



A new theoretical error estimate of the method of fundamental solutions applied to reduced wave problems in the exterior region of a disk

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ABSTRACT

In this paper, we present a mathematical study of the method of fundamental solutions (MFS) applied to reduced wave problems with Dirichlet boundary conditions in the exterior domain of a disk. A theorem in this paper shows that the MFS with N source points in equi-distantly equally phased arrangement with assignment parameter q ($0 < q < 1$), which characterizes the position of the source points and the collocation points, gives an approximate solution with error of $O(q^N)$ if the Fourier coefficients of the boundary data decay exponentially. This error estimate is an extension of the results of the previous studies. Numerical examples make good agreements with the results of the theoretical study.

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1. Introduction

In this paper, we examine the accuracy of the method of fundamental solutions (MFS, in abbreviation) applied to Dirichlet boundary value problems of two-dimensional reduced wave equation, that is, the two-dimensional Helmholtz equation in the exterior region of a disk.

The MFS is a numerical solver for boundary value problems of homogeneous partial differential equations, where the solution is approximated by a linear combination of the fundamental solutions of the partial differential equation with singularities outside the problem domain. This method has the advantages that (i) it is easy to program, (ii) its computational cost is low and (iii) it achieves high accuracy such as exponential convergence under some conditions, and, due to these advantages, it is widely used especially for potential problems, where the method is usually called the *charge simulation method*.

Previous studies related to the MFS are as follows. The MFS was first proposed by Steinbigler for electric field analysis with cyclic symmetry [1], where the electric field is simulated by posing fictitious point, line, and ring charges. Singer et al. also applied the MFS to electric field studies [2]. Studies on the MFS for elliptic boundary value problems are reviewed in [3]. As regards theoretical studies on the MFS, Katsurada and Okamoto presented theoretical error estimates of the MFS applied to two-dimensional potential problems in a disk [4,5], where it is shown that the approximate solution by the MFS converges to the exact solution at an exponential rate if the boundary data is an analytic function. The theoretical study presented in this paper is essentially based on these studies of Katsurada and Okamoto. The MFS is applied to studies in various fields,

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for example, Amano et al. proposed a method of numerical conformal mappings of various types of complex domains using the MFS [6,7], where the mapping problems are reduced to the Dirichlet problems of the Laplace equation and approximate mapping functions are obtained by solving these problems by the MFS. As another application of the MFS, Ogata et al. used the MFS for the studies of various periodic problems, namely, two-dimensional potential problems with one-dimensional periodicity [8], two- or three-dimensional Stokes flow problems with two- or three-dimensional periodicity [9–11], and two-dimensional elasticity problems with one-dimensional periodicity [12], where the solutions are approximated by linear combinations of the periodic fundamental solutions, that is, the fundamental solutions with singularities in a periodic array. Regarding the MFS for wave problems, the works of Sánchez-Sesma [13] and Sánchez-Sesma and Rosenblueth [14] were the earliest ones in which the MFS were applied as far as the authors know. In [13], two-dimensional elastic scattering problems are examined and the solution is approximated by the MFS which uses a linear combination of the fundamental solution of the Helmholtz equation. In [14], problem of ground motion, more exactly, SH-wave diffraction in canyons is examined and, by expressing the solution as a single layer potential, the problem is formulated as a Fredholm integral equation of the first kind, whose discretization leads to a solution by the MFS. See also [15] as a review of the MFS for scattering and radiation problems. We examine a two-dimensional Dirichlet problem of the Helmholtz equation for $u = u(\mathbf{x})$

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathcal{D} \\ u = f & \text{on } \partial \mathcal{D} \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0 \end{cases} \quad (1)$$

where k is a given positive constant, \mathcal{D} is an exterior simply connected domain with boundary $\partial \mathcal{D}$ of a closed Jordan curve, f is a function given on the boundary $\partial \mathcal{D}$, and the third condition is the Sommerfeld outgoing radiation condition. This problem models two-dimensional waves scattered by an object, where k corresponds to the wave number. In the MFS applied to the problem (1), we approximate the solution by

$$u(\mathbf{x}) \simeq u_N(\mathbf{x}) = \sum_{j=1}^N Q_j H_0^{(1)}(k\|\mathbf{x} - \xi_j\|), \quad (2)$$

where $H_0^{(1)}$ is the Hankel function of order 0 of the first kind, $\|\cdot\|$ is the two-dimensional Euclidean norm, ξ_j ($j = 1, 2, \dots, N$) are source points given in $\mathbb{R}^2 \setminus \overline{\mathcal{D}^3}$ and Q_j ($j = 1, 2, \dots, N$) are complex coefficients. We choose the coefficients Q_j ($j = 1, 2, \dots, N$) so that $u_N(\mathbf{x})$ satisfies the Dirichlet boundary condition collocationally.⁴ Namely, we choose points $\mathbf{x}_i \in \partial \mathcal{D}$ ($i = 1, 2, \dots, N$) which we call the *collocation points* and we determine Q_j such that $u_N(\mathbf{x})$ satisfies the collocation condition

$$u_N(\mathbf{x}_i) = \sum_{j=1}^N Q_j H_0^{(1)}(k\|\mathbf{x}_i - \xi_j\|) = f(\mathbf{x}_i) \quad (i = 1, 2, \dots, N). \quad (3)$$

Eq. (3) form a linear system of equations

$$\mathbf{G} \vec{Q} = \vec{f}, \quad (4)$$

where \mathbf{G} is the $N \times N$ matrix with the (i, j) -element

$$G_{ij} = H_0^{(1)}(k\|\mathbf{x}_i - \xi_j\|) \quad (i, j = 1, 2, \dots, N) \quad (5)$$

and

$$\vec{Q} = [Q_1, Q_2, \dots, Q_N]^t, \quad \vec{f} = [f(\mathbf{x}_1), f(\mathbf{x}_2), \dots, f(\mathbf{x}_N)]^t. \quad (6)$$

It is crucial in general how to arrange the source points ξ_j and the collocation points \mathbf{x}_i for obtaining the approximate solution $u_N(\mathbf{x})$ with high accuracy.

In this paper, we confine ourselves to the case where the domain \mathcal{D} is the exterior to a disk

$$\mathcal{D} = \mathcal{D}_\rho = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| > \rho\}, \quad (7)$$

where ρ is a given positive constant corresponding to the radius of the disk, and establish a theorem on the convergence of the MFS. This is a very special case but is important from a viewpoint of applications, for example, to the FEM–MFS combined

³ Throughout this paper, we denote the set of all positive integers by \mathbb{N} , the set of all integers by \mathbb{Z} , the set of all real numbers by \mathbb{R} and the set of all complex numbers by \mathbb{C} .

⁴ In the methods shown in [3], the coefficients Q_j are determined by the least square matching on the boundary.

method, which is a kind of domain decomposition method, for two-dimensional wave problem in the exterior of a general scattering body [16]. It is natural for this case to arrange the collocation points \mathbf{x}_i and the source points ξ_j as

$$\left. \begin{aligned} \mathbf{x}_i &= \rho \left(\cos \frac{2\pi(i-1)}{N}, \sin \frac{2\pi(i-1)}{N} \right) \\ \xi_i &= q\rho \left(\cos \frac{2\pi(i-1)}{N}, \sin \frac{2\pi(i-1)}{N} \right) \end{aligned} \right\} \quad (i = 1, 2, \dots, N) \quad (8)$$

with a constant q such that $0 < q < 1$. We call the arrangement of the points (8) the *equi-distant equally phased arrangement* and call q the *assignment parameter*. Ushijima and Chiba showed in [17] the unique solvability and the convergence of the MFS applied to the reduced wave problem (1) in \mathcal{D}_ρ with the points in equi-distantly equally phased arrangement (8) as in the following theorem.⁵

Theorem 1. We assume that the parameters k , ρ and q satisfy the condition that

$$J_n(qk\rho) \neq 0 \quad (\forall n \in \mathbb{Z}). \quad (9)$$

1. For sufficiently large N , the MFS determines an approximate solution $u_N(\mathbf{x})$ uniquely, that is, the linear system of Eq. (4) has a unique solution \bar{Q} .
2. We further assume that the boundary data f is a trigonometric polynomial or the plane wave $f(\mathbf{x}) = \exp(ikx)$. Then, we have

$$\sup_{\mathbf{x} \in \partial \mathcal{D}_\rho} |u(\mathbf{x}) - u_N(\mathbf{x})| = O(N^{-m}) \quad \text{as } N \rightarrow \infty \quad (10)$$

for an arbitrary positive integer m . \square

Recently, Chiba and Ushijima [19] have improved the above theorem on the convergence of the MFS as in the following theorem.

Theorem 2. We assume that the parameters k , ρ and q satisfy the condition (9) and that the Fourier coefficients of the boundary data

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(\rho \cos \theta, \rho \sin \theta) e^{-in\theta} d\theta \quad (n \in \mathbb{Z})$$

satisfy

$$|f_n| = O(|H_n^{(1)}(k\rho)J_n(qk\rho)|) \quad \text{as } n \rightarrow \pm\infty.$$

Then, we have

$$\sup_{\mathbf{x} \in \mathcal{D}_\rho} |u(\mathbf{x}) - u_N(\mathbf{x})| = O(q^{N/2}) \quad \text{as } N \rightarrow \infty. \quad \square \quad (11)$$

In this paper, we further improve Chiba and Ushijima's result on the convergence of the MFS (11), namely, we claim that

$$\sup_{\mathbf{x} \in \mathcal{D}_\rho} |u(\mathbf{x}) - u_N(\mathbf{x})| = O(q^N) \quad \text{as } N \rightarrow \infty \quad (12)$$

under different conditions from the conditions in Theorem 2. The following facts should be remarked.

- (1) The above estimate cannot be derived from Theorem 2.
- (2) For cases satisfying the conditions in Theorem 2, our main theorem still gives error estimates of the same order as the error estimate by Theorem 2.
- (3) Our main theorem gives theoretical error estimates for cases to which the results by the previous works cannot do.

In addition, for problems with plane waves as the boundary data, our theorem gives the error estimate of order $O(q^N)$, which is an improvement of the theoretical error estimate of order $O(q^{N/2})$ by Theorem 2. The numerical examples included in this paper support the above theoretical error estimate (12).

Related to the results of this paper mentioned above, we also remark the results of Chiba and Ushijima's study on the MFS applied to reduced wave Neumann problems in the exterior region of a disk [20,21]. In the papers [20,21], the MFS with points in equi-distantly equally phased arrangement with assignment parameter q is estimated theoretically to give an error decay of $O(q^{N/2})$ but numerical examples in the paper show that the error decays of the MFS are of $O(q^N)$. We expect that these results, though they are for Neumann problems, support the result (12) in this paper.

The contents of this paper are as follows. In Section 2, we define some notations and present the main theorem of this paper, which claims the exponential error decay of $O(q^N)$ of the MFS. In Section 3, we present numerical examples for some typical cases, which support the main theorem. In Section 4, we prove the main theorem. Some lemmas used for the proof are proved in the Appendix. In Section 5, we give concluding remarks and refer to some problems for future studies.

⁵ In [18], the condition of Theorem 1 is given as $H_n^{(1)}(k\rho)J_n(qk\rho) \neq 0$ ($\forall n \in \mathbb{Z}$). However, since we have $H_n^{(1)}(k\rho) \neq 0$ ($\forall n \in \mathbb{Z}$) for any k and ρ , this condition is equivalent to the condition (9).

2. Main theorem

We equalize a point in the two-dimensional Euclidean plane $\mathbf{x} = (x, y) \in \mathbb{R}^2$ to the complex number $z = x + iy \in \mathbb{C}$. Then, the collocation points and the source points (8) are respectively rewritten as

$$z_i = \rho \omega^{i-1}, \quad \zeta_i = q \rho \omega^{i-1} \quad (i = 1, 2, \dots, N) \quad (13)$$

with $\omega = e^{2\pi i/N}$, where z_i and ζ_i ($i = 1, 2, \dots, N$) correspond to the collocation points \mathbf{x}_i and the source points ξ_i respectively, and the approximate solution is rewritten as

$$u_N(z) = \sum_{j=1}^N Q_j H_0^{(1)}(k|z - q\rho\omega^{j-1}|). \quad (14)$$

The main theorem of this paper is as follows.

Theorem 3. We assume that the parameters k , ρ and q ($0 < q < 1$) satisfy the condition (9) and that the Fourier coefficients of the boundary data

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) e^{-in\theta} d\theta \quad (n \in \mathbb{Z})$$

decay exponentially, i.e., there exist a constant a satisfying $0 < a < 1$ and a positive constant A_f depending on f such that

$$|f_n| \leq A_f a^{|n|} \quad (n \in \mathbb{Z}). \quad (15)$$

Then the error of the approximate solution $u_N(z)$ of the MFS (14) for the problem (1) with the collocation points and the source points given by (8) is bounded in the closure of the domain \mathcal{D}_ρ by

$$\sup_{z \in \overline{\mathcal{D}_\rho}} |u(z) - u_N(z)| \leq A_f C(k, \rho, a, q) \times \begin{cases} q^N & \text{if } q > \sqrt{a} \\ Nq^N & \text{if } q = \sqrt{a} \\ a^{N/2} & \text{if } q < \sqrt{a} \end{cases} \quad (16)$$

if N is sufficiently large for given k , ρ and q , where $C(k, \rho, a, q)$ is a positive constant depending on k , ρ , a and q only. \square

Remark 1. The following is noted in [18]: for given k (> 0) and ρ (> 0), the condition (9) is satisfied for every $q \in (0, 1)$ except for a finite number of $q \in (0, 1)$ depending on k and ρ . Especially, if $0 < k\rho < j_{0,1}$, where $j_{0,1}$ is the smallest positive zero of the Bessel function $J_0(x)$, the condition (9) is satisfied for every $q \in (0, 1)$. \square

Remark 2. The condition (15) is satisfied if and only if the boundary data is an analytic function in a neighborhood of the boundary. In fact, if $f(z)$ is analytic on the annulus $r_0 \leq |z| \leq \rho^2/r_0$ with r_0 such that $0 < r_0 < \rho$, we have

$$|f_n| \leq \|f\|_{r_0, \infty} \left(\frac{r_0}{\rho} \right)^{|n|} \quad (n \in \mathbb{Z})$$

with

$$\|f\|_{r_0, \infty} = \sup_{r_0 \leq |z| \leq \rho^2/r_0} |f(z)|$$

and Theorem 3 holds by taking $a = r_0/\rho$. Conversely, if the condition (15) holds, let

$$\mathcal{F}_+(z) = \sum_{n=0}^{\infty} f_n z^n, \quad \mathcal{F}_-(z) = \sum_{n=1}^{\infty} f_{-n} z^{-n}.$$

The functions \mathcal{F}_+ , and \mathcal{F}_- , are analytic in a neighborhood of the closed unit disk in z -plane having the origin as its center, and analytic in a neighborhood of the closure of the exterior of the unit disk, respectively. Hence the function $\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_-$ is certainly analytic in a neighborhood of the unit circle in z -plane. Let

$$F(z) = \mathcal{F} \left(\frac{z}{\rho} \right).$$

Then $F(z)$ is analytic in a neighborhood of the circle $\partial\mathcal{D}_\rho$ in z -plane, with radius ρ having the origin as its center. $F(z)$ coincides with the given boundary data $f(\rho e^{i\theta})$ on the boundary $\partial\mathcal{D}_\rho$. \square

Remark 3. Theorem 2 is a special case of Theorem 3. In fact, we have

$$H_n^{(1)}(k\rho)J_n(qk\rho) = O\left(\frac{q^{|n|}}{|n|}\right) \quad \text{as } n \rightarrow \pm\infty,$$

which will be shown in Lemma 7 in the Appendix. Therefore, if f satisfies the condition of Theorem 2, we can take $a = q$ in Theorem 3 and, remarking that $q < \sqrt{q}$, we have $\epsilon_N = O(q^{N/2})$ where $\epsilon_N = \sup_{z \in \mathcal{D}_\rho} |u(z) - u_N(z)|$. \square

It is interesting to compare the above theorem with the convergence theorem of the MFS applied to the Laplace equation problem in the exterior to a disk [4]. We consider the Dirichlet boundary value problem of the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{D}_\rho \\ u = f & \text{on } \partial\mathcal{D}_\rho \\ \sup_{z \in \mathcal{D}_\rho} |u(z)| < \infty, \end{cases} \quad (17)$$

where f is a given function on $\partial\mathcal{D}_\rho$. The MFS⁶ with the source points and the collocation points given by (8) approximates the solution of the problem (17) by

$$u(z) \simeq u_N(z) = Q_0 + \sum_{j=1}^N Q_j \log |z - q\rho\omega^{j-1}|, \quad (18)$$

where the coefficients Q_j ($j = 0, 1, 2, \dots, N$) are determined by the collocation condition

$$u_N(z_i) = Q_0 + \sum_{j=1}^N Q_j \log |z_i - q\rho\omega^{j-1}| = f(z_i) \quad (i = 1, 2, \dots, N) \quad (19)$$

with collocation points $z_i = \rho\omega^{i-1}$ ($i = 1, 2, \dots, N$) and the constraint

$$\sum_{j=1}^N Q_j = 0. \quad (20)$$

For the MFS shown above, we have the following theorem, which is proved by a way similar to that in [4].

Theorem 4. 1. We can determine uniquely the approximate solution u_N of the MFS applied to the Dirichlet problem of the Laplace equation (17), that is, the coefficient matrix of the simultaneous linear equations for Q_j ($j = 0, 1, \dots, N$) formed by (19) and (20) is non-singular.

2. We further assume that the Fourier coefficients of the boundary data

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) e^{-in\theta} d\theta$$

decay exponentially, i.e., there exist a constant a satisfying $0 < a < 1$ and a positive constant A_f depending on f such that the condition (15) is satisfied. Then, we have the inequality

$$\sup_{z \in \mathcal{D}_\rho} |u(z) - u_N(z)| \leq A_f \tilde{C}(\rho, a, q) \times \begin{cases} q^N & \text{if } q > \sqrt{a} \\ Nq^N & \text{if } q = \sqrt{a} \\ a^{N/2} & \text{if } q < \sqrt{a} \end{cases} \quad (21)$$

for sufficiently large N , where $\tilde{C}(\rho, a, q)$ is a positive constant depending on ρ , a and q only. \square

We remark that the convergence rate of the MFS applied to the Helmholtz equation problem (1) given by (16) is the same as the one of the MFS applied to the Laplace equation problem (17) given by (21).

3. Numerical examples

In this section, we show numerical examples for some typical cases, which support our theoretical result. All the computations were performed on a DELL Precision 380 workstation with Intel Pentium 4 CPU 3.80 GHz and 1 GB memory using programs coded in C++ with 100 decimal digit precision working except for the problem with $k\rho = 10$ in Example 3, where the computations are worked out in 200 decimal digit precision, by the multiple precision arithmetic library *exflib* [24].

⁶ As a scheme of the MFS for potential problems, we consider the invariant scheme proposed in [22,23] which remains invariant with respect to affine transformations due to the constraint (20).

Example 1. The first example is the problem with the boundary data of single mode cosine function.

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathcal{D}_\rho \\ u = \cos m\theta & \text{on } \partial \mathcal{D}_\rho \ (m = 1, 8, 16) \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0. \end{cases} \quad (22)$$

The exact solution $u(z)$ of this problem is

$$u(z) = \frac{H_m^{(1)}(kr)}{H_m^{(1)}(k\rho)} \cos m\theta. \quad (23)$$

We computed the approximate solution u_N by obtaining the coefficients Q_j which are the solution of the linear system of equations (4) rewritten as

$$G' \vec{Q}' = \vec{f}' \quad (24)$$

with

$$G' = \begin{bmatrix} \operatorname{Re} G_{11} & -\operatorname{Im} G_{11} & \cdots & \operatorname{Re} G_{1N} & -\operatorname{Im} G_{1N} \\ \operatorname{Im} G_{11} & \operatorname{Re} G_{11} & \cdots & \operatorname{Im} G_{1N} & \operatorname{Re} G_{1N} \\ \vdots & \vdots & & \vdots & \vdots \\ \operatorname{Re} G_{N1} & -\operatorname{Im} G_{N1} & \cdots & \operatorname{Re} G_{NN} & -\operatorname{Im} G_{NN} \\ \operatorname{Im} G_{N1} & \operatorname{Re} G_{N1} & \cdots & \operatorname{Im} G_{NN} & \operatorname{Re} G_{NN} \end{bmatrix}, \quad (25)$$

$$\vec{Q}' = [\operatorname{Re} Q_1 \operatorname{Im} Q_1 \cdots \operatorname{Re} Q_N \operatorname{Im} Q_N]^t,$$

and

$$\vec{f}' = [\operatorname{Re} f_1 \operatorname{Im} f_1 \cdots \operatorname{Re} f_N \operatorname{Im} f_N]^t,$$

and substituting Q_j into (2). From Theorem 3, we expect that the error estimate

$$\epsilon_N = \sup_{z \in \mathcal{D}_\rho} |u_N(z) - u(z)| \quad (26)$$

obeys

$$\epsilon_N = O(q^N) \quad (27)$$

since we can take the constant a in (15) as an arbitrary small positive number less than 1 for this example. In order to examine this theoretical error estimate, we computed the value⁷

$$\tilde{\epsilon}_N = \max_{i=1, \dots, 1000} |u_N(\tilde{z}_i) - u(\tilde{z}_i)|, \quad (28)$$

where \tilde{z}_i ($i = 1, 2, \dots, 1000$) are random points distributed on the set

$$\tilde{\mathcal{D}}_\rho = \{z \in \overline{\mathcal{D}_\rho} \mid |\operatorname{Re} z| \leq R, |\operatorname{Im} z| \leq R\} \quad (R = 10), \quad (29)$$

instead of ϵ_N . The points \tilde{z}_i are chosen different in each computation. Fig. 1 shows the behavior of $\tilde{\epsilon}_N$, as a function of N , for $m = 1, 8, 16$ and $k\rho = 1$, and 10. From Fig. 1, we find that, if N is sufficiently large, $\tilde{\epsilon}_N$ obeys the relation

$$\tilde{\epsilon}_N \simeq (\text{positive constant}) \times \alpha^N, \quad (30)$$

where α is a constant such that $0 < \alpha < 1$. Table 1 shows the constant α in (30) for each $m = 1, 8, 16$ and each $k\rho = 1$ and 10 estimated by the least squares fitting using the *fit* command of the software *gnuplot*. From the table, we find that $\alpha \simeq q$ for each m and each $k\rho$. These results are in agreement with the theoretical error estimate (27) obtained from Theorem 3.

Besides, we remark that, in the cases of $m = 8$ and 16, the error estimates $\tilde{\epsilon}_N$ become very small at $N = 2m$. The reason why this phenomena occur will be given in the Appendix.

⁷ In the computations of the Hankel functions $H_n^{(1)}(\cdot) = J_n(\cdot) + iY_n(\cdot)$ ($n = 0, 1, 2, \dots$) which appear in (23) and so on, we computed the Bessel functions $J_n(\cdot)$ and $Y_n(\cdot)$ using the programs in the header file "exfut11.h" included in the *exflib* library.

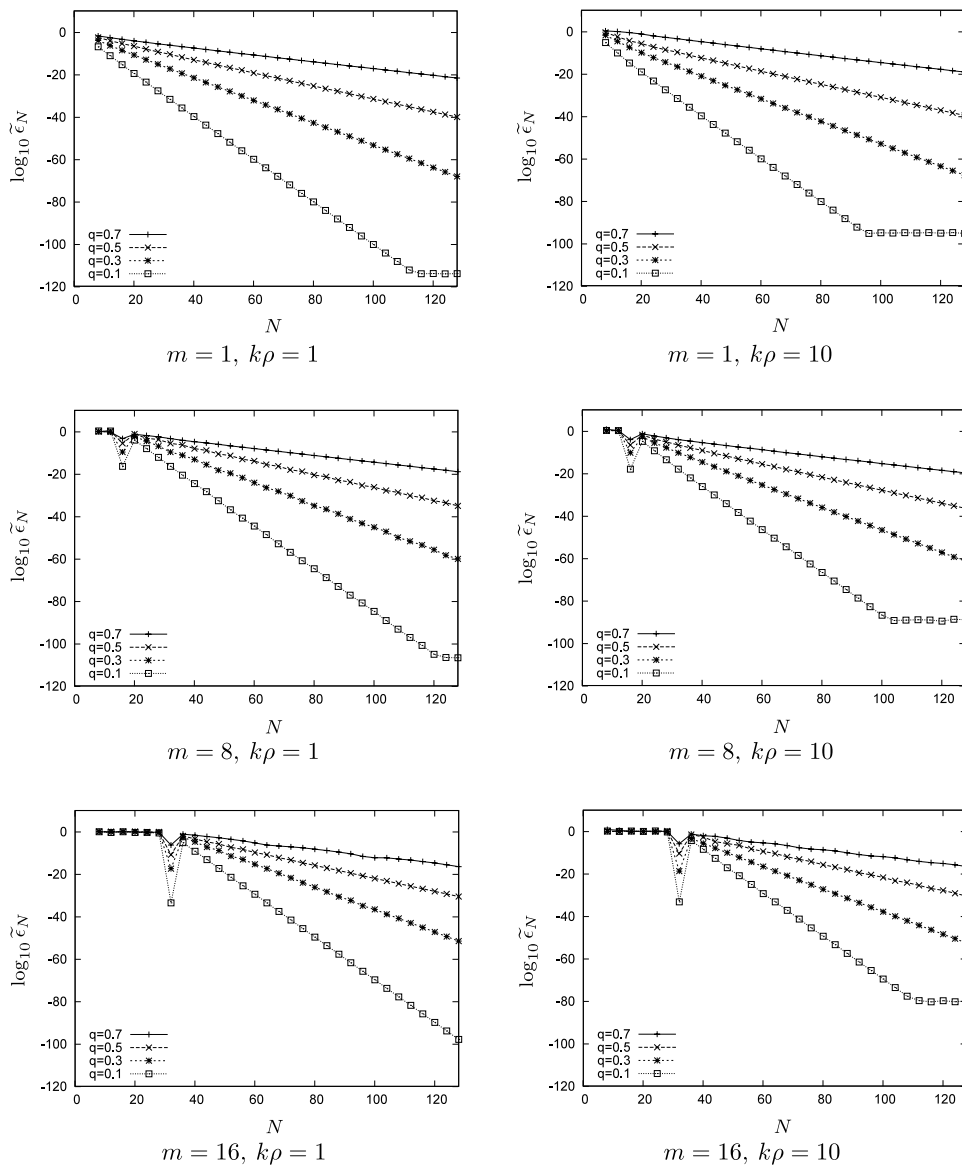


Fig. 1. The error estimate $\tilde{\epsilon}_N$ of the MFS applied to [Example 1](#).

Table 1

The constant α in the experimental error estimate $\tilde{\epsilon}_N \simeq (\text{positive constant}) \times \alpha^N$ of (30).

q		0.1	0.3	0.5	0.7
α	$m = 1$				
	$k\rho = 1$	0.098	0.29	0.49	0.69
	$k\rho = 10$	0.094	0.29	0.46	0.68
	$m = 8$				
	$k\rho = 1$	0.098	0.29	0.49	0.69
	$k\rho = 10$	0.096	0.29	0.49	0.68
	$m = 16$				
	$k\rho = 1$	0.098	0.29	0.48	0.68
	$k\rho = 10$	0.097	0.29	0.49	0.68

We also find that, in the case of $k\rho = 10$, the error estimates $\tilde{\epsilon}_N$ do not decrease if N is large and remain of order 10^{-80} – 10^{-100} . From these results, it seems that the problems are ill-conditioned if $k\rho$ is large. [Fig. 2](#) shows the L^∞ -condition numbers $\kappa(G')$ of the matrices G' of (25) for $k\rho = 1$ and 10 computed through a program included in the *exflib* library. From [Fig. 2](#), it can be seen that condition numbers $\kappa(G')$ are very large and increase exponentially, as functions of N , but there are not much differences between the cases $k\rho = 1$ and 10.

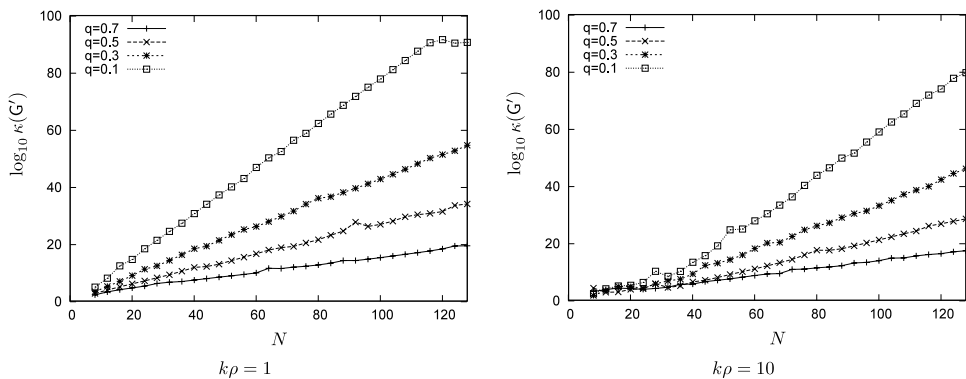


Fig. 2. The L^∞ -condition number $\kappa(G')$ of the matrix G' of (25).

Example 2. The second example is the problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathcal{D}_\rho \\ u = \frac{1 - b \cos \theta}{1 - 2b \cos \theta + b^2} & \text{on } \partial \mathcal{D}_\rho \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0 \end{cases} \quad (31)$$

with a constant b such that $0 < b < 1$ which is taken as $b = 0.16$ here. The exact solution of this problem is

$$u(z) = \frac{H_0^{(1)}(kr)}{H_0^{(1)}(k\rho)} + \sum_{n=1}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} b^n \cos n\theta. \quad (32)$$

This problem may be artificial, but, as explained below, the authors believe that it is a good example for examining whether our theoretical result in Theorem 3 gives error estimates which agree with numerical results. The boundary data f is expanded as

$$f(\rho e^{i\theta}) = 1 + \frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} b^{|n|} e^{in\theta},$$

which implies that the order of the Fourier coefficients are given as $f_n = O(b^{|n|})$ (as $n \rightarrow \pm\infty$). Therefore, we can take the constant a in (15) as $a = b = 0.16$ and, from Theorem 3, we find that

$$\epsilon_N = \begin{cases} O(q^N) & \text{if } q > \sqrt{b} = 0.4 \\ O(b^{N/2}) = O(0.4^N) & \text{if } q < \sqrt{b} = 0.4. \end{cases} \quad (33)$$

To examine this theoretical error estimate, we computed the value $\tilde{\epsilon}_N$ defined by Eq. (28).⁸ The points \tilde{z}_i , $i = 1, 2, \dots, 1000$ used in computing $\tilde{\epsilon}_N$ by (28) are different from the points used in Example 1 and are chosen different in each computation. Fig. 3 shows the behavior of the error estimates $\tilde{\epsilon}_N$ of the MFS as functions of N . From Fig. 3, we find that $\tilde{\epsilon}_N$ obeys the relation (30) if N is large. Table 2 shows the constant α in (30) estimated by the least squares fitting. From Table 2, we find that we have

$$\alpha \simeq \begin{cases} q & \text{if } q > 0.4 \\ 0.4 & \text{if } q < 0.4, \end{cases}$$

that is,

$$\tilde{\epsilon}_N = \begin{cases} O(q^N) & \text{if } q > 0.4 \\ O(0.4^N) & \text{if } q < 0.4 \end{cases}$$

and these results are in agreement with the theoretical error estimate (33).

⁸ In the computations of $\tilde{\epsilon}_N$, we computed the exact solution $u(z)$ in (32) by truncating the infinite sum on the right-hand side at the v th term such that $u^{(v)}(z) \simeq u^{(v-1)}(z)$ numerically with

$$u^{(v)}(z) = \frac{H_0^{(1)}(kr)}{H_0^{(1)}(k\rho)} + \sum_{n=1}^v \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} b^n \cos n\theta.$$

Table 2

The constant α in the experimental error estimate $\tilde{\epsilon}_N \simeq (\text{positive constant}) \times \alpha^N$ of (30).

q	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
α	$k\rho = 1$	0.38	0.40	0.39	0.40	0.49	0.59	0.69
	$k\rho = 10$	0.39	0.38	0.39	0.39	0.48	0.58	0.68

Table 3

The constant α in the experimental error estimate of (30).

q	0.1	0.3	0.5	0.7
α	$k\rho = 1$	0.10	0.29	0.49
	$k\rho = 10$	0.21	0.30	0.49

Example 3. The last example is the problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathcal{D}_\rho \\ u = -\exp(ik(\operatorname{Re} z)) & \text{on } \partial \mathcal{D}_\rho \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \end{cases} \quad (34)$$

which illustrate the scattering of a plane wave by a cylinder. The exact solution of this problem is

$$u(z) = - \left\{ J_0(k\rho) \frac{H_0^{(1)}(kr)}{H_0^{(1)}(k\rho)} + 2 \sum_{n=1}^{\infty} i^n J_n(k\rho) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} \cos n\theta \right\}. \quad (35)$$

The boundary data f is expanded as

$$f(\rho e^{i\theta}) = - \sum_{n \in \mathbb{Z}} i^n J_n(k\rho) e^{in\theta},$$

and, by the asymptotic formula for fixed $x > 0$ (see Section 9.3 of [25])

$$J_m(x) \sim \frac{1}{\sqrt{2\pi m}} \left(\frac{ex}{2m} \right)^m$$

and the formula $J_{-m}(x) = (-1)^m J_m(x)$, we can take the constant a in (15) as an arbitrary small positive number less than 1. Therefore, from Theorem 3, we expect that

$$\epsilon_N = O(q^N). \quad (36)$$

In order to examine this theoretical error estimate, we computed the value $\tilde{\epsilon}_N$ defined by Eq. (28).⁹ We computed $\tilde{\epsilon}_N$ with 200 decimal digit precision only in the case of $k\rho = 10$. Fig. 4 shows the behavior of the numerical error estimates $\tilde{\epsilon}_N$ as functions of N . From Fig. 4, we find that $\tilde{\epsilon}_N$ obeys the relation (30) if N is large. Table 3 shows the constant α in (30) estimated by the least squares fitting. From Table 3, we find that, if $k\rho = 1$, $\alpha \simeq q$, that is,

$$\tilde{\epsilon}_N = O(q^N), \quad (37)$$

which agrees with the theoretical error estimate (36), except for $q = 0.1$. The reason that the error estimate $\tilde{\epsilon}_N$, for the case $q = 0.1$ and $k\rho = 10$, does not obey the relation (37) may be that N taken in this example is not “sufficiently” large in the sense of Theorem 3.

Besides, we find from Fig. 4 that the decay of the error $\tilde{\epsilon}_N$ in the case $k\rho = 10$, as N increases, is slow compared with the error in the case $k\rho = 1$. To understand this experimental convergence behavior thoroughly, the following points need to be examined.

- (1) The necessity of fine discretization for the case of large wave number k , that is, small wave length.
- (2) The effect of round-off error for the ill-conditioned linear system of Eq. (4).

The examinations for the above points remain our future works.

⁹ In the computations of $\tilde{\epsilon}_N$, we computed the exact solution $u(z)$ in (35) by truncating the infinite sum on the right-hand side at the v th term such that $u^{(v)}(z) \simeq u^{(v-1)}(z)$ numerically with

$$u^{(v)}(z) = - \left\{ J_0(kr) \frac{H_0^{(1)}(kr)}{H_0^{(1)}(k\rho)} + 2 \sum_{n=1}^v i^n J_n(k\rho) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} \cos n\theta \right\}.$$

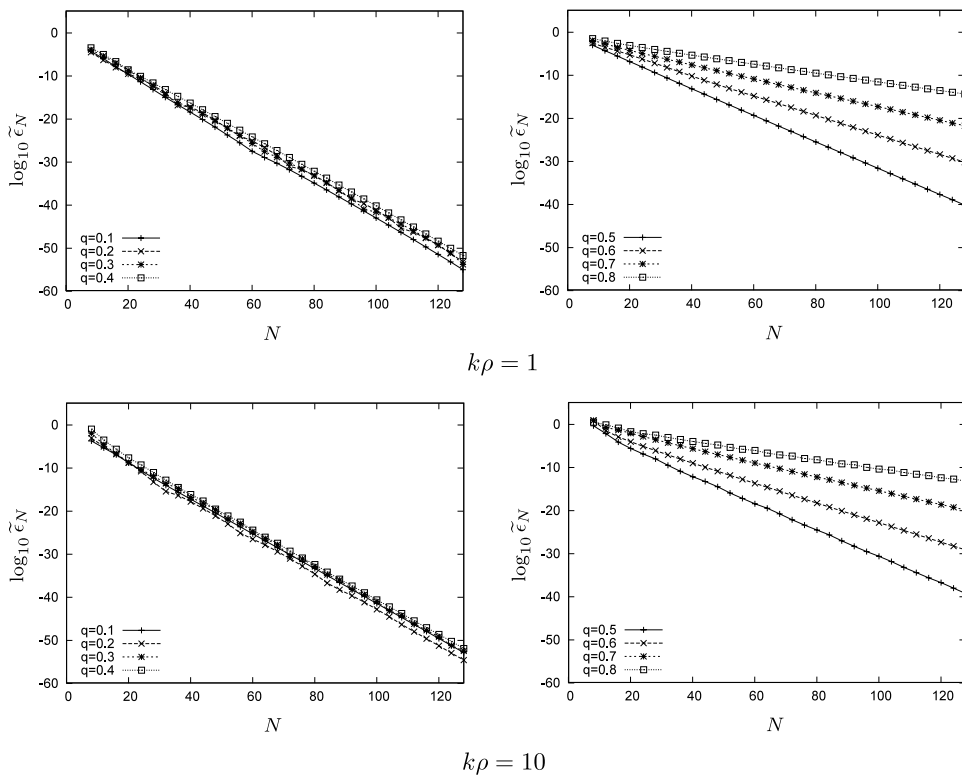


Fig. 3. The error estimate $\tilde{\epsilon}_N$ of the MFS applied to Example 2.

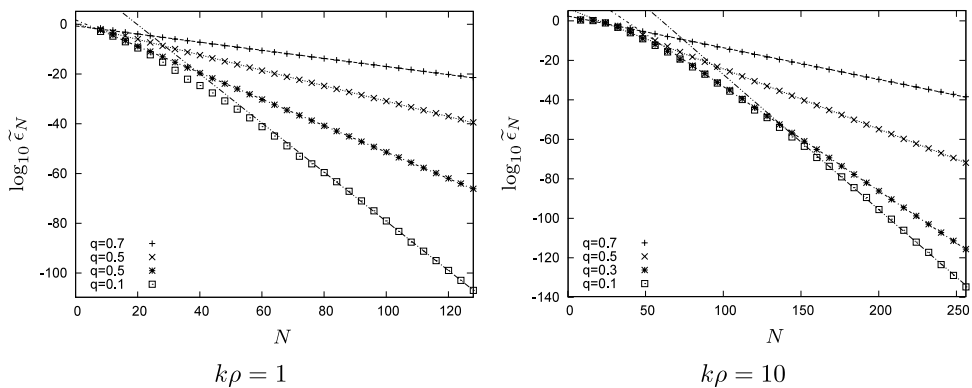


Fig. 4. The error estimate $\tilde{\epsilon}_N$ of the MFS applied to Example 3. The least squares fitting lines are also included.

4. Proof of the main theorem

We prove Theorem 3 in two steps. In the first step, we obtain explicit expressions of the approximate solution $u_N(z)$ and the error function $e_N(z) \equiv u(z) - u_N(z)$. In the second step, we obtain an upper bound of $\sup_{z \in \overline{\mathcal{D}_\rho}} |e_N(z)|$.

Step 1. In the problem considered in this study, the coefficient matrix G of the linear system of Eq. (4) is given by

$$G = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix}, \quad G_{ij} = H_0^{(1)}(k\rho|1 - q\omega^{j-i}|) \quad (1 \leq i, j \leq N). \quad (38)$$

Remarking that G is a cyclic matrix, G is diagonalized explicitly as

$$W^{-1}GW = N \begin{bmatrix} g_0^{(N)}(\rho) & & & \\ & g_1^{(N)}(\rho) & & \\ & & \ddots & \\ & & & g_{N-1}^{(N)}(\rho) \end{bmatrix}, \quad (39)$$

where W is the $N \times N$ matrix with the (i, j) -element

$$W_{ij} = \frac{1}{\sqrt{N}} \omega^{(i-1)(j-1)} \quad (40)$$

and

$$g_n^{(N)}(z) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{nj} H_0^{(1)}(k|z - q\rho\omega^j|), \quad z \in \overline{\mathcal{D}_\rho}, \quad n \in \mathbb{Z}. \quad (41)$$

The symbol $g_n^{(N)}(z)$ will play an important role in the proof of the theorem.

From (39), we have

$$G^{-1} = \frac{1}{N} W \begin{bmatrix} g_0^{(N)}(\rho)^{-1} & & & \\ & g_1^{(N)}(\rho)^{-1} & & \\ & & \ddots & \\ & & & g_{N-1}^{(N)}(\rho)^{-1} \end{bmatrix} W^{-1}$$

and

$$\text{the } (j, l)\text{-element of } G^{-1} = \frac{1}{N^2} \sum_{m=0}^{N-1} \frac{\omega^{m(j-l)}}{g_m^{(N)}(\rho)}.$$

Here, we remark that

$$g_n^{(N)}(\rho) \neq 0, \quad n = 0, 1, \dots, N-1. \quad (42)$$

In fact, from the assumption (9), we have $g_n^{(N)}(\rho) \neq 0$ ($0 \leq |n| \leq N/2$) by Lemma 8 in the Appendix, and we have (42) by the obvious relation

$$g_n^{(N)}(z) = g_m^{(N)}(z) \quad \text{if } n \equiv m \pmod{N}. \quad (43)$$

Then, we obtain the coefficients Q_j ($j = 1, 2, \dots, N$), i.e., the solution of the linear system of Eq. (4) as

$$Q_j = \frac{1}{N^2} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \omega^{m(j-l-1)} \frac{f(\rho\omega^l)}{g_m^{(N)}(\rho)}. \quad (44)$$

Substituting this into (14), we obtain the expression of the approximate solution $u_N(z)$ as

$$\begin{aligned} u_N(z) &= \frac{1}{N^2} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \omega^{(j-l)m} \frac{f(\rho\omega^l)}{g_m^{(N)}(\rho)} H_0^{(1)}(k|z - q\rho\omega^j|) \\ &= \frac{1}{N} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \omega^{-lm} f(\rho\omega^l) \frac{g_m^{(N)}(z)}{g_m^{(N)}(\rho)}. \end{aligned} \quad (45)$$

Further, we rewrite the expression (45) using the Fourier expansion of the boundary data $f(\rho e^{i\theta}) = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}$. Substituting $f(\rho\omega^l) = \sum_{n \in \mathbb{Z}} f_n \omega^{nl}$ into (45), we have

$$\begin{aligned} u_N(z) &= \frac{1}{N} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n \in \mathbb{Z}} \omega^{(n-m)l} f_n \frac{g_m^{(N)}(z)}{g_m^{(N)}(\rho)} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n \in \mathbb{Z}} \left\{ \sum_{l=0}^{N-1} \omega^{(n-m)l} \right\} f_n \frac{g_m^{(N)}(z)}{g_m^{(N)}(\rho)} \\ &= \sum_{m=0}^{N-1} \sum_{n \equiv m \pmod{N}} f_n \frac{g_m^{(N)}(z)}{g_m^{(N)}(\rho)} = \sum_{m=0}^{N-1} \sum_{n \equiv m \pmod{N}} f_n \frac{g_n^{(N)}(z)}{g_n^{(N)}(\rho)}, \end{aligned}$$

where we used the relation

$$\sum_{l=0}^{N-1} \omega^{(n-m)l} = \begin{cases} N & \text{if } n \equiv m \pmod{N} \\ 0 & \text{otherwise} \end{cases} \quad (46)$$

on the third equality, and we used (43) on the fourth equality. Therefore, we obtain the expression of $u_N(z)$ as

$$u_N(z) = \sum_{n \in \mathbb{Z}} f_n \frac{g_n^{(N)}(z)}{g_n^{(N)}(\rho)}. \quad (47)$$

On the other hand, the exact solution of our problem (1) is written as

$$u(re^{i\theta}) = \sum_{n \in \mathbb{Z}} f_n \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} e^{in\theta}. \quad (48)$$

Finally, the error function $e_N(z) \equiv u(z) - u_N(z)$ is written as

$$e_N(re^{i\theta}) = \sum_{n \in \mathbb{Z}} f_n \phi_n^{(N)}(r, \theta), \quad (49)$$

where

$$\phi_n^{(N)}(r, \theta) = \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} e^{in\theta} - \frac{g_n^{(N)}(re^{i\theta})}{g_n^{(N)}(\rho)}. \quad (50)$$

Step 2. We estimate the error of the MFS using the inequality

$$|e_N(re^{i\theta})| \leq \sum_{n \in \mathbb{Z}} |f_n| |\phi_n^{(N)}(r, \theta)|, \quad r \geq \rho, \quad 0 \leq \theta < 2\pi \quad (51)$$

obtained by (49). For this purpose, we need upper bounds for $|f_n|$ and $|\phi_n^{(N)}(r, \theta)|$. An upper bound of $|f_n|$ is given by the assumption (15) of Theorem 3. Regarding an upper bound of $|\phi_n^{(N)}(r, \theta)|$, we have the following lemma.

Lemma 5. We assume that the constants k , ρ and q satisfy the condition (9). Then there exist a positive integer $N'(k\rho, q)$ and positive constants $C'(k\rho, q)$ and $C''(k\rho, q)$ which depend on $k\rho$ and q only such that, if $N \geq N'(k\rho, q)$,

$$|\phi_n^{(N)}(r, \theta)| \leq C'(k\rho, q) q^{N-2|n|} \quad \left(0 \leq |n| \leq \frac{N}{2}, r \geq \rho, 0 \leq \theta < 2\pi \right), \quad (52)$$

$$|\phi_n^{(N)}(r, \theta)| \leq C''(k\rho, q) \quad (n \in \mathbb{N}, r \geq \rho, 0 \leq \theta < 2\pi). \quad (53)$$

This lemma will be proved in the Appendix.

We are now ready to prove Theorem 3. We divide the infinite sum on the right-hand side of (51) into three terms as

$$|e_N(re^{i\theta})| \leq |f_0| |\phi_0^{(N)}(r, \theta)| + \sum_{1 \leq |n| \leq N/2} |f_n| |\phi_n^{(N)}(r, \theta)| + \sum_{|n| > N/2} |f_n| |\phi_n^{(N)}(r, \theta)|. \quad (54)$$

We assume that $N \geq N'(k\rho, q)$. For the first term, we have

$$|f_0| |\phi_0^{(N)}(r, \theta)| \leq C'(k\rho, q) A_f q^N \quad (55)$$

by (15) and (52). For the third term, we have

$$\begin{aligned} \sum_{|n| > N/2} |f_n| |\phi_n^{(N)}(r, \theta)| &\leq 2C''(k\rho, q) A_f \sum_{n=\lfloor N/2 \rfloor + 1}^{\infty} a^n \\ &\leq 2C''(k\rho, q) A_f \frac{a^{\lfloor N/2 \rfloor + 1}}{1-a} \leq \frac{2C''(k\rho, q)}{1-a} A_f a^{N/2}. \end{aligned} \quad (56)$$

by the inequalities (15) and (53), where $\lfloor N/2 \rfloor$ is the largest integer less than or equal to $N/2$. For the second term, by (15) and (52), we have

$$\begin{aligned} \sum_{1 \leq |n| \leq N/2} A_f |\phi_n^{(N)}(r, \theta)| &\leq 2C'(k\rho, q) A_f \sum_{n=1}^{\lfloor N/2 \rfloor} q^{N-2n} a^n \\ &\leq 2C'(k\rho, q) A_f q^N \sum_{n=1}^{\lfloor N/2 \rfloor} \left(\frac{a}{q^2} \right)^n \\ &\leq (\text{constant depending on } k, \rho, a \text{ and } q \text{ only}) \times A_f \begin{cases} q^N & \text{if } q > \sqrt{a} \\ Nq^N & \text{if } q = \sqrt{a} \\ a^{N/2} & \text{if } q < \sqrt{a}, \end{cases} \end{aligned} \quad (57)$$

where we used

$$\sum_{n=1}^{\lfloor N/2 \rfloor} \left(\frac{a}{q^2} \right)^n \leq \begin{cases} \frac{a/q^2}{1 - a/q^2} & \text{if } a/q^2 < 1 \\ N/2 & \text{if } a/q^2 = 1 \\ \frac{(a/q^2)^{N/2+1}}{a/q^2 - 1} & \text{if } a/q^2 > 1 \end{cases}$$

on the third inequality.

By (55), (56) and (57), we obtain the inequality (16).

5. Concluding remarks

In this paper, we presented a theoretical error estimate of the MFS to the Dirichlet problem of the Helmholtz equation in the exterior of a disk. In the Theorem 3 of this paper, we have shown that the error of the MFS with equi-distantly equally phased arrangement of the collocation and source points with assignment parameter q is of order $O(q^N)$. This theorem includes the results of the previous studies. We also showed numerical examples for some problems, and the results are in good agreement with the theoretical error estimate.

Problems related to future studies are as follows.

- Theoretical error estimates of the MFS applied to Neumann problems of the Helmholtz equation in the exterior of a disk and boundary value problems of the Helmholtz equation in a general exterior domain such as the exterior of an ellipse.
- Improvement of the scheme of the MFS for reduced wave problems with large wave numbers k .

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Appendix A. Proof of Lemma 5

In order to prove Lemma 5, we prepare the following four lemmas, Lemmas 6–9.

Lemma 6. The symbol $g_n^{(N)}(z)$ defined in (41) is also expressed as

$$g_n^{(N)}(z) = \sum_{m \equiv n \pmod{N}} H_m^{(1)}(kr) J_m(qk\rho) e^{im\theta} \quad (z = re^{i\theta}). \quad (\text{A.1})$$

This lemma is proved in Section 3 of [19], where Graf's addition theorem for the Bessel functions (see Section 11.3 of [26]) is used.

Lemma 7. We assume that k , ρ and q ($0 < q < 1$) satisfy the condition (9).

1. There exists a positive integer $N_1(k\rho, q)$ depending on $k\rho$ and q only such that

$$\left| H_m^{(1)}(k\rho) J_m(qk\rho) - \frac{q^{|m|}}{i\pi|m|} \right| \leq \frac{q^{|m|}}{2\pi|m|} \quad \text{if } m \in \mathbb{Z}, |m| \geq N_1(k\rho, q). \quad (\text{A.2})$$

2. There exist positive constants $C_1^{(L)}(k\rho, q)$ and $C_1^{(U)}(k\rho, q)$ depending on $k\rho$ and q such that

$$C_1^{(L)}(k\rho, q) \frac{q^{|m|}}{|m|} \leq |H_m^{(1)}(k\rho) J_m(qk\rho)| \leq C_1^{(U)}(k\rho, q) \frac{q^{|m|}}{|m|} \quad (\forall m \in \mathbb{Z} \setminus \{0\}). \quad (\text{A.3})$$

This lemma will be proved later.

Lemma 8. We assume the condition (9).

1. There exist positive constants $C_2^{(U)}(k\rho, q)$ and $C_3^{(U)}(k\rho, q)$ which depend on $k\rho$ and q ($0 < q < 1$) only such that

$$|g_n^{(N)}(z)| \leq C_2^{(U)}(k\rho, q) \frac{q^{|n|}}{|n|} \quad \text{if } |z| \geq \rho, 1 \leq |n| \leq \frac{N}{2}, \quad (\text{A.4})$$

$$|g_0^{(N)}(z)| \leq C_3^{(U)}(k\rho, q) \quad \text{if } |z| \geq \rho. \quad (\text{A.5})$$

2. There exist a positive integer $N_2(k\rho, q)$ and positive constants $C_2^{(L)}(k\rho, q)$ and $C_3^{(L)}(k\rho, q)$ which depend on $k\rho$ and q ($0 < q < 1$) only such that, if $N \geq N_2(k\rho, q)$,

$$|g_n^{(N)}(\rho)| \geq C_2^{(L)}(k\rho, q) \frac{q^{|n|}}{|n|} \quad \text{if } 1 \leq |n| \leq \frac{N}{2}, \quad (\text{A.6})$$

$$|g_0^{(N)}(\rho)| \geq C_3^{(L)}(k\rho, q). \quad (\text{A.7})$$

This lemma will be proved later.

Lemma 9. $|H_n^{(1)}(kr)|$ is a decreasing function of $r (\geq \rho)$.

This lemma is proved in the proof of Proposition 6 of [19].

Using the above lemmas, we prove Lemma 5 as follows.

First we prove inequality (52). In the case where $1 \leq |n| \leq N/2$, by (A.1), we express $\phi_n^{(N)}(r, \theta)$ as

$$\begin{aligned} \phi_n^{(N)}(r, \theta) &= \left\{ g_n^{(N)}(\rho) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} e^{in\theta} - g_n^{(N)}(re^{i\theta}) \right\} / g_n^{(N)}(\rho) \\ &= \sum_{\substack{m \equiv n \\ \text{mod } N}} J_m(qk\rho) \left\{ H_m^{(1)}(k\rho) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} e^{in\theta} - H_m^{(1)}(kr) e^{im\theta} \right\} / g_n^{(N)}(\rho). \end{aligned} \quad (\text{A.8})$$

For the denominator, by (A.6), we have

$$|g_n^{(N)}(\rho)| \geq C_2^{(L)}(k\rho, q) \frac{q^{|n|}}{|n|} \geq 2C_2^{(L)}(k\rho, q) \frac{q^{|n|}}{N} \quad (\text{A.9})$$

if $N \geq N_2(k\rho, q)$. For the numerator, remarking that the sum does not include the term of $m = n$, we have

$$\begin{aligned} &\left| \sum_{m \equiv n} J_m(qk\rho) \left\{ H_m^{(1)}(k\rho) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} e^{in\theta} - H_m^{(1)}(kr) e^{im\theta} \right\} \right| \\ &\leq \sum_{m \equiv n (m \neq n)} |J_m(qk\rho) H_m^{(1)}(k\rho)| \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} \right| + \sum_{m \equiv n (m \neq n)} |H_m^{(1)}(kr) J_m(qk\rho)| \\ &\leq 2 \sum_{m \equiv n (m \neq n)} |H_m^{(1)}(k\rho) J_m(qk\rho)| \\ &\leq 2 \sum_{l=1}^{\infty} |H_{n+Nl}^{(1)}(k\rho) J_{n+Nl}(qk\rho)| + 2 \sum_{l=1}^{\infty} |H_{n-Nl}^{(1)}(k\rho) J_{n-Nl}(qk\rho)| \\ &\leq 2C_1^{(U)}(k\rho, q) \left\{ \sum_{l=1}^{\infty} \frac{q^{n+Nl}}{n+Nl} + \sum_{l=1}^{\infty} \frac{q^{-n+Nl}}{-n+Nl} \right\} \\ &\leq 2C_1^{(U)}(k\rho, q) \left\{ \frac{q^{N+|n|}}{(N+|n|)(1-q^N)} + \frac{q^{N-|n|}}{(N-|n|)(1-q^N)} \right\} \\ &\leq \frac{8C_1^{(U)}(k\rho, q)}{N(1-q^N)} q^{N-|n|} \leq \frac{8C_1^{(U)}(k\rho, q)}{N(1-q)} q^{N-|n|}, \end{aligned} \quad (\text{A.10})$$

where we used Lemma 9 on the second inequality and (A.3) on the fourth inequality. Consequently, by (A.9) and (A.10), we have (52) for the case that $1 \leq |n| \leq N/2$. The inequality for the case $n = 0$ is proved similarly.

Second, we prove (53). Since $g_n^{(N)}(z) = g_{n'}^{(N)}(z)$ if $n \equiv n' \pmod{N}$, we only have to prove the inequality for the case that $|n| \leq N/2$. If $1 \leq |n| \leq N/2$, we have by (A.4) and (A.6)

$$|\phi_n^{(N)}(r, \theta)| \leq 1 + \frac{|g_n^{(N)}(re^{i\theta})|}{|g_n^{(N)}(\rho)|} \leq 1 + \frac{C_2^{(U)}(k\rho, q)}{C_2^{(L)}(k\rho, q)}$$

if $N \geq N_2(k\rho, q)$. Consequently, we obtain inequality (53) if we take N sufficiently large for given $k\rho$ and q . Inequality (53) for $n = 0$ is obtained similarly.

We still have to prove the lemmas used in this section. Among these, Lemmas 6 and 9 are proved in [19] as mentioned above. Therefore, we only have to prove Lemmas 7 and 8.

Proof of Lemma 7.

1. By the asymptotic formulas for fixed $x > 0$ (see Section 9.3 of [25])

$$H_m^{(1)}(x) \sim -i\sqrt{\frac{2}{\pi m}} \left(\frac{2m}{ex}\right)^m, \quad J_m(x) \sim \frac{1}{\sqrt{2\pi m}} \left(\frac{ex}{2m}\right)^m \quad \text{as } m \rightarrow \infty,$$

we have

$$H_m^{(1)}(k\rho)J_m(qk\rho) \sim \frac{q^m}{i\pi m} \quad \text{as } m \rightarrow \infty.$$

This implies that there exists a positive integer $N_1(k\rho, q)$ depending on $k\rho$ and q only such that

$$\left| \frac{H_m^{(1)}(k\rho)J_m(qk\rho)}{q^m/(i\pi m)} - 1 \right| \leq \frac{1}{2} \quad \text{if } m \in \mathbb{N}, m \geq N_1(k\rho, q).$$

Therefore, remarking $H_{-m}^{(1)}(x) = (-1)^m H_m^{(1)}(x)$, $J_{-m}(x) = (-1)^m J_m(x)$, we obtain the inequality (A.2).

2. From (A.2), we have

$$\frac{q^{|m|}}{2\pi|m|} \leq |H_m^{(1)}(k\rho)J_m(qk\rho)| \leq \frac{3q^{|m|}}{2\pi|m|} \quad \text{if } m \in \mathbb{Z}, |m| \geq N_1(k\rho, q).$$

Then, we obtain the inequality (A.3) by putting

$$C_1^{(L)}(k\rho, q) = \min \left\{ \frac{1}{2\pi}, \frac{|m|}{q^{|m|}} |H_m^{(1)}(k\rho)J_m(qk\rho)| \mid 0 < |m| < N_1(k\rho, q) \right\}, \quad (\text{A.11})$$

$$C_1^{(U)}(k\rho, q) = \max \left\{ \frac{3}{2\pi}, \frac{|m|}{q^{|m|}} |H_m^{(1)}(k\rho)J_m(qk\rho)| \mid 0 < |m| < N_1(k\rho, q) \right\}, \quad (\text{A.12})$$

where we remark that $C_1^{(L)}(k\rho, q) > 0$ since we assume the condition (9). \square

Proof of Lemma 8.

1. By (A.1) and (A.3), we have

$$|g_n^{(N)}(z)| \leq \sum_{m \equiv n \pmod{N}} |H_m^{(1)}(kr)J_m(qk\rho)| \leq \sum_{m \equiv n \pmod{N}} |H_m^{(1)}(k\rho)J_m(qk\rho)|.$$

In the case $1 \leq |n| \leq N/2$, we have

$$\begin{aligned} |g_n^{(N)}(z)| &\leq |H_n^{(1)}(k\rho)J_n(qk\rho)| + \sum_{l=1}^{\infty} |H_{n+Nl}^{(1)}(k\rho)J_{n+Nl}(qk\rho)| + \sum_{l=1}^{\infty} |H_{n-Nl}^{(1)}(k\rho)J_{n-Nl}(qk\rho)| \\ &\leq C_1^{(U)}(k\rho, q) \left\{ \frac{q^{|n|}}{|n|} + \sum_{l=1}^{\infty} \frac{q^{n+Nl}}{n+Nl} + \sum_{l=1}^{\infty} \frac{q^{-n+Nl}}{-n+Nl} \right\} \\ &\leq C_1^{(U)}(k\rho, q) \left\{ \frac{q^{|n|}}{|n|} + \frac{q^{N+|n|}}{(N+|n|)(1-q^N)} + \frac{q^{N-|n|}}{(N-|n|)(1-q^N)} \right\} \\ &\leq C_1^{(U)}(k\rho, q) \left\{ \frac{q^{|n|}}{|n|} + \frac{q^{N+|n|}}{3|n|(1-q^N)} + \frac{q^{N-|n|}}{|n|(1-q^N)} \right\} \\ &\leq C_1^{(U)}(k\rho, q) \frac{q^{|n|}}{|n|} \left\{ 1 + \frac{1}{1-q^N} \left(\frac{q^N}{3} + q^{N-2|n|} \right) \right\} \\ &\leq C_1^{(U)}(k\rho, q) \frac{7-3q}{3(1-q)} \frac{q^{|n|}}{|n|}, \end{aligned}$$

where we used $N-2|n| \geq 0$ on the last inequality. Consequently, we obtain (A.4). Inequality (A.5) is obtained similarly.

2. We prove (A.6) dividing the problem into two cases, namely, the case $1 \leq |n| \leq N/4$ and the case $N/4 < |n| \leq N/2$.

First, we consider the case where $1 \leq |n| \leq N/4$. In this case, we have

$$|g_n^{(N)}(\rho)| \geq |H_n^{(1)}(k\rho)J_n(qk\rho)| - \sum_{l=1}^{\infty} |H_{n+Nl}^{(1)}(k\rho)J_{n+Nl}(qk\rho)| - \sum_{l=1}^{\infty} |H_{n-Nl}^{(1)}(k\rho)J_{n-Nl}(qk\rho)|$$

$$\begin{aligned}
&\geq C_1^{(L)}(k\rho, q) \frac{q^{|n|}}{|n|} - C_1^{(U)}(k\rho, q) \left\{ \sum_{l=1}^{\infty} \frac{q^{n+nl}}{n+nl} + \sum_{l=1}^{\infty} \frac{q^{-n+nl}}{-n+nl} \right\} \\
&\geq C_1^{(L)}(k\rho, q) \frac{q^{|n|}}{|n|} - C_1^{(U)}(k\rho, q) \left\{ \frac{q^{N+|n|}}{(N+|n|)(1-q^N)} + \frac{q^{N-|n|}}{(N-|n|)(1-q^N)} \right\} \\
&\geq C_1^{(L)}(k\rho, q) \frac{q^{|n|}}{|n|} - C_1^{(U)}(k\rho, q) \left\{ \frac{q^{N+|n|}}{5|n|(1-q^N)} + \frac{q^{N-|n|}}{3|n|(1-q^N)} \right\} \\
&\geq C_1^{(L)}(k\rho, q) \frac{q^{|n|}}{|n|} \left\{ 1 - \frac{C_1^{(U)}(k\rho, q)}{C_1^{(L)}(k\rho, q)(1-q^N)} \left(\frac{q^N}{5} + \frac{q^{N-2|n|}}{3} \right) \right\} \\
&\geq C_1^{(L)}(k\rho, q) \frac{q^{|n|}}{|n|} \left\{ 1 - \frac{C_1^{(U)}(k\rho, q)}{C_1^{(L)}(k\rho, q)(1-q^N)} \left(\frac{q^N}{5} + \frac{q^{N/2}}{3} \right) \right\},
\end{aligned}$$

where we used (A.1) on the first inequality and (A.3) on the second inequality. Remarking that the underlined part on the right-hand side can be arbitrarily small by taking N sufficiently large, we obtain inequality (A.6) in the case $1 \leq |n| \leq N/4$.

Second, we consider the case $N/4 < |n| \leq N/2$. In this case, we only have to consider the case $N/4 < n \leq N/2$ since $g_{-n}^{(N)}(\rho) = g_n^{(N)}(\rho)$. If $N/4 < n \leq N/2$, we have

$$|g_n^{(N)}(\rho)| \geq |H_n^{(1)}(k\rho)J_n(qk\rho) + H_{N-n}^{(1)}(k\rho)J_{N-n}(qk\rho)| - \sum_{l=1}^N |H_{n+nl}^{(1)}(k\rho)J_{n+nl}(qk\rho)| - \sum_{l=2}^N |H_{n-nl}^{(1)}(k\rho)J_{n-nl}(qk\rho)|.$$

For the first term on the right-hand side, by (A.2), we have

$$\left| H_n^{(1)}(k\rho)J_n(qk\rho) + H_{N-n}^{(1)}(k\rho)J_{N-n}(qk\rho) - \frac{1}{i\pi} \left(\frac{q^n}{n} + \frac{q^{N-n}}{N-n} \right) \right| \leq \frac{1}{2\pi} \left(\frac{q^n}{n} + \frac{q^{N-n}}{N-n} \right),$$

which implies

$$\begin{aligned}
|H_n^{(1)}(k\rho)J_n(qk\rho) + H_{N-n}^{(1)}(k\rho)J_{N-n}(qk\rho)| &\geq \frac{1}{2\pi} \left(\frac{q^n}{n} + \frac{q^{N-n}}{N-n} \right) \\
&\geq \frac{q^n}{2\pi n} \left(1 + \frac{q^{N-2n}}{3} \right) \geq \frac{2q^n}{3\pi n}.
\end{aligned}$$

By the above inequality and (A.3), we have

$$\begin{aligned}
|g_n^{(N)}(\rho)| &\geq \frac{2q^n}{3\pi n} - C_1^{(U)}(k\rho, q) \left\{ \sum_{l=1}^{\infty} \frac{q^{n+nl}}{n+nl} + \sum_{l=2}^{\infty} \frac{q^{-n+nl}}{-n+nl} \right\} \\
&\geq \frac{2q^n}{3\pi n} - C_1^{(U)}(k\rho, q) \left\{ \frac{q^{n+N}}{(N+n)(1-q^N)} + \frac{q^{2N-n}}{(2N-n)(1-q^N)} \right\} \\
&\geq \frac{q^n}{n} \left\{ \frac{2}{3\pi} - \frac{C_1^{(U)}(k\rho, q)}{1-q^N} \left(\frac{4nq^N}{5N} + \frac{2nq^{2(N-n)}}{3N} \right) \right\} \\
&\geq \frac{q^n}{n} \left(\frac{2}{3\pi} - \frac{11}{15} \frac{C_1^{(U)}(k\rho, q)}{1-q^N} q^N \right),
\end{aligned}$$

where the underlined part can be arbitrarily small by taking N sufficiently large for given $k\rho$ and q .

Consequently, (A.6) is obtained for n such that $1 \leq |n| \leq N/2$.

Inequality (A.7) is proved similarly. \square

Appendix B. On the error estimate for problems with single mode cosine functions as boundary data

We give a reason why the error estimates ϵ_N for Example 1 become small if $N = 2m$. In this case, we have $f_m = f_{-m} = 1/2$ and $f_n = 0$ ($n \neq \pm m$), and then we have

$$\begin{aligned}
e_N(re^{i\theta}) &= \frac{1}{2} \phi_m^{(N)}(r, \theta) + \frac{1}{2} \phi_{-m}^{(N)}(r, \theta) \\
&= \frac{1}{2g_m^{(N)}(\rho)} \left\{ g_m^{(N)}(\rho) \frac{H_m^{(1)}(kr)}{H_m^{(1)}(k\rho)} e^{im\theta} - g_m^{(N)}(re^{i\theta}) \right\} + \frac{1}{2g_{-m}^{(N)}(\rho)} \left\{ g_{-m}^{(N)}(\rho) \frac{H_{-m}^{(1)}(kr)}{H_{-m}^{(1)}(k\rho)} e^{-im\theta} - g_{-m}^{(N)}(re^{i\theta}) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{g_m^{(N)}(\rho)} \left\{ g_m^{(N)}(\rho) \frac{H_m^{(1)}(kr)}{H_m^{(1)}(k\rho)} \cos m\theta - \frac{1}{2} g_m^{(N)}(re^{i\theta}) - \frac{1}{2} g_m^{(N)}(re^{-i\theta}) \right\} \\
&= \frac{1}{g_m^{(N)}(\rho)} \left\{ \sum_{l=1}^{\infty} h_{l,m}^{(N)}(r, \theta) + \sum_{l=1}^{\infty} h_{-l,m}^{(N)}(r, \theta) \right\},
\end{aligned} \tag{B.1}$$

where

$$h_{l,m}^{(N)}(r, \theta) = J_{lN+m}(qk\rho) \left\{ H_{lN+m}^{(1)}(k\rho) \frac{H_m^{(1)}(kr)}{H_m^{(1)}(k\rho)} \cos m\theta - H_{lN+m}^{(1)}(kr) \cos(lN+m)\theta \right\} \tag{B.2}$$

and we used (A.8) on the second equality, the formula $H_{-m}^{(1)}(x) = (-1)^m H_m^{(1)}(x)$ and the equality $g_{-m}^{(N)}(z) = g_m^{(N)}(\bar{z})$ obtained from (41) on the third equality and (A.1) on the last equality. The magnitude of each term $h_{l,m}^{(N)}(re^{i\theta})$ is of order $O(q^{|lN+m|})$ since we have

$$\begin{aligned}
|h_{l,m}^{(N)}(r, \theta)| &\leq |H_{lN+m}^{(1)}(k\rho) J_{lN+m}(qk\rho)| \frac{|H_{lN+m}^{(1)}(kr)|}{|H_{lN+m}^{(1)}(k\rho)|} + |H_{lN+m}^{(1)}(kr)| |J_{lN+m}(qk\rho)| \\
&\leq 2|H_{lN+m}^{(1)}(k\rho) J_{lN+m}(qk\rho)| \leq 2C_1^{(U)}(k\rho, q) \frac{q^{|lN+m|}}{|lN+m|},
\end{aligned}$$

where we used Lemma 9 on the second inequality and (A.3) on the last inequality. Therefore, the term $h_{-1,m}^{(N)}(r, \theta)$, which is of order $O(q^{N-m})$, is the most dominant term in the error $e_N(z)$ of (B.1).

In the case $N > 2m$, we obtain again the error estimate $\epsilon_N = O(q^N)$, which is the result of Theorem 3. In fact, by the inequalities

$$\begin{aligned}
\left| \sum_{l=1}^{\infty} h_{l,m}^{(N)}(r, \theta) \right| &\leq 2C_1^{(U)}(k\rho, q) \sum_{l=1}^{\infty} \frac{q^{lN+m}}{lN+m} \leq 2C_1^{(U)}(k\rho, q) \sum_{l=1}^{\infty} \frac{q^{lN+m}}{N+m} \\
&= 2C_1^{(U)}(k\rho, q) \frac{q^{N+m}}{(N+m)(1-q^N)} \leq \frac{2C_1^{(U)}(k\rho, q)}{N(1-q)} q^{N+m},
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\left| \sum_{l=1}^{\infty} h_{-l,m}^{(N)}(r, \theta) \right| &\leq 2C_1^{(U)}(k\rho, q) \sum_{l=1}^{\infty} \frac{q^{lN-m}}{lN-m} \leq 2C_1^{(U)}(k\rho, q) \sum_{l=1}^{\infty} \frac{q^{lN-m}}{N-m} \\
&= 2C_1^{(U)}(k\rho, q) \frac{q^{N-m}}{(N-m)(1-q^N)} \leq \frac{4C_1^{(U)}(k\rho, q)}{N(1-q)} q^{N-m},
\end{aligned} \tag{B.4}$$

and by the inequality

$$|g_m^{(N)}(\rho)| \geq 2C_2^{(L)}(k\rho, q) \frac{q^m}{m} \geq 4C_2^{(L)}(k\rho, q) \frac{q^m}{N}, \tag{B.5}$$

which is obtained by (A.6), we have

$$\begin{aligned}
|e_N(re^{i\theta})| &\leq \frac{1}{|g_m^{(N)}(\rho)|} \left\{ \sum_{l=1}^{\infty} |h_{l,m}^{(N)}(r, \theta)| + \sum_{l=1}^{\infty} |h_{-l,m}^{(N)}(r, \theta)| \right\} \\
&\leq \frac{(1+2q^{-2m})C_1^{(U)}(k\rho, q)}{2(1-q)C_2^{(L)}(k\rho, q)} q^N \equiv E_m(N).
\end{aligned} \tag{B.6}$$

However, in the case $N = 2m$, we remark that the most dominant term $h_{-1,m}^{(N)}(r, \theta)$ in (B.1) vanishes since

$$\begin{aligned}
h_{-1,m}^{(2m)}(r, \theta) &= J_{-m}(qk\rho) \left\{ H_{-m}^{(1)}(k\rho) \frac{H_m^{(1)}(kr)}{H_m^{(1)}(k\rho)} \cos m\theta - H_{-m}^{(1)}(kr) \cos(-m\theta) \right\} \\
&= J_m(qk\rho) \left\{ H_m^{(1)}(k\rho) \frac{H_m^{(1)}(kr)}{H_m^{(1)}(k\rho)} \cos m\theta - H_m^{(1)}(kr) \cos m\theta \right\} = 0,
\end{aligned}$$

where we used $H_{-m}^{(1)}(x) = (-1)^m H_m^{(1)}(x)$ and $J_{-m}(x) = (-1)^m J_m(x)$ on the second equality. Therefore, the error estimate ϵ_N becomes small if $N = 2m$. In fact, if $N = 2m$, we have

$$|e_N(re^{i\theta})| \leq \frac{1}{|g_m^{(N)}(\rho)|} \left\{ \sum_{l=1}^{\infty} |h_{l,m}^{(N)}(r, \theta)| + \sum_{l=2}^{\infty} |h_{-l,m}^{(N)}(r, \theta)| \right\}$$

$$\begin{aligned}
&\leq \frac{N}{4C_2^{(L)}(k\rho, q)q^m} \left\{ 2C_1^{(U)}(k\rho, q) \sum_{l=1}^{\infty} \frac{q^{lN+m}}{lN+m} + 2C_1^{(U)}(k\rho, q) \sum_{l=2}^{\infty} \frac{q^{lN-m}}{lN-m} \right\} \\
&\leq \frac{NC_1^{(U)}(k\rho, q)}{2C_2^{(L)}(k\rho, q)} \left\{ \frac{1}{N+m} \sum_{l=1}^{\infty} q^{lN} + \frac{1}{N-m} \sum_{l=2}^{\infty} q^{lN-2m} \right\} \\
&\leq \frac{NC_1^{(U)}(k\rho, q)}{2C_2^{(L)}(k\rho, q)} \left\{ \frac{q^N}{N(1-q)} + \frac{2q^{2N-2m}}{N(1-q)} \right\} = \frac{3C_1^{(U)}(k\rho, q)q^{2m}}{2(1-q)C_2^{(L)}(k\rho, q)} < E_m(2m),
\end{aligned}$$

where E_m is defined in (B.6).

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