beta+\gamma L)+\beta+\gamma\right]}{1-\frac{h(\beta+\gamma L)c_\pi Bc_\alpha}{1-\gamma}}\tilde{X}_{n+k-1} \\ &\le \frac{1+\nu h+\frac{hB}{1-\gamma}\left[c_\pi c_\alpha(\beta+\gamma L)+\beta+\gamma\right]}{1-\frac{h(\beta+\gamma L)c_\pi Bc_\alpha}{1-\gamma}}\tilde{X}_{n+k-1} \\ &\le (1+c^*h)\tilde{X}_{n+k-1} \end{aligned}$$

and

$$\tilde{X}_{n+k} \le (1+c^*h)^{n+1}\tilde{X}_{k-1} \le \exp(c^*(t_{n+k}-t_{k-1}))\max\left\{\max_{0\le i\le k-1}G_i(h\beta_k), M_1, M_2\right\}$$

where $\nu = \frac{\alpha(\beta_k-\min_{i\in I_1}\beta_i)}{1+\alpha\lambda^*-\xi}$ and $c^* = \frac{2c_\pi c_\alpha B(\beta+\gamma L)+B(\beta+\gamma)+(1-\gamma)\nu}{(1-\gamma)(1-\zeta^*)}$. This completes the proof of the theorem. □

In the same way, the following result can be easily obtained.

Theorem 4.7. *Under the assumptions of Theorem 4.6, if $\beta + \gamma L \neq 0$, then there exists a constant $c_2^* > 0$, which only depends on the constants β, γ, λ^*, γL and the method, for any $h \in \tilde{H}_{\alpha,\lambda^*}$ and $n \geq k$, such that the following inequality holds:*

$$\|y_n - z_n\| \leq c_\alpha \exp(c_2^*(t_n - t_{k-1})) \max\left\{\max_{0\leq i\leq k-1} G_i(h\beta_k), M_1\right\} + \frac{c_\alpha B[\exp(c_2^*(t_n - t_{k-1})) - 1]}{2(\beta + \gamma L)c_\pi c_\alpha B + (1-\gamma)\nu}(\beta M_1 + \gamma M_2). \quad (4.16)$$

It should be pointed out that Li [22] and Wen and Li [24] have studied the numerical stability of the linear multistep methods (3.1) for ODEs with $\alpha \leq 0$ and DDEs with $\alpha \leq 0$, respectively. However, from the proof of Theorem 4.6, we have the following results on ODEs and DDEs with $\alpha > 0$.

Corollary 4.8. *Apply the method (3.1) to problem (2.1) belonging to class $D(\alpha, \lambda^*)$ with $\alpha > 0$, and assume that the set*

$$\tilde{H}_{O,\alpha,\lambda^*} = \left\{h \in R | h > 0;\ \lambda^* \leq h\beta_i \leq h\beta_k, \forall i \in I_1;\ \alpha h\beta_k \leq \xi < 1 + \alpha\lambda^*\right\}$$

is nonempty. Then there exists a constant $c_O > 0$, which depends only on α and the method, for any $h \in \tilde{H}_{O,\alpha,\lambda^*}$ and $n \geq k$, such that the following inequality holds:

$$\|y_n - z_n\| \leq c_\alpha \exp(c_O(t_n - t_{k-1})) \max_{0\leq i\leq k-1} G_i(h\beta_k). \quad (4.17)$$

Corollary 4.9. *Apply the method (3.2) with interpolation Π^h to problem (2.3) belonging to class $D(\alpha, \beta, \lambda^*)$ with $\alpha > 0$, and assume that the set*

$$\tilde{H}_{D,\alpha,\lambda^*} = \left\{h \in R | h > 0;\ \lambda^* \leq h\beta_i \leq h\beta_k, \forall i \in I_1;\ \alpha h\beta_k \leq \xi < 1 + \alpha\lambda^*;\ h\beta \leq \frac{\zeta^*}{Bc_\pi c_\alpha}, \exists \zeta^* \in (0,1)\right\}$$

is nonempty. Then there exists a constant $c_D > 0$, which depends only on α, β and the method, for any $h \in \tilde{H}_{D,\alpha,\lambda^*}$ and $n \geq k$, such that the following inequality holds:

$$\|y_n - z_n\| \leq c_\alpha \exp(c_D(t_n - t_{k-1})) \max\left\{\max_{0\leq i\leq k-1} G_i(h\beta_k), M_1\right\}. \quad (4.18)$$

Remark 4.1. From the above results, we know that for some linear multistep methods, if we properly choose the step size, the numerical solutions obtained by these methods are stable.

Remark 4.2. For problem class $D(\alpha, \beta, \gamma, \sigma, \lambda^*)$, we can obtain the same results as for problem class $D(\alpha, \beta, \gamma, L, \lambda^*)$ in a natural way by replacing $\beta + \gamma L$ with σ in Lemma 4.1 and Theorems 4.4–4.7. And, in general, $\sigma \leq \beta + \gamma L$. In particular, when $\sigma = 0$, (4.11) and (4.16) should be replaced by

$$\|y_n - z_n\| \leq \max\left\{\max_{0\leq i\leq k-1} G_i(h\mu), M_1\right\} + C_B(t_n - t_{k-1})(\beta M_1 + \gamma M_2)$$

and

$$\|y_n - z_n\| \leq c_\alpha \max\left\{\max_{0\leq i\leq k-1} G_i(h\mu), M_1\right\} + c_\alpha C_B(t_n - t_{k-1})(\beta M_1 + \gamma M_2),$$

respectively. However, for problem class $D(\alpha, \beta, \gamma, L, \lambda^*)$, we have the following results.

4.2. Numerical methods based on the interpolation operator $\tilde{\Pi}^h$

If we consider the behavior of the numerical solution when $T \gg 0$, numerical method (3.2) based on direct evaluation does not seem to be applicable in practice because it requires one to trace back the recursion until the initial interval is reached. Now we therefore discuss the stability of numerical method (3.2) based on interpolation operator $\tilde{\Pi}^h$.

Note that for any $0 \leq t \leq t_{n+k}$,

$$\begin{aligned}\|F(t)\| &\leq \beta\|\omega^h(\eta(t))\| + \gamma\|\bar{\omega}^h(\eta(t))\| \\ &\leq \beta c_\pi \max\left\{\max_{1\leq i\leq n+k}\|\omega_i\|, M_1\right\} + \gamma c_{\tilde{\pi}} \max\left\{\max_{1\leq i\leq n+k}\|\bar{\omega}_i\|, M_2\right\} \\ &\leq \beta c_\pi \max\left\{\max_{1\leq i\leq n+k}\|\omega_i\|, M_1\right\} + \gamma c_{\tilde{\pi}}\left[L\max_{1\leq i\leq n+k}\|\omega_i\| + \beta c_\pi \max\left\{\max_{1\leq i\leq n+k}\|\omega_i\|, M_1\right\}\right.\end{aligned}$$

$$+\ \gamma c_{\bar{\pi}} \max\left\{\max_{1\le i\le n+k}\|\bar{\omega}_i\|, M_2\right\}\Bigg] + \gamma c_{\bar{\pi}} M_2$$
$$\le \frac{c^{\pi}(\beta+\gamma L)}{1-\gamma c_{\bar{\pi}}}\max\left\{\max_{1\le i\le n+k}\|\omega_i\|, M_1\right\} + \frac{1}{1-\gamma c_{\bar{\pi}}}M_2 \tag{4.19}$$

whenever $\gamma c_{\bar{\pi}} < 1$, where $c^{\pi} = \max\{c_{\pi}, c_{\bar{\pi}}\}$. Then it is easy to obtain some results which are similar to the results obtained in Section 4.1. Since the difference is only that c_{π} and $\gamma < 1$ are replaced by c^{π} and $\gamma c_{\bar{\pi}} < 1$ respectively, we do not state these results here. However, it should be pointed out that we cannot show that the numerical solution produced by the method (3.2) with interpolations Π^h and $\bar{\Pi}^h$ is stable for problem class $D(\alpha, \beta, \gamma, \sigma, \lambda^*)$.

5. Examples and numerical experiments

In this section, we consider the applications of the results obtained in Section 4.

Example 5.1. Apply the two-step second-order method (see [45])

$$\frac{2}{3}y_{n+2} - y_{n+1} + \frac{1}{3}y_n = h\left(\frac{11}{12}f_{n+2} - f_{n+1} + \frac{5}{12}f_n\right) \tag{5.1}$$

to problem (2.6). In contrast to the case for the method (3.1), $\alpha_2 = \frac{2}{3}, \alpha_1 = -1, \alpha_0 = \frac{1}{3}, \beta_2 = \frac{11}{8}, \beta_1 = 1, \beta_0 = \frac{5}{4}$. From Example 2.1, we get $\alpha = 0.73$. Thus, from Corollary 4.2 we know that when $h < \frac{8}{8.03}$, $C_h \le \frac{16(1-0.73h)}{8-8.03h}$ and (4.6) holds with $\mu \in [\frac{5}{4}, \frac{11}{8}]$.

Example 5.2. Consider the two-step method (see [46])

$$\begin{aligned} y_{n+2} - (1-h^2)y_{n+1} - h^2 y_n = {} & \frac{1}{2}[(\exp(h)-1)f(t_{n+2}, y_{n+2}, y^h(\eta(t_{n+2}))) \\ & + (1-\exp(-h))f(t_{n+1}, y_{n+1}, y^h(\eta(t_{n+1})))], \end{aligned} \tag{5.2}$$

which is of order 2, together with a piecewise Lagrangian linear interpolation procedure

$$\begin{aligned} y^h(t) &= \Pi^h(t; \phi, y_1, y_2, \ldots, y_{n+1}) \\ &= \begin{cases} \frac{1}{h}[(t_{i+1}-t)y_i + (t-t_i)y_{i+1}], & \text{if } t_i \le t \le t_{i+1}, i = 0, \ldots, n+1, \\ \phi(t), & \text{if } t_{-1} \le t \le 0. \end{cases} \end{aligned} \tag{5.3}$$

It is easy to verify that the interpolation operator Π^h satisfies the canonical condition (3.3) with $c_{\pi} = 1$. In contrast to the case for the method (3.2), $\alpha_2 = 1, \alpha_1 = -(1-h^2), \alpha_0 = -h^2, \beta_2 = \frac{\exp(h)-1}{2h}, \beta_1 = \frac{1-\exp(-h)}{2h(1-h^2)}, \beta_0 = 0$. And we have $B = \frac{\exp(h)-\exp(-h)}{2h}$. Apply the method (5.2)–(5.3) to problem (2.7) with $\alpha = 1, \beta = 0.125, \lambda^* = 0$. Then $\tilde{H}_{D,\alpha,\lambda^*} = \{h \in R | 0 < h \le 1, \exp(h) - \exp(-h) < 8(3-\exp(h))\}$ is nonempty. Therefore, there exists a constant c_D such that (4.18) holds.

Example 5.3. As an illustration of the comprehensive application of the results obtained in this paper, consider the problem (2.19) with $|b| < 1$. First, since it belongs to the class $D(-a, 0, |b|, a, 0)$, we apply the third-order backward differentiation formula

$$\frac{11}{6}y_{n+3} - 3y_{n+2} + \frac{3}{2}y_{n+1} - \frac{1}{3}y_n = hf(t_{n+3}, y_{n+3}, y^h(\eta(t_{n+3})), \bar{y}^h(\eta(t_{n+3}))) \tag{5.4}$$

together with linear interpolation (5.3) and direct evaluation (3.4), or with linear interpolation (5.3) and piecewise linear interpolation

$$\begin{aligned} \bar{y}^h(t) &= \bar{\Pi}^h(t; \phi', \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n+1}) \\ &= \begin{cases} \frac{1}{h}[(t_{i+1}-t)\bar{y}_i + (t-t_i)\bar{y}_{i+1}], & \text{if } t_i \le t \le t_{i+1}, i = 0, \ldots, n+1, \\ \phi'(t), & \text{if } t_{-1} \le t \le 0, \end{cases} \end{aligned} \tag{5.5}$$

to solve the problem. It is also easy to verify that the interpolation operator $\bar{\Pi}^h$ satisfies the canonical condition (3.6) with $c_{\bar{\pi}} = 1$. Then we have

$$A = \frac{11}{29}, \qquad B = \frac{6}{11}, \qquad \mu_0 = 0.$$

Therefore, from Theorem 4.4 we know that for any $\zeta \in (0, 1)$, when $\frac{3}{a} \le h \le \frac{11(1-|b|)\zeta}{6|ba|}$, there exists a constant c such that (4.8) holds.

On the other hand, from Proposition 2.2, we know that when $0 < a < 2$ or $2 < a \le 4$, problem (2.19) belongs to $D(2 - \frac{4}{|a-2|}, 0, |b|, |a|, -\frac{1}{2})$. Then consider applying the two-step second-order method

$$\begin{aligned} 8y_{n+2} &= (8+2h)y_{n+1} - 2hy_n + 4hf(t_{n+2}, y_{n+2}, y^h(\eta(t_{n+2})), \bar{y}^h(\eta(t_{n+2}))) \\ &\quad + (4h - h^2)f(t_{n+1}, y_{n+1}, y^h(\eta(t_{n+1})), \bar{y}^h(\eta(t_{n+1}))) - h^2 f(t_n, y_n, y^h(\eta(t_n)), \bar{y}^h(\eta(t_n))), \end{aligned} \tag{5.6}$$

together with linear interpolation (5.3) and direct evaluation (3.4), or with two linear interpolations (5.3), (5.5) to solve this problem. In contrast to the case for the method (3.2), $\alpha_2 = 8, \alpha_1 = -(8+2h), \alpha_0 = 2h, \beta_2 = \frac{1}{2}, \beta_1 = -\frac{4-h}{8+2h}, \beta_0 = -\frac{1}{2}$. And when $h \le 4$, we have

$$A = \frac{2}{2+h}, \qquad B = 1, \qquad \mu_0 = 0.$$

Therefore, $H_{\lambda^*} = \{h \in R | 0 < h \le 1\}$ is nonempty, and for any $\mu \in [0, \frac{1}{2}]$, (4.1) is satisfied with

$$C_h = \frac{2 - \frac{2|a-2|-4}{|a-2|}\left[\left(\frac{h^2}{8+2h} - \frac{|a-2|}{2|a-2|-4} - \frac{1}{2}\right)h + 1\right]}{2 - \frac{2|a-2|-4}{|a-2|}(h+1)}.$$

Further when $|a-2| < \frac{3}{2}$, for any $\zeta \in (0, 1)$,

$$H_{\alpha,\lambda^*} = \left\{h \in R | 0 < h \le \min\left\{1, \frac{4|a-2|-6}{|a-2|-2}, \frac{(1-|b|)\zeta}{|ba|}\right\}\right\} \subset H_{\lambda^*}$$

is nonempty, then $C_h < 1$, and (4.8) holds with $c = \frac{2|ba|+|b|}{(1-|b|)(1-\zeta)}$. In particular, when $\mu = 0$, from Theorem 4.4 it is easy to see that

$$\|y_n - z_n\| \le \exp(c(t_n - t_1)) \max\{\|w_1\|, M_1, M_2\}, n \ge 1.$$

When $a > 4$, from Proposition 2.2, problem (2.19) belongs to $D(\frac{2a-8}{(a-2)}, 0, |b|, |a|, -\frac{1}{2})$. Then applying a one-step method (see [46])

$$y_{n+1} - y_n = \tan(h/2)[f(t_{n+1}, y_{n+1}, y^h(\eta(t_{n+1})), \bar{y}^h(\eta(t_{n+1}))) + f(t_n, y_n, y^h(\eta(t_n)), \bar{y}^h(\eta(t_n)))], \tag{5.7}$$

which is of order 2, together with linear interpolation procedure (5.3) and direct evaluation (3.4), or with linear interpolations (5.3), (5.5) to solve problem (2.19), we have $\alpha_1 = 1, \alpha_0 = -1, \beta_1 = \frac{1}{h}\tan(\frac{h}{2}), \beta_0 = -\frac{1}{h}\tan(\frac{h}{2})$ and $A = 1, B = \frac{2}{h}\tan(\frac{h}{2})$. Therefore, $H_{\lambda^*} = \{h \in R | 0 < h \le 2\arctan(\frac{1}{2})\}$ is nonempty, and (4.1) is satisfied with

$$C_h = \frac{(a-2) - (2a-8)(\frac{1}{2} - \tan(\frac{h}{2}))}{(a-2) - (2a-8)(\frac{1}{2} + \tan(\frac{h}{2}))}.$$

Further, for any $\zeta^* \in (0, 1)$ and $\xi \in (0, \frac{2}{a-2})$,

$$\begin{aligned} \tilde{H}_{\alpha,\lambda^*} = \Bigg\{h \in R | 0 < h \le \min\Bigg\{&2\arctan\left(\frac{1}{2}\right), 2\arctan\left(\frac{(a-2)\xi}{2a-8}\right), \\ &2\arctan\left(\frac{(1-|b|)\zeta^*[2-(a-2)\xi]}{4|ba|}\right)\Bigg\}\Bigg\} \subset H_{\lambda^*} \end{aligned}$$

is nonempty. Then, there exists a constant $c^* > 0$ such that (4.13) holds for all $h \in H_{\alpha,\lambda^*}$ and $n \ge k$.

This example is a comprehensive application and illustrates a simple fact: that we need choose different step sizes for different methods to obtain a stable numerical solution.

Example 5.4. Consider the backward differentiation formula (BDF) together with the linear interpolation procedure (5.3) and direct evaluation (3.4). Here $\mu_0 = 0, B = \beta_k$. Apply the method to the $D(\alpha, \beta, \gamma, \sigma, \lambda^*)$ problem with $\alpha < 0, \lambda^* \le 0, \gamma < 1$. When the set

$$H_{\alpha,\lambda^*} = \left\{h \in R | 0 < \frac{(1+\alpha\lambda^*)(1-A)}{-\alpha A\beta_k} \le h \le \frac{\zeta(1-\gamma)}{\beta_k\sigma}, \exists\zeta \in (0, 1)\right\}$$

is nonempty, there exists a constant $c > 0$, which only depends on the constants β, γ, σ and the method, such that (4.8) holds for all $h \in H_{\alpha,\lambda^*}$, $\mu \in [0, \beta_k]$ and $n \ge k$.

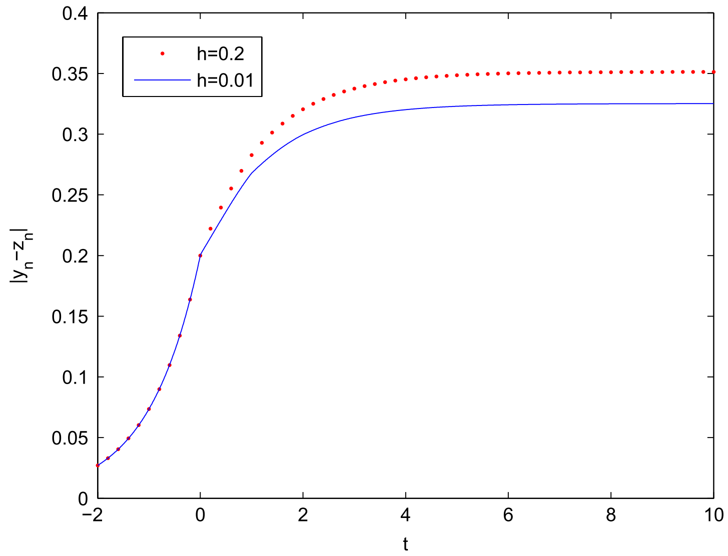

Fig. 1. $|y_n - z_n|$ given by the method (5.8), (5.3), (3.4) with step size $h = 0.2$ and $h = 0.01$ for problem (2.18) with different initial values $\phi(t) = t\exp(t) + 0.1\exp(t)$, $\psi(t) = t\exp(t) - 0.1\exp(t)$.

Example 5.5. As a numerical example, consider Example 2.3 with $\tau = 1$, $a(t) = a > 0$, $b(t) = b$, $|c(t)| = |c| < 1$. Apply the second-order BDF

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(t_{n+2}, y_{n+2}, y^h(\eta(t_{n+2})), \bar{y}^h(\eta(t_{n+2}))) \tag{5.8}$$

together with linear interpolation (5.3) and direct evaluation (3.4) to solve the problem (2.18). Then it is easily obtained that

$$\beta_2 = \frac{2}{3}, \qquad A = \frac{3}{5}, \qquad B = \frac{2}{3}, \qquad c_\pi = 1, \qquad \mu_0 = 0.$$

From Theorem 4.6, we know that for any $\zeta \in (0, 1)$, when $h < \min\{\frac{1}{\alpha\beta_2}, \frac{\zeta(1-\gamma)}{\sigma\beta_2}\}$, (4.13) holds for $n \geq 1$. For example, let $a(t) = 0.5$, $b(t) = -0.5$, $c(t) = 0.1$, $T = 10$ and $g(t) = t\exp(t) + 0.1\exp(t)$. Then $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.1$, $\sigma = 0.45$. Let y_n and z_n denote the numerical solutions produced by the method (5.8), (5.3), (3.4) applied to (2.18) with different initial values $\phi(t) = t\exp(t) + 0.1\exp(t)$, $\psi(t) = t\exp(t) - 0.1\exp(t)$, respectively. In Fig. 1 we have plotted the absolute value of the difference of the two numerical solutions given by the method (5.8), (5.3), (3.4) with step size $h = 0.2$ and $h = 0.01$ to problem (2.18). From Fig. 1, it can be easily seen that the difference $|y_n - z_n|$ increases with time. Fortunately it is less than the bound we gave in Theorem 4.6. We also applied the sixth-order BDF to the problem (2.18) and observed the same phenomenon.

From the theoretical analysis given in this paper and the numerical results shown in Fig. 1, we come to the following remark:

Under a step size restriction given in this paper, the methods discussed in this paper are stable for NDDEs. We also note that the condition on the step size is sufficient but not necessary for the stability of these methods.

To further confirm our conclusions, we consider applying BDF (5.8) together with linear interpolation (5.3) and direct evaluation (3.4) to solve the following problem:

$$\begin{cases} y'(t) = y(t) - 0.2y^3 + 0.1y^3(t-1) + 0.5y'(t-1) + \dfrac{3y(t-1)}{1 + y^2(t-1)}, & t \geq 0, \\ y(t) = \phi(t), & t \leq 0. \end{cases} \tag{5.9}$$

Then $\alpha = 1$, $\gamma = 0.5$, $\sigma = 3.5$. Let y_n and z_n denote the numerical solutions produced by the method (5.8), (5.3), (3.4) applied to (5.9) with different initial values $\phi(t) = 21$, $\psi(t) = 20$, respectively. Take $h = 0.2$ and $h = 0.01$. The numerical results are presented in Fig. 2. These results further confirm our conclusions. On the other hand, from Fig. 2, we can observe that the system (5.9) is stable although $\alpha > 0$.

6. Conclusions

Under the condition $\alpha < 0$, the asymptotic stability of one-leg methods and linear multistep methods for NDDEs with constant delay in Hilbert space have been studied in [16,47], respectively. But these methods are confined to A-stable

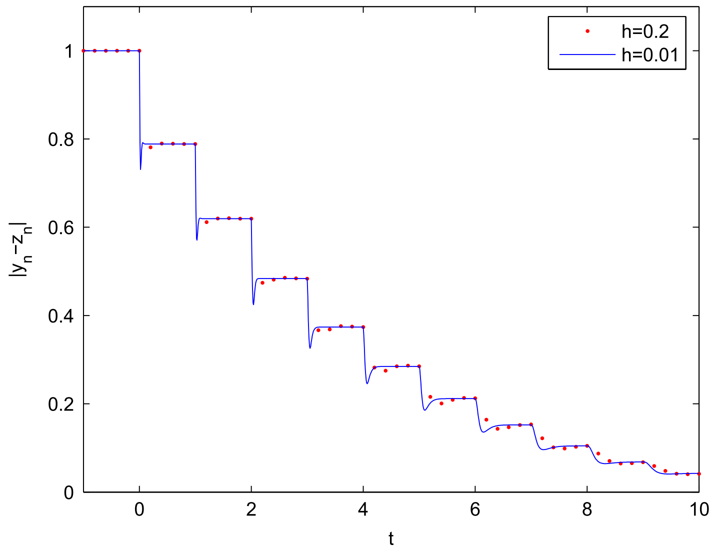

Fig. 2. $|y_n - z_n|$ given by the method (5.8), (5.3), (3.4) with step size $h = 0.2$ and $h = 0.01$ for problem (5.9) with different initial values $\phi(t) = 21$, $\psi(t) = 20$.

methods. In this paper, the bounded stability of a class of linear multistep methods, including high order BDF, for solving NDDEs with any variable delay has been studied.

On the other hand, it is known that when applied to contractive linear autonomous ODEs in Banach space, a linear multistep method or a Runge–Kutta method is contractive on any mesh only if it has order 1 (see [48]). This means that the search for linear multistep methods or Runge–Kutta methods for NDDEs in Banach space, which preserve the contractivity property on any mesh, has to be confined to methods of order 1. In this paper, we have studied the stability of a class of variable coefficient linear multistep methods for solving NDDEs with any variable delay. The results obtained in this paper provide some criteria for choosing the step size such that the numerical method is stable.

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