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Nearest matrix with prescribed eigenvalues and its applications

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ABSTRACT

Consider an $n \times n$ matrix A and a set Λ consisting of $k \leq n$ prescribed complex numbers. Lippert (2010) in a challenging article, studied geometrically the spectral norm distance from A to the set of matrices whose spectra included specified set Λ and constructed a perturbation matrix Δ with minimum spectral norm such that $A+\Delta$ had Λ in its spectrum. This paper presents an easy practical computational method for constructing the optimal perturbation Δ by improving and extending the methodology, necessary definitions and lemmas of previous related works. Also, some conceivable applications of this issue are provided.

1. Introduction

Let A be an $n \times n$ complex matrix and let L be the set of complex $n \times n$ matrices that have $\lambda \in \mathbb{C}$ as a prescribed multiple eigenvalue. In 1999, Malyshev [1] obtained the following formula for the spectral norm distance from A to L:

$$\min_{B \in L} \|A - B\|_2 = \max_{\gamma \geq 0} s_{2n-1}\left(\begin{bmatrix} A - \lambda I & \gamma I_n \\ 0 & A - \lambda I \end{bmatrix}\right),$$

where $\|\cdot\|_2$ denotes the spectral matrix norm and $s_1(\cdot) \geq s_2(\cdot) \geq s_3(\cdot) \geq \cdots$ are the singular values of the corresponding matrix in nonincreasing order. Also he constructed a perturbation, Δ, of matrix A such that $A + \Delta$ belonged to the L and Δ was the optimal perturbation of the matrix A. Malyshev's work can be considered as a solution to Wilkinson's problem, that is, the computation of the distance from a matrix $A \in \mathbb{C}^{n \times n}$ which has only simple eigenvalues, to the set of $n \times n$ matrices with multiple eigenvalues. Wilkinson introduced this distance in [2] and some bounds were computed for it by Ruhe [3], Wilkinson [4–7] and Demmel [8].

However, in a non-generic case, if A is a normal matrix then Malyshev's formula is not directly applicable. Ikramov and Nazari [9] showed this point and they obtained an extension of Malyshev's formula for normal matrices. Furthermore, Malyshev's formula was extended by them [10] for the case of a spectral norm distance from A to matrices with a prescribed triple eigenvalue. In 2011, under some conditions, a perturbation Δ of matrix A was constructed by Mengi [11] such that

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Δ had minimum spectral norm and $A+\Delta$ belonged to the set of matrices that had a prescribed eigenvalue of prespecified algebraic multiplicity. Moreover, Malyshev's work also was extended by Lippert [12] and Gracia [13]. They computed a spectral norm distance from A to the matrices with two prescribed eigenvalues.

Recently, Lippert [14] provided a geometric basis for the results obtained in [13,12] and also, he introduced a geometric motivation for the smallest perturbation in the spectral norm such that the perturbed matrix had some given eigenvalues.

Denote by $\mathcal{M}_k$ the set of $n\times n$ matrices that have $k\le n$ prescribed eigenvalues. This article, motivated by the above spectrum updating problems, studies and considers the spectral norm distance from A to $\mathcal{M}_k$. In particular, it describes a clear and intelligible computational technique for construction of Δ having minimum spectral norm and satisfying $A+\Delta\in\mathcal{M}_k$. Expanding and improving the methodology used in [1,13,12,10,11] for the case of $k\le n$ fixed eigenvalues, and presenting a general solution for the matrix nearness problem in an evident computational manner are the main goals considered herein. This paper is also intended to provide definitions and proofs in a broad way and give sufficient conditions with the intention of encompassing the aforesaid works. On the other hand, in [10,14, Section 5] and [12, Section 6] it was mentioned that the optimal perturbations are not always computable for the case of fixing three or more distinct eigenvalues. This paper, provides two assumptions such that the optimal perturbation is always computable when these assumptions hold. It is noticeable that if one or both conditions are not satisfied then still $A+\Delta\in\mathcal{M}_k$, but Δ has not necessary minimum spectral norm. In this case we can have some lower bounds and $\|\Delta\|_2$ as an upper bound for the spectral norm distance from A to $A+\Delta\in\mathcal{M}_k$. Meanwhile, a selection of possible applications of this topic is considered.

Note that if, in a special case, A is a normal matrix, i.e., $A^*A=AA^*$, then we cannot use the method described in this paper for the computation of the perturbation, immediately. In this case, by following the analysis performed in [15,9,16] one can derive a refinement of our results for the case of normal matrices. Therefore, throughout of this paper, it is assumed that A is not a normal matrix. Suppose now that an $n\times n$ matrix A and a set of complex numbers $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_k\}$ in which $k\le n$, are given. Now for

$$\gamma=\{\gamma_{1,1},\gamma_{2,1},\ldots,\gamma_{k-1,1},\gamma_{1,2},\gamma_{2,2},\ldots,\gamma_{k-2,2},\ldots,\gamma_{1,k-1}\}\in\mathbb{C}^{\frac{k(k-1)}{2}},$$

define the $nk\times nk$ upper triangular matrix $Q_A(\gamma)$ as

$$Q_A(\gamma)=\begin{bmatrix} B_1 & \gamma_{1,1}I_n & \gamma_{1,2}I_n & \cdots & \gamma_{1,k-1}I_n \\ 0 & B_2 & \gamma_{2,1}I_n & \cdots & \gamma_{2,k-2}I_n \\ \vdots & \ddots & B_3 & \ddots & \vdots \\ & & & \ddots & \gamma_{k-1,1}I_n \\ 0 & \cdots & & 0 & B_k \end{bmatrix}_{nk\times nk}, \tag{1}$$

where $\gamma_{i,1}$, $(i=1,\ldots,k-1)$ are real variables and $B_j=A-\lambda_j I_n,\ (j=1,\ldots,k)$.

Clearly, $Q_A(\gamma)$ can be assumed as a matrix function of complex variables $\gamma_{i,j}$ for $i=1,2,\ldots,k-1,\ j=1,2,\ldots,k-i$. Hereafter, for the sake of simplicity, the positive integer $nk-(k-1)$ is denoted by κ. Assume that the spectral norm distance from A to $\mathcal{M}_k$ is denoted by $\rho_2(A,\mathcal{M}_k)$, i.e,

$$\rho_2(A,\mathcal{M}_k)=\|\Delta\|_2=\min_{M\in\mathcal{M}_k}\|A-M\|_2\,.$$

Generally, this paper follows the plan including two main phases: Computing a lower bound, say α, for $\|\Delta\|_2$ and next constructing a perturbation matrix Δ such that $\|\Delta\|_2=\alpha$, (or as close as possible to α) and $A+\Delta\in\mathcal{M}_k$. Due to this, some lower bounds for 2-norm of the optimal perturbation Δ are obtained in the next section. In Section 3, selected properties of κth singular value of $Q_A(\gamma)$, i.e., $s_\kappa\left(Q_A(\gamma)\right)$, and its corresponding singular vectors are studied. These properties will be used in Section 4, with the purpose of computing the optimal perturbation $\|\Delta\|_2$, having minimum 2-norm and satisfying $A+\Delta\in\mathcal{M}_k$. Finally, in Section 6, some numerical examples and imaginable implementation of the topic of the matrix nearness problem are given to illustrate the validation and application of the presented method.

2. Lower bounds for the optimal perturbation

Let us begin by considering $s_\kappa\left(Q_A(\gamma)\right)$ which is the κth singular value of $Q_A(\gamma)$. First, note that $s_\kappa\left(Q_A(\gamma)\right)$ is a continuous function of variable γ. Also, if we define the unitary matrix U of the form

$$U=\begin{bmatrix} I_n & & & & 0 \\ & -I_n & & & \\ & & I_n & & \\ & & & \ddots & \\ 0 & & & & (-1)^{k-1}I_n \end{bmatrix}_{nk\times nk},$$

where I_n is the $n \times n$ identity matrix, then it is straightforward to see that

$$UQ_A(\gamma)U^* = \begin{bmatrix} B_1 & -\gamma_{1,1}I_n & \gamma_{1,2}I_n & \dots & \gamma_{1,k-1}I_n \\ 0 & B_2 & -\gamma_{2,1}I_n & \dots & \gamma_{2,k-2}I_n \\ \vdots & \ddots & B_3 & \ddots & \vdots \\ & & & \ddots & -\gamma_{k-1,1}I_n \\ 0 & \dots & & 0 & B_k \end{bmatrix}_{nk\times nk}.$$

Since under a unitary transformation the singular values of a square matrix are invariant, it follows that $s_\kappa\left(Q_A(\gamma)\right)$ is an even function with respect to all its real variables $\gamma_{i,1}$, $(i = 1, \dots, k-1)$. Therefore, without loss of generality, in the remaining part of the paper, we assume that $\gamma_{i,1} \geq 0$, for every $i = 1, \dots, k-1$.

Lemma 2.1. *If $A \in \mathbb{C}^{n\times n}$ has $\lambda_1, \lambda_2, \dots, \lambda_k$ as some of its eigenvalues, then for all $\gamma \in \mathbb{C}^{\frac{k(k-1)}{2}}$, it holds that $s_\kappa\left(Q_A(\gamma)\right) = 0$.*

Proof. Suppose that $\lambda_1, \lambda_2, \dots, \lambda_k$ are some of the eigenvalues of A corresponding to the associated eigenvectors $e_1, e_2, \dots, e_k$, respectively. Evidently,

$$Ae_i = \lambda_i e_i, \quad \text{and} \quad B_j e_i = \left(\lambda_i - \lambda_j\right) e_i, \quad i, j = 1, \dots, k. \tag{2}$$

We have shown that $Q_A(\gamma)$ has k linearly independent eigenvectors corresponding to zero as one of its eigenvalues. This implies that $s_\kappa\left(Q_A(\gamma)\right) = 0$. Two cases are now considered.

Case 1. Let $\lambda_1, \dots, \lambda_k$ be k distinct eigenvalues of $Q_A(\gamma)$. Consequently, $e_1, \dots, e_k$ are k linearly independent eigenvectors. Consider now the $nk \times 1$ vectors $v^1, v^2, \dots, v^k$ such that $v^1 = [e_1, 0, \dots, 0]^T$ and introduce the remaining vectors by the following formula

$$v_j^m = \begin{cases} 0 & j > m \\ e_m & j = m \\ \dfrac{1}{\lambda_j - \lambda_m} \displaystyle\sum_{p=1}^{m-j} \left(\gamma_{j,p} v_{p+j}^m\right) e_m & j = m-1, \dots, 1, \end{cases} \tag{3}$$

where v_j^m denotes the jth component of v^m. The elements of each vector v^m should be computed recursively, starting from the last component. Clearly, $v^1, v^2, \dots, v^k$ are k linearly independent vectors. Using (2) and (3), straightforward calculation yields

$$\begin{aligned} \left(Q_A(\gamma)\, v^m\right)_i &= B_i v_i^m + \sum_{p=1}^{m-i} \left(\gamma_{i,p} v_{p+i}^m\right) \\ &= \frac{1}{\lambda_i - \lambda_m} \sum_{p=1}^{m-i} \left(\gamma_{i,p} v_{p+i}^m\right) B_i e_m + \sum_{p=1}^{m-i} \left(\gamma_{i,p} v_{p+i}^m\right) \\ &= -\sum_{p=1}^{m-i} \left(\gamma_{i,p} v_{p+i}^m\right) + \sum_{p=1}^{m-i} \left(\gamma_{i,p} v_{p+i}^m\right) \\ &= 0. \end{aligned}$$

Thus, the vectors $v^1, v^2, \dots, v^k$ satisfy $Q_A(\gamma)v^m = 0$, $(m = 1, \dots, k)$. Consequently, the dimension of the null space of $Q_A(\gamma)$ is at least κ, which means that $s_\kappa\left(Q_A(\gamma)\right) = 0$.

Case 2. Assume that some of the complex numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ are equal. Without loss of generality, we can assume that λ_1 is an eigenvalue of algebraic multiplicity l, i.e., $\lambda_1 = \lambda_2 = \dots = \lambda_l$. In this case, a new method is provided for constructing the vectors associated with λ_1, i.e., $v^1, v^2, \dots, v^l$. Obviously, it is enough to construct a set of linearly independent vectors $\{v^1, v^2, \dots, v^l, v^{l+1}, \dots, v^k\}$, for which $Q_A(\gamma)v^m = 0$, $(m = 1, \dots, l)$. To do this, let e_1 be the right eigenvector for λ_1 and let $e_1, \bar{e}_1, \bar{e}_2, \dots, \bar{e}_{l-1}$ form a chain of generalized eigenvectors of length l associated with λ_1. So, we have

$$(A - \lambda_1 I)\, e_1 = 0, \qquad (A - \lambda_1 I)\, \bar{e}_1 = e_1, \quad \text{and} \quad (A - \lambda_1 I)\, \bar{e}_j = \bar{e}_{j-1}, \quad (j = 2, \dots, l).$$

Define the operator $\mathcal{V}$ as follows:

$$\mathcal{V}[e_1] = \bar{e}_1 \quad \text{and} \quad \mathcal{V}[\bar{e}_i] = \bar{e}_{i+1}, \quad i = 1, \dots, l-1. \tag{4}$$

Now, let $v^1 = [e_1, 0, \dots, 0]^T$ and define the vectors , $v^2, \dots, v^l$ by

$$v_j^m = \begin{cases} 0 & j > m \\ e_1 & j = m \\ -\displaystyle\sum_{p=1}^{m-j} \left(\gamma_{j,p} \mathcal{V}\left[v_{p+j}^m\right]\right) & j = m-1, \dots, 1 \end{cases},$$

where v_j^m denotes the jth component of v^m and analogous to Case 1 elements of each vector v^m, can now be computed by the above recursive relation, beginning from the last element. The remaining vectors associated with simple eigenvalues are constructed by the method presented in Case 1. Linearity independence of vectors v^i, $(i = 1, \ldots, l)$ is concluded by their definition. Thus, $v^1, \ldots, v^k$ are linearly independent vectors. Also, calculations similar to what was performed in Case 1, conclude that $v^1, \ldots, v^l$ are the l eigenvectors associated to zero as an eigenvalue of $Q_A(\gamma)$, i.e., $Q_A(\gamma)v^i = 0$, for every $i = 1, \ldots, l$. □

Following corollary can be concluded by similar calculations of the above lemma.

Corollary 2.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be some eigenvalues of $A \in \mathbb{C}^{n\times n}$. Then for all $\gamma \in \mathbb{C}^{\frac{k(k-1)}{2}}$ we have*

$$\begin{aligned}
&s_n\left(A-\lambda_i I\right)=0, \quad i=1,\ldots,k,\\
&s_{2n-1}\left(\begin{bmatrix} A-\lambda_i I & \gamma_{1,1} I\\ 0 & A-\lambda_j I\end{bmatrix}\right)=0, \quad i,j=1,\ldots,k,\\
&s_{3n-2}\left(\begin{bmatrix} A-\lambda_i I & \gamma_{1,1} I & \gamma_{1,2} I\\ 0 & A-\lambda_j I & \gamma_{2,1} I\\ 0 & 0 & A-\lambda_l I\end{bmatrix}\right)=0, \quad i,j,l=1,\ldots,k,\\
&\vdots\\
&s_\kappa\left(Q_A(\gamma)\right)=0.
\end{aligned} \tag{5}$$

Lemma 2.3. *Suppose that $A \in \mathbb{C}^{n\times n}$ has $\lambda_1, \lambda_2, \ldots, \lambda_k$ as some of its eigenvalues. If Δ is the minimum norm perturbation such that $A + \Delta \in \mathcal{M}_k$, then for every γ,*

$$\|\Delta\|_2 \geq s_\kappa\left(Q_A(\gamma)\right).$$

Proof. By Lemma 2.1, we know that $s_\kappa\left(Q_{A+\Delta}(\gamma)\right) = 0$. Applying the Weyl inequalities for singular values (for example, see Corollary 5.1 of [17]) to the relation $Q_{A+\Delta}(\gamma) = Q_A(\gamma) + I_k \otimes \Delta$, yields

$$s_\kappa\left(Q_A(\gamma)\right) = \left|s_\kappa\left(Q_A(\gamma)\right) - s_\kappa\left(Q_{A+\Delta}(\gamma)\right)\right| \leq \|I_k \otimes \Delta\|_2 = \|\Delta\|_2. \quad \square$$

Similar calculations as performed above for the singular value $s_\kappa\left(Q_A(\gamma)\right)$, can be considered for remainder of the singular values appearing in the left hand side of Eqs. (5). Thus, the following relations are deduced:

$$\begin{aligned}
&\|\Delta\|_2 \geq s_n\left(A-\lambda_i I\right), \quad i=1,\ldots,k,\\
&\|\Delta\|_2 \geq s_{2n-1}\left(\begin{bmatrix} A-\lambda_i I & \gamma_{1,1} I\\ 0 & A-\lambda_j I\end{bmatrix}\right), \quad i,j=1,\ldots,k,\\
&\|\Delta\|_2 \geq s_{3n-2}\left(\begin{bmatrix} A-\lambda_i I & \gamma_{1,1} I & \gamma_{1,2} I\\ 0 & A-\lambda_j I & \gamma_{2,1} I\\ 0 & 0 & A-\lambda_l I\end{bmatrix}\right), \quad i,j,l=1,\ldots,k,\\
&\vdots\\
&\|\Delta\|_2 \geq s_\kappa\left(Q_A(\gamma)\right).
\end{aligned}$$

Next corollary gives the main result of this section.

Corollary 2.4. *Assume that $A \in \mathbb{C}^{n\times n}$ and a set Λ consisting of $k \leq n$ complex numbers are given. Then for all $\gamma \in \mathbb{C}^{\frac{k(k-1)}{2}}$, the optimal perturbation Δ, satisfies*

$$\|\Delta\|_2 \geq \max\left\{\alpha_1, \alpha_2, \ldots, \alpha_k\right\}, \tag{6}$$

where

$$\begin{aligned}
&\alpha_1 = \max\left\{s_n\left(A-\lambda_i I\right),\ i=1,\ldots,k\right\},\\
&\alpha_2 = \max\left\{s_{2n-1}\left(\begin{bmatrix} A-\lambda_i I & \gamma_{1,1} I\\ 0 & A-\lambda_j I\end{bmatrix}\right),\ i,j=1,\ldots,k\right\},\\
&\alpha_3 = \max\left\{s_{3n-2}\left(\begin{bmatrix} A-\lambda_i I & \gamma_{1,1} I & \gamma_{1,2} I\\ 0 & A-\lambda_j I & \gamma_{2,1} I\\ 0 & 0 & A-\lambda_l I\end{bmatrix}\right)=0,\ i,j,l=1,\ldots,k\right\},\\
&\vdots\\
&\alpha_k = s_\kappa\left(Q_A(\gamma)\right).
\end{aligned}$$

From this point, construction of an optimal perturbation is considered. This is completely dependent on dominance of α_i, $(i = 1, \ldots, k)$. First, let maximum of right hand side of (6) occur in $\alpha_k = s_\kappa\,(Q_A(\gamma))$.

3. Properties of $s_\kappa(Q_A(\gamma))$ and its corresponding singular vectors

In this section, we obtain further properties of $s_\kappa\,(Q_A(\gamma))$ and its associated singular vectors. In the next section, this properties are applied to construct the optimal perturbation Δ. For our discussion, it is necessary to reform some definitions and lemmas of [1,13,12,11].

Definition 3.1. Suppose that vectors

$$u(\gamma) = \begin{bmatrix} u_1(\gamma) \\ \vdots \\ u_k(\gamma) \end{bmatrix}, \qquad v(\gamma) = \begin{bmatrix} v_1(\gamma) \\ \vdots \\ v_k(\gamma) \end{bmatrix} \in \mathbb{C}^{nk} \quad (u_j(\gamma), v_j(\gamma) \in \mathbb{C}^n, j = 1, \ldots, k),$$

is a pair of left and right singular vectors of $s_\kappa\,(Q_A(\gamma))$, respectively. Define

$$U(\gamma) = [u_1(\gamma), \ldots, u_k(\gamma)]_{n\times k}, \quad \text{and} \quad V(\gamma) = [v_1(\gamma), \ldots, v_k(\gamma)]_{n\times k}.$$

Considering definition of the vectors $u(\gamma)$ and $v(\gamma)$, we have the following relations

$$Q_A(\gamma)v(\gamma) = s_\kappa\,(Q_A(\gamma))\,u(\gamma), \tag{7}$$

$$Q_A(\gamma)^* u(\gamma) = s_\kappa\,(Q_A(\gamma))\,v(\gamma), \tag{8}$$

also without loss of generality, assume that $u(\gamma)$ and $v(\gamma)$ are unit vectors.

The following lemma, which can be verified by considering Lemma 3.5 of [12] (see also Lemma 3 of [13]), concludes that there exists a finite point $\gamma \in \mathbb{C}^{\frac{k(k-1)}{2}}$ where the function $s_\kappa\,(Q_A(\gamma))$ attains its maximum value.

Lemma 3.2. *$s_\kappa\,(Q_A(\gamma)) \to 0$ as $|\gamma| \to \infty$.*

Definition 3.3. Let γ_* be a point where the singular value $s_\kappa\,(Q_A(\gamma))$ attains its maximum value such that $\prod_{i=1}^{k-1} \gamma_{*i,1} > 0$ at this point. We set $\alpha_k^* = s_\kappa\,(Q_A(\gamma_*))$.

It is easy to verify that if $\alpha_k^* = 0$, then $\lambda_1, \lambda_2, \ldots, \lambda_k$ are some eigenvalues of $Q_A(\gamma)$. Therefore, in what follows we assume that $\alpha_k^* > 0$. Moreover, suppose that α_k^* is a simple (not repeated) singular value of $Q_A(\gamma_*)$.

Investigating other properties of $s_\kappa\,(Q_A(\gamma))$ and its associated singular vectors requires results deduced in Theorems 2.10 and 2.11 of [11].

Theorem 3.4. *Let $A(t)\colon \mathbb{R} \to \mathbb{C}^{n\times n}$ be an analytic matrix-valued function. There exists a decomposition $A(t) = U(t)S(t)V(t)^*$, where $U(t)\colon \mathbb{R} \to \mathbb{C}^{n\times n}$, $V(t)\colon \mathbb{R} \to \mathbb{C}^{n\times n}$ are unitary and analytic, $S(t)$ is diagonal and analytic for all t. Assume that $u_l(t)$, $v_l(t)$ are the lth columns of $U(t)$ and $V(t)$, respectively, and $s_l(t)$ is the signed singular value at the lth diagonal entry of $S(t)$. Using the product rule and the fact that $A(t)v_l(t) = s_l(t)u_l(t)$, it is straightforward to deduce*

$$\frac{ds_l(t)}{dt} = Re\left(u_l(t)^* \frac{dA(t)}{dt} v_l(t)\right). \tag{9}$$

In particular, if $s_l(t_*) > 0$ at a local extremum t_*, and $s_l(t_*)$ is a simple singular value, then

$$\frac{ds_l(t_*)}{dt} = Re\left(u_l(t_*)^* \frac{dA(t_*)}{dt} v_l(t_*)\right) = 0. \tag{10}$$

Sun [18] has shown that a simple singular value has an analytic expansion for the derivative formula similar to what is mentioned in (9). In this case, we can differentiate a singular value along the tangent $\frac{dA(t)}{dt}$ from $A(t)$ and find a singular value decomposition that expresses the derivative as right hand side of (9). The first part of the above theorem (Theorem 2.10 of [11]) implies that for one parameter $A(t)$'s, Sun's formulas will still work, as long as U's and V's are chosen correctly. Moreover, the second part of Theorem 3.4 (Theorem 2.11 of [11]) assures that singular vectors $u_l(t_*)$ and $v_l(t_*)$ for which Eq. (10) is satisfied when $s_l(t_*) > 0$, can be found. On the other hand, suppose we are given an n parameter family $A(t_1, \ldots, t_n)$, and it is desired to find singular vectors u's and v's such that $\frac{\partial s}{\partial t_i} = \mathrm{Re}(u^* \frac{\partial A}{\partial t_i} v)$, $(i = 1, \ldots, n)$. In this case, Theorem 3.4 cannot be always applied. The k eigenvalue case needs to deal with this failure analyticity. Fortunately, we can cope with this essential flaw by assuming that α_k^* is an isolated singular value (as this qualification is considered). Now, it should be noted that $\gamma_{i,1}$, $(i = 1, \ldots, k-1)$ are pure real variables, while other components of the vector γ are complex variables. If we set $\gamma_{i,j} = \gamma_{i,j,R} + i\gamma_{i,j,I}$, for $j > 1$, then it can be assumed that γ has $(k-1)^2$ real components. Following lemma is deduced from applying Theorem 3.4 to $Q_A(\gamma)$.

Lemma 3.5. *Let γ_* and α_k^* be as defined in Definition 3.3, and let $\alpha_k^* > 0$ be a simple singular value of $Q_A(\gamma_*)$. Then there exists a pair*

$$\begin{bmatrix} u_1(\gamma_*) \\ \vdots \\ u_k(\gamma_*) \end{bmatrix}, \begin{bmatrix} v_1(\gamma_*) \\ \vdots \\ v_k(\gamma_*) \end{bmatrix} \in \mathbb{C}^{nk} \quad (u_j(\gamma_*), v_j(\gamma_*) \in \mathbb{C}^n, j = 1, \dots, k),$$

of left and right singular vectors of α_k^*, respectively, such that

1. $u_i(\gamma_*)^* v_{i+j}(\gamma_*) = 0,\ i = 1, \dots, k-1,\ j = 1, \dots, k-i$, *and*
2. $u_i(\gamma_*)^* u_j(\gamma_*) = v_i(\gamma_*)^* v_j(\gamma_*),\ i, j = 1, \dots, k$.

Proof. To prove first part of the lemma, two cases are considered.

Case 1. At first assume that $j > 1$.

By applying Lemma 2.10 of [11], for both of the real and imaginary parts of $\gamma_{*i,j} = \gamma_{*i,j,R} + i\gamma_{*i,j,I}$ we have $\mathrm{Re}(u_i(\gamma_*)^* v_{i+j}(\gamma_*)) = 0$, and $\mathrm{Re}(iu_i(\gamma_*)^* v_{i+j}(\gamma_*)) = 0$, respectively. This means that $u_i(\gamma_*)^* v_{i+j}(\gamma_*) = 0$.

Case 2. Consider the case for which $j = 1$.

Applying Lemma 2.10 of [11] for real numbers $\gamma_{*i,1}$, $(i = 1, \dots, k-1)$, concludes that $\mathrm{Re}(u_i(\gamma_*)^* v_{i+1}(\gamma_*)) = 0$. So, first part of lemma is proved if we show that $u_i(\gamma_*)^* v_{i+1}(\gamma_*)$ is a real number. Let us introduce the $1 \times nk$ vectors $\mathfrak{u}_{i,i}(\gamma_*) = [0, \dots, 0, u_i(\gamma_*)^*, 0, \dots, 0]$ and $\mathfrak{v}_{i,i}(\gamma_*) = [0, \dots, 0, v_i(\gamma_*)^*, 0, \dots, 0]$ where $u_i(\gamma_*)^*$ and $v_i(\gamma_*)^*$ are the ith components. By multiplying (7) by $\mathfrak{u}_{i,i}(\gamma_*)$ and (8) by $\mathfrak{v}_{i,i}(\gamma_*)$ both from the left, following results are obtained, respectively,

$$u_i(\gamma_*)^* B_i v_i(\gamma_*) + \sum_{p=1}^{k-i} \left(\gamma_{*i,p} u_i(\gamma_*)^* v_{i+p}(\gamma_*)\right) = \alpha_k^* u_i(\gamma_*)^* u_i(\gamma_*), \tag{11}$$

$$v_i(\gamma_*)^* B_i^* u_i(\gamma_*) + \sum_{p=1}^{i-1} \left(\bar{\gamma}_{*p,i-p} v_i(\gamma_*)^* u_p(\gamma_*)\right) = \alpha_k^* v_i(\gamma_*)^* v_i(\gamma_*), \tag{12}$$

Conjugating (11), subtracting it from (12) and considering results obtained in Case 1 leads to the following relation

$$\gamma_{*i,1} u_i(\gamma_*)^* v_{i+1}(\gamma_*) = \alpha_k^* \left(u_i(\gamma_*)^* u_i(\gamma_*) - v_i(\gamma_*)^* v_i(\gamma_*)\right), \tag{13}$$

clearly, the right hand side of (13) is a real number which implies that $u_i(\gamma_*)^* v_{i+1}(\gamma_*)$ is also real. Thus first part of the lemma is proved completely.

Now attempt to prove second part of the lemma. Multiplying (7) by $\mathfrak{u}_{i,j}(\gamma_*)$ from the left where $u_i(\gamma_*)^*$ is in the jth place, and multiplying (8) by $\mathfrak{v}_{j,i}(\gamma_*)$ from the left where $v_j(\gamma_*)^*$ is in the ith place, leads to similar results as in (11) and (12), respectively. Performing similar calculations as first part of the proof, leads to

$$u_i(\gamma_*)^* B_j v_i(\gamma_*) - u_i(\gamma_*)^* B_i v_j(\gamma_*) = \alpha_k^* \left(u_i(\gamma_*)^* u_i(\gamma_*) - v_i(\gamma_*)^* v_i(\gamma_*)\right),$$

which can be written as

$$\left(\lambda_i - \lambda_j\right) u_i(\gamma_*)^* v_j(\gamma_*) = \alpha_k^* \left(u_i(\gamma_*)^* u_i(\gamma_*) - v_i(\gamma_*)^* v_i(\gamma_*)\right).$$

The left hand side of this equation is equal to zero. In fact, the assertion is obvious when $i = j$. For $i \neq j$, assume $i < j$ then considering first part of the proof. Thus proof is completed by keeping in mind that $\alpha_k^* > 0$. □

Corollary 3.6. *Let suppositions of Lemma 3.5 hold. Then the matrices $U(\gamma_*)$ and $V(\gamma_*)$ (recall Definition 3.1 but for $\gamma = \gamma_*$), satisfy $U(\gamma_*)^* U(\gamma_*) = V(\gamma_*)^* V(\gamma_*)$.*

Lemma 3.7. *Suppose that assumptions of Lemma 3.5 are satisfied. Then the two matrices $U(\gamma_*)$ and $V(\gamma_*)$ have full rank.*

Proof. This result can be verified by considering Lemma 4.4 of [12]. □

4. Construction of the optimal perturbation

Let γ_* and α_k^* be as defined in Definition 3.3, and let $\alpha_k^* > 0$ be a simple singular value of $Q_A(\gamma_*)$. In this section, a perturbation $\Delta \in \mathbb{C}^{n\times n}$ is constructed such that it satisfies $\|\Delta\|_2 = \alpha_k^*$, and perturbed matrix $A + \Delta$ has k prescribed eigenvalues. Suppose that the matrices $U(\gamma_*)$ and $V(\gamma_*)$ are as in Corollary 3.6 and define

$$\Delta = -\alpha_k^* U(\gamma_*) V(\gamma_*)^\dagger, \tag{14}$$

where $V(\gamma_*)^\dagger$ denotes the Moore–Penrose pseudoinverse of $V(\gamma_*)$. From Corollary 3.6 it can be deduced that the two matrices $U(\gamma_*)$ and $V(\gamma_*)$ have the same nonzero singular values. So, there exists a unitary matrix $W \in \mathbb{C}^{n\times n}$ such that $U(\gamma_*) = WV(\gamma_*)$. Notice that Lemma 3.7 implies $V(\gamma)^\dagger V(\gamma) = I_k$. Therefore,

$$\|\Delta\|_2 = \left\|-\alpha_k^* U(\gamma_*)V(\gamma_*)^\dagger\right\|_2 = \alpha_k^* \left\|WV(\gamma_*)V(\gamma_*)^\dagger\right\|_2 = \alpha_k^*,$$

and

$$\Delta V(\gamma_*) = -\alpha_k^* U(\gamma_*) \Leftrightarrow \Delta v_i(\gamma_*) = -\alpha_k^* u_i(\gamma_*), \quad i = 1, \dots, k. \tag{15}$$

Using (7) and (15) for every $i = 1, \dots, k$, yields

$$(B_i + \Delta)\, v_i(\gamma_*) = -\sum_{p=1}^{k-i} \gamma_{*i,p} v_{i+p}(\gamma_*). \tag{16}$$

By Lemma 3.7, we know that $v_i(\gamma_*)$, $(i = 1, \dots, k)$ are linearly independent vectors. Consequently, $(B_i + \Delta)\, v_j(\gamma_*)$ is a nonzero vector for every $i, j \in \{1, \dots, k\}$, except for the case $i = j = k$. Now, define the k vectors $\psi_k, \psi_{k-1}, \dots, \psi_1$ by

$$\psi_k = v_k(\gamma_*), \quad \text{and} \quad \psi_i = \left(\prod_{p=i+1}^{k} \left(B_p + \Delta\right)\right) v_i(\gamma_*), \quad i = k-1, \dots, 1, \tag{17}$$

note that again applying (16) concludes $\psi_k, \psi_{k-1}, \dots, \psi_1$ as k nonzero vectors. It can be easily verified that $(B_i + \Delta)$ and $\left(B_j + \Delta\right)$ are commuting matrices, i.e.,

$$(B_i + \Delta)\left(B_j + \Delta\right) = \left(B_j + \Delta\right)(B_i + \Delta), \quad i, j = 1, \dots, k. \tag{18}$$

Let us now, show that $A + \Delta \in \mathcal{M}_k$. First, form (7) (but for $\gamma = \gamma_*$) and (15) we have

$$(B_k + \Delta)\, \psi_k = (B_k + \Delta)\, v_k(\gamma_*) = \alpha_k^* u_k(\gamma_*) - \alpha_k^* u_k(\gamma_*) = 0,$$

next by considering Eqs. (7), (16) and (18) we can derive that

$$\begin{aligned}(B_{k-1} + \Delta)\, \psi_{k-1} &= (B_{k-1} + \Delta)(B_k + \Delta)\, v_{k-1}(\gamma_*)\\ &= (B_k + \Delta)(B_{k-1} + \Delta)\, v_{k-1}(\gamma_*)\\ &= (B_k + \Delta)\left(-\gamma_{*1,k-1} v_k(\gamma_*)\right)\\ &= 0,\end{aligned}$$

using the above results and performing analogous calculations, concludes

$$\begin{aligned}(B_{k-2} + \Delta)\, \psi_{k-2} &= (B_{k-2} + \Delta)(B_{k-1} + \Delta)(B_k + \Delta)\, v_{k-2}(\gamma_*)\\ &= (B_k + \Delta)(B_{k-1} + \Delta)(B_{k-2} + \Delta)\, v_{k-2}(\gamma_*)\\ &= (B_k + \Delta)(B_{k-1} + \Delta)\left(-\gamma_{*1,k-2} v_{k-1}(\gamma_*) - \gamma_{*2,k-2} v_k(\gamma_*)\right)\\ &= (B_k + \Delta)(B_{k-1} + \Delta)\left(-\gamma_{*1,k-2} v_{k-1}(\gamma_*)\right) + (B_{k-1} + \Delta)(B_k + \Delta)\left(-\gamma_{*2,k-2} v_k(\gamma_*)\right)\\ &= 0.\end{aligned}$$

By following similar computations it is straightforward to verify that

$$(B_i + \Delta)\, \psi_i = 0, \Leftrightarrow (A + \Delta)\psi_i = \lambda_i \psi_i, \quad i = k, \dots, 1,$$

This means that each λ_i, $(i = 1, \dots, k)$ is an eigenvalue of $A + \Delta$ corresponding to each ψ_i as an associated eigenvector.

The main results of this section are summarized in the next theorem.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ and $k \le n$ complex numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ be given. If γ_* is a point where the singular value $s_\kappa\left(Q_A(\gamma)\right)$ attains its maximum value, such that $\prod_{i=1}^{k-1} \gamma_{*i,1} > 0$, and if $\alpha_k^* > 0$ is a simple singular value, then $A + \Delta \in \mathcal{M}_k$ and $\|\Delta\|_2 = \alpha_k^*$, where Δ is given by (14).*

Remark 4.2. In this section, our discussion concerned the construction of an optimal perturbation Δ, benefiting the results obtained in the previous section. In particular, one of the most significant issues in Section 3, is defining the two quantities γ_* and α_k^* (see Definition 3.3) and using Theorem 3.4 which results in further consequences. Actually, we should follow analogous method, similar to what is described in Sections 3 and 4, if construction of the optimal perturbation corresponding to every α_i, $(i = 1, \dots, k)$ (see Corollary 2.4) is desired. However, many numerical experiments were tried and noticed that α_k^* is greater than other α_i^* except in very rarely examples, and with an insignificant difference. Where α_i^*, $(i = 1, \dots, k)$ are defined similar to Definition 3.3. Nevertheless, if α_k^* is considered it will do and will include the majority of cases.

Remark 4.3. In the method described above, two sufficient conditions are provided such that the optimal perturbation is computable if the assumptions hold. Nevertheless, in some experiments, especially for large enough k, one may observe that one or both of the two conditions are not satisfied. In this situation, it is clear that the method introduced in Section 4 still works and $A + \Delta \in \mathcal{M}_k$, whereas $\|\Delta\|_2$ may not necessarily be minimum. Here, we can consider lower bounds computed in Section 2 and $\|\Delta\|_2$ as an upper bound for the distance $\rho_2(A, \mathcal{M}_k)$.

5. Conclusion

Lippert [14] in a hard challenging paper established a geometric motivation for spectral norm distance from $A \in \mathbb{C}^{n\times n}$ to the set $\mathcal{M}_k$ of matrices that have k prescribed eigenvalues. In this work, a clear computational method for obtaining the optimal perturbation Δ such that $A + \Delta \in \mathcal{M}_k$ was presented. For the spectral norm distance from A to $\mathcal{M}_k$, some lower bounds were obtained. Furthermore, an optimal perturbation to A, such that the perturbed matrix has the same k eigenvalues, was constructed under two qualifications, that were, α_k^* be a simple singular value and $\prod_{i=1}^{k-1} \gamma_{*i,1} \neq 0$. We presented a numerical technique which can be considered as a generalization of results obtained in [13,12,11]. Finally, it was pointed out that if one or both of the two assumptions are violated then the procedure described in Section 4 still works, but the perturbation Δ is not necessarily optimal. In this situation, $\|\Delta\|_2$ is as an upper bound for $\rho_2(A, \mathcal{M}_k)$ whereas, we have lower bounds obtained in Section 2 for this distance. However, finding the minimum norm perturbation Δ when $\prod_{i=1}^{k-1} \gamma_{*i,1} = 0$, or α_k^* has multiplicity greater than one, remains for a future investigation.

6. Applications and numerical experiment

In this section, we are interested to present some conceivable applications of the topic of the paper. In addition, a numerical experiment is applied to illustrate the validity of the method described in previous sections and clarify our aim. All computations were performed in MATLAB with 16 significant digits, however, for simplicity all numerical results are shown with 4 decimal places. Also, OCTAVE can perform similar computations. Let us review concisely the main topic of the article. In this paper, we computed the distance for $A \in \mathbb{C}^{n\times n}$ to $n \times n$ matrices that had $k \leq n$ given complex numbers $\lambda_1, \ldots \lambda_k$ in their spectrum, while constructing an associated perturbation of A was also considered. Nevertheless, another aspect of the problem may be investigated. Let us now concentrate on the subject of *finding a matrix with some ordered eigenvalues*, and consider some topics and utilizations of matrices, that at the moment of this writing we find important for our aim and may be of general interest to the readers as well. However, the matrix nearness problems have received much attention of several researchers and has met many applications. A basic reference for the theory and applications of the problem is [19] and references therein.

Inverse eigenvalue problem. One of the straightforward usage of aforesaid viewpoint is computing a matrix that has some prespecified eigenvalues, which is known as *inverse eigenvalue problem*. An inverse eigenvalue problem (IEP) concerns the reconstruction of a matrix from prescribed spectral data. The spectral data involved may consist of the complete or only partial information of eigenvalues or eigenvectors. Inverse eigenvalue problem has a long list of applications in areas such as control theory, mechanics, signal processing and numerical analysis. Two acceptable references for theory and application of inverse eigenvalue problem are [20,21]. Assume now, we are asked to find a matrix having given scalars $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ where $l \leq n$, as its eigenvalues. To do this, by using the technique explained in this paper, one can consider an arbitrary matrix, namely, A in the craved size. Next, by following procedure described in Section 4, the desired matrix which has $\lambda_1, \ldots, \lambda_l$ as some of its eigenvalues is computable.

Example 6.1. Assume that we have been asked to provide a matrix having five given scalars $\{1 - i, 0, 2, -3, \sqrt{2}i\}$, as some of its eigenvalues. For doing this, we can consider an arbitrary n-square matrix A for $n \geq 5$. Next, by following the procedure described in Section 4, we obtain a matrix which possesses the desired property.

For instance, let $n = 6$ and consider the 6×6 matrix

$$A = \begin{bmatrix} 10 & -2 & 1 & -2 & 8 & 10 \\ -1 & -4 & -1 & 7 & -4 & 10 \\ 4 & -5 & -5 & 8 & -2 & 7 \\ 6 & -3 & 8 & 8 & 8 & 4 \\ 2 & 3 & 5 & 2 & 1 & 9 \\ 8 & 3 & 2 & 10 & 2 & 2 \end{bmatrix},$$

that is randomly generated by MATLAB. Employing the MATLAB function `fmincon` which finds a minimum of a constrained nonlinear multi-variable function, for positive values α_i, $(i = 1, 2, 3, 4)$ introduced in Corollary 2.4, we have

$$\alpha_1 = 0.1589, \qquad \alpha_2 = 0.3559, \qquad \alpha_3 = 0.6898, \qquad \alpha_4 = 1.2636; \quad \gamma \in \mathbb{C}^{\frac{k(k-1)}{2}}.$$

In addition, we derive that $s_\kappa(Q_A(\gamma))$ attains its maximum value at

$$\begin{aligned}\gamma_* = \{&7.6550, 11.9794, 13.6277, 9.4679, -0.1266 - 1.2906\,\mathrm{i}, -3.5116 + 0.3821\,\mathrm{i}, \\ &-1.9574 + 1.0271\,\mathrm{i}, -4.1763 - 0.0478\,\mathrm{i}, 1.2353 - 0.3886\,\mathrm{i}, -1.6664 - 0.6315\,\mathrm{i}\},\end{aligned}$$

where, the κth singular value of $Q_A(\gamma_*)$ is isolated and

$$\alpha_5^* = s_\kappa\left(Q_A\left(\gamma_*\right)\right) = s_{26}\left(Q_A\left(\gamma_*\right)\right) = 2.6089.$$

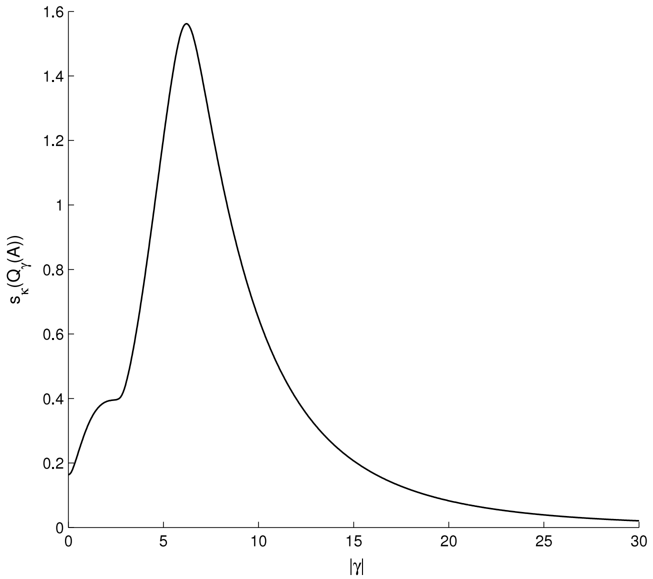

Fig. 1. The graph of the $s_\kappa\left(Q_A(\gamma)\right)$.

Clearly, α_5^* is strictly greater than α_i, for every $i = 1, 2, 3, 4$. Also, in Fig. 1 the graph of the $s_\kappa\left(Q_A(\gamma)\right)$ is plotted for $|\gamma| \in [0, 30]$ that can confirm Lemma 3.2, where $|\gamma| = \left(\sum_{i,j}\left(\gamma_{i,j}\right)^2\right)^{\frac{1}{2}}$.

Now, by pursuing the method explained in Sections 3 and 4 we are able to compute the full rank matrices $U(\gamma_*)$ and $V(\gamma_*)$ satisfying

$$\left\|U(\gamma_*)^*U(\gamma_*) - V(\gamma_*)^*V(\gamma_*)\right\|_2 = 7.9700 \times 10^{-4},$$

which verifies Corollary 3.6. Moreover, we obtain the perturbation matrix Δ in (14) such that $\|\Delta\|_2 = 2.6089$ and the spectrum of $A + \Delta$ is

$$\sigma(A + \Delta) = \{23.9851 - 0.0094\,\mathrm{i}, -3, 1 - 1\,\mathrm{i}, 2, 0, 0.4142\,\mathrm{i}\},$$

where

$$\Delta = \begin{bmatrix} 1.7737 & -0.1429 & -0.3737 & -1.2738 & -0.4630 & -0.6863 \\ -0.0726 & 2.3853 & 0.4469 & 0.0497 & -0.4936 & 0.4190 \\ -0.4187 & -0.8046 & 2.1643 & -0.5331 & -0.6578 & 0.3661 \\ -0.0537 & -0.3145 & 0.00645 & 1.8435 & -0.17917 & -1.0578 \\ -0.6451 & 0.1944 & 0.1928 & -0.7446 & 2.1703 & -0.3585 \\ -0.7070 & -0.5467 & -1.1922 & -0.2208 & -0.4246 & 1.6481 \end{bmatrix}$$
$$+ \begin{bmatrix} 0.0512 & -0.0085 & -0.0171 & -0.0277 & -0.0115 & -0.0199 \\ -0.0116 & 0.0682 & 0.0111 & -0.0100 & -0.0141 & 0.0283 \\ -0.0215 & -0.0272 & 0.0759 & -0.0261 & -0.0274 & 0.0273 \\ 0.0060 & -0.0151 & -0.0170 & 0.0628 & -0.0041 & -0.0441 \\ -0.0206 & 0.0172 & 0.0135 & -0.0314 & 0.0913 & -0.0218 \\ -0.0206 & -0.0059 & -0.022 & -0.0033 & -0.0302 & 0.0553 \end{bmatrix} \mathrm{i}.$$

Approximating a matrix with another one that has prescribed eigenvalues. In many applications of matrices, particularly in optimization problems, we are concerned with a matrix that must have some assumptions on its eigenvalues. Suppose, for instance, that $f(x)$ is a function that we want to maximize or minimize it. In the various method for optimizing $f(x)$, it is necessary that $\nabla^2 f$ be a positive definite matrix. A matrix M is positive definite if and only if any eigenvalue of M is greater than zero. However, $\nabla^2 f$ may not be positive definite for some cases. In such problems, some approaches will be applied to overcome this difficulty which all of them benefit the constructing of a correction matrix ΔM that ensures all eigenvalues of $M + \Delta M$ are greater enough than zero. For a symmetric matrix M, we can obtain the correction matrix ΔM with minimum Euclidean norm and Frobenius norm by using the techniques performed in Section 3.4 of [22]. Now, our procedure for a general non-positive definite matrix M, can be applied. First, construct the matrix ΔM and then replace M by a positive definite approximation $M + \Delta M$, in which all negative eigenvalues in M are replaced by arbitrary (large enough) positive numbers, while keeping its other (positive) eigenvalues unchanged.

Example 6.2. Consider the non-positive definite matrix

$$A = \begin{bmatrix} 3 & 6 & 9 & 10 \\ 4 & 1 & -1 & -2 \\ 7 & 5 & 0 & -4 \\ 4 & -3 & -1 & 6 \end{bmatrix},$$

which is a random matrix generated by MATLAB and its eigenvalues are 12.9377, 7.0550, −0.3641 and −9.6286. Assume now the set $\Lambda = \{12.9377, 7.0550, \varepsilon, \varepsilon\}$, in order to construct a positive definite approximation of A where $\varepsilon = 1 \times 10^{-4}$, is chosen arbitrarily. Using the MATLAB function `fmincon`, one can derive that the function $s_{13}(Q_A(\gamma))$ attains its maximum value at the finite point

$$\gamma_* = \{6.6686, 8.2009, 7.9580, 0.5925 - 0.0021\,\mathrm{i}, -2.3778 - 0.0009\,\mathrm{i}, 1.4475 - 0.0009\,\mathrm{i}\},$$

where the 13th singular value of $Q_A(\gamma_*)$ is isolated and $\alpha_4^* = s_{13}(Q_A(\gamma_*)) = 5.1231$. By applying the technique presented in the paper, the matrix Δ in (14) is computed as follows:

$$\Delta = \begin{bmatrix} 3.4650 & -1.1376 & 1.2052 & 3.3916 \\ 0.2301 & 1.4468 & -4.5382 & 1.8719 \\ -0.2479 & 4.6130 & 1.9171 & 1.1181 \\ -3.7579 & -1.2664 & 0.7100 & 3.1641 \end{bmatrix} + \begin{bmatrix} -0.0011 & -0.0004 & -0.0006 & -0.0025 \\ -0.0022 & -0.0012 & -0.0019 & -0.0046 \\ 0.0006 & -0.0005 & 0.0005 & 0.0035 \\ 0.0001 & -0.0010 & 0.0005 & 0.0018 \end{bmatrix} \mathrm{i},$$

which satisfies $\|\Delta\| \simeq \alpha_4^*$ and $12.9377, 7.0550, 10^{-4}, 10^{-4}$ are eigenvalues of $A + \Delta$. Clearly, $A + \Delta$ is a positive definite matrix.

Our next numerical experiment is borrowed from Section 9.2 of [14] with the aim of comparing the presented method with other related existing methods.

Example 6.3. The second numerical example in [14, Section 9.2] concerns a spectral norm distance from the Frank matrix of order 12, F_{12}, to the matrices with four fixed eigenvalues $\Lambda = \{0.1, 0.1i, -0.1, -0.1i\}$. The Frank matrix of order n, denoted by F_n, is an n-square upper Hessenberg matrix with determinant 1. The MATLAB command `gallery('frank', n)` returns F_n whose all eigenvalues are real, positive and very ill-conditioned.

The procedure established in [14] concludes that the optimal 2-norm distance from F_{12} to the matrices having Λ in their spectrum is 6.9×10^{-4}. Now, applying the presented method gives us that

$$\gamma_* = \{3.7874, 2.7734, 1.6624, -0.3556 + 0.0022\,\mathrm{i}, -0.1644 - 0.1657\,\mathrm{i}, -0.0149 + 0.0214\,\mathrm{i}\},$$

is the finite point in which the several variable function $s_\kappa\left(Q_{F_{12}}(\gamma)\right)$ attains its maximum value. Also, $\alpha_4^* = s_{45}(Q_A(\gamma_*)) = 6.8970 \times 10^{-4}$, is an isolated singular value of constant matrix $s_\kappa\left(Q_{F_{12}}(\gamma_*)\right)$. Furthermore, the 12×4 matrices $U(\gamma_*)$ and $V(\gamma_*)$ are full rank and satisfy $\|U(\gamma_*)^*U(\gamma_*) - V(\gamma_*)^*V(\gamma_*)\| = 1.5710 \times 10^{-6}$, which verifies Corollary 3.6. Additionally, it is straightforward to confirm that the perturbation Δ in (14) satisfies $\|\Delta\|_2 = 6.8970 \times 10^{-4}$ and the perturbed matrix $A + \Delta$ has Λ as its spectrum.

Moreover, this idea can be implemented in areas such as solving a singular or ill condition system of linear equations, i.e., $Ax = b$ for some $A \in \mathbb{C}^{n\times n}$, and studying the solutions of fast–slow systems. These two issues are reviewed briefly in the following.

Notice that, singular square matrices are a thin subset of the space of all square matrices and adding a tiny random perturbation to a singular matrix makes it nonsingular. But the perturbed matrix, namely, $\tilde{A}$ may not be useful in practice. In other words, we will able to compute a solution for $\tilde{A}x = b$, but that solution may be meaningless. This failure, in general, is caused mainly by changing all eigenvalues of A and being large enough the condition number of $\tilde{A}$ in which the smallest eigenvalue of $\tilde{A}$ has a significant effect. It seems that we can cope simultaneously with these failures by applying the method explained in Section 4. For the case of a singular (ill condition) system we construct a perturbation in which all zero (small enough) eigenvalues in A are replaced by nonzero (large enough) numbers. Nevertheless, it should be noted that the accuracy and credibility of the solution depends extremely on the problem we are trying to solve. This topic is known as *regularization*, however, it may have different names in different contexts.

On the other hand, consider the fast–slow systems. In order to apply some methods for studying the solutions of fast–slow systems, we must have a matrix that has no eigenvalues on the imaginary axis [23]. We can overcome this problem, by constructing a perturbation of the matrix such that all its eigenvalues are the same and changing only the eigenvalues which are on the imaginary axis. See [24–26] and references therein for theory and applications of fast–slow systems.

Low-rank approximation problem. Suppose that $A \in \mathbb{C}^{m\times n}$ and a positive integer $r < \min\{m, n\}$ are given. In mathematics, *low-rank approximation problem* concerns finding a matrix $D \in \mathbb{C}^{m\times n}$ such that $\mathrm{rank}(D) = r$ and D is nearest matrix to A, i.e., $\|A - D\| \geq 0$ is as small as possible and $\|\cdot\|$ can in general be any matrix norm. The problem is used for mathematical modeling, data and image compression, noise reduction, system and control theory, computer algebra and so on. Many different techniques are provided for computing a matrix with another one of the same size that has reduced rank. See [27–30] as the suggested references on the low-rank approximation problem and its applications. However, we can use the results obtained in this paper for computing a low-rank approximation of a given square matrix. To do this, by applying the procedure described, we can set $\lambda_1 = \cdots = \lambda_m = 0$ where $m \leq k$. Now if its corresponding eigenvectors $\psi_1, \ldots, \psi_m$ introduced in (17) satisfy rank $([\psi_1, \ldots, \psi_m]) = \tau$, then it is clear that rank of the perturbed matrix $A + \Delta$ is almost $n - \tau$.

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