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Complete Hermite–Birkhoff interpolation on scattered data by combined Shepard operators

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ABSTRACT

The problem of Hermite–Birkhoff interpolation on scattered data under certain conditions of completeness is considered by using Shepard basis functions in combination with local interpolating polynomials based on the vertices of triangles. This approach allows the construction of interpolation operators with nontrivial polynomial reproduction and a good accuracy of approximation.

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1. Introduction

Let Ω be an open, convex and bounded subset of $\mathbb{R}^2$ and let $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \Omega$, $\boldsymbol{x}_i = (x_i, y_i)$, $i = 1, \ldots, N$ be a set of distinct points (the interpolation nodes or *data sites* or *sample points*). We assume that the nodes are arbitrarily distributed or scattered, i.e. they do not lie in a uniform or in a regular grid. The Hermite interpolation problem on scattered data consists of determining a continuous function $S_f : \Omega \to \mathbb{R}$ such that its values and derivatives up to order r_i at each interpolation node $\boldsymbol{x}_i$ (the order may depend on the node) match the values assumed at that node by an unknown continuous function $f : \Omega \to \mathbb{R}$ and its derivatives, respectively. The Birkhoff interpolation problem on scattered data is a generalization of the Hermite one, which differs only in the way of selecting the functional or derivative values to be interpolated. In fact, in the Birkhoff interpolation not all the functional or derivative values for all points are supplied: hence this kind of interpolation is also called *lacunary interpolation*. We refer to a function that solves a Hermite–Birkhoff interpolation problem in an opportune space of function $\mathcal{F}$ with data taken from a function f as a Hermite–Birkhoff interpolant of f in $\mathcal{F}$.

The problem of Hermite–Birkhoff interpolation on scattered data has already been considered by Wu [1], Sun [2], Narcowich and Ward [3] and others in the framework of radial basis functions (see [4, Ch. 36] or [5, Ch. 16.2] and the references therein). In particular, in [5, Ch. 16.2] it is shown that it is always possible to reconstruct a function from Hermite–Birkhoff data by an interpolant which uses C^{2k} functions in order to interpolate C^k data. This is the price one need to pay to ensure invertibility of the interpolation matrix [4, Ch. 36, p. 334], which can be also very large in the applications. In any case, whenever large datasets are considered, above methods can be combined with fast evaluation methods like, for instance, partition of unity methods [5, Ch. 15.4].

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In line with other authors, here we approach the Hermite–Birkhoff interpolation problem on scattered data by using moving least squares method (see for example [6, Appendix B] or [5, Ch. 4]), in particular Shepard method combined with local interpolation polynomials [7]. In a wider context, this approach has been already considered by Coman [8] with the aim of constructing Shepard-type operators with algebraic degree of exactness greater than zero. Coman's approach requires an appropriate order of the sample points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ and, starting from each $\boldsymbol{x}_k$, it is necessary to specify an appropriate subsequence of consecutive points $\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{k+\nu_k-1}$, $\nu_k \in \mathbb{N}$, that guarantees the existence and uniqueness of local polynomial interpolants on these points. Although this procedure is formally correct, it may be very difficult to perform in practice, especially in the presence of a large number of nodes. In the recent paper [9] Allasia and Bracco give a theoretical solution to the Hermite–Birkhoff interpolation problem on scattered data, by combining Shepard basis functions with Taylor–Birkhoff polynomials, which are nothing but Taylor polynomials of f of a certain order, centered at each $\boldsymbol{x}_k$, (possibly) deprived of certain terms (see Section 4). The resulting combination has a simple explicit representation, interpolates on all the data, but may have a low degree of algebraic precision and may even not have the property of reproducing polynomials and this badly affects the approximation performance of the operator.

In this paper, under certain conditions of completeness of the Hermite–Birkhoff data, we introduce an interpolation scheme which removes the weaknesses and holds the strengths of the aforesaid general methods (see Section 3). In fact, the proposed method reproduces polynomials just as well as the Coman method and is as simple to implement as the Allasia and Bracco method. The main idea is to use the local interpolants on suitable triangles. Due to the structure of the data to be interpolated, one can replace cartesian coordinates by barycentric coordinates. This allows us to determine the regularity of a Hermite–Birkhoff interpolation – in principle – by mapping it to one of a finite number of reference triangles for which the question of regularity has already been settled. The advantage of this approach is threefold: firstly, it only requires the solution of a relatively small number of small linear systems to solve local Hermite–Birkhoff interpolation problems in polynomial spaces; secondly, the resulting Hermite–Birkhoff interpolant of f is *local*, that is its value at a point $\boldsymbol{x} = (x, y) \in \Omega$ depends only on the values of f at a small number of neighboring nodes; thirdly, it allows to reconstruct a function from Hermite–Birkhoff data by an interpolant which uses C^{k+1} functions in order to interpolate C^k data. The numerical results show the effectiveness of the method, its feasibility and its accuracy for very general approximation data.

2. Hermite–Birkhoff interpolation by Shepard operators

In the papers [8,9] Hermite–Birkhoff interpolants are obtained by combining Classic or Local Shepard Basis Functions with local interpolating polynomials. Classic Shepard Basis Functions (CSBF) are defined as follows:

$$A_{\mu,i}(\boldsymbol{x}) = \frac{|\boldsymbol{x} - \boldsymbol{x}_i|^{-\mu}}{\sum_{j=1}^{N} |\boldsymbol{x} - \boldsymbol{x}_j|^{-\mu}}, \quad \mu > 0,\ i = 1, \ldots, N, \tag{1}$$

where $|\cdot|$ denotes the euclidean distance. CSBF satisfy the following properties:

P1 CSBF are continuous functions for each $\mu > 0$ and differentiable functions for each $\mu > 1$; in particular [10]

$$\begin{cases} A_{\mu,i}(\boldsymbol{x}) \in C^{[\mu]} & \text{if } \mu > 1 \text{ is not an integer,} \\ A_{\mu,i}(\boldsymbol{x}) \in C^{\mu-1} & \text{if } \mu > 1 \text{ is an integer,} \end{cases}$$

(if μ is even then $A_{\mu,i}(\boldsymbol{x})$ are rational functions with no poles in $\mathbb{R}^2$);

P2 CSBF are non negative functions for each $\mu > 0$: $A_{\mu,i}(\boldsymbol{x}) \geq 0$;

P3 CSBF constitute a partition of unity: $\sum_{i=1}^{N} A_{\mu,i}(\boldsymbol{x}) \equiv 1$;

P4 CSBF are cardinal functions: $A_{\mu,i}(\boldsymbol{x}_j) = \delta_{i,j}$ for each $i, j = 1, \ldots, N$ where $\delta_{i,j}$ is the Kronecker's delta;

P5 CSBF have the property of vanishing derivatives at the interpolation nodes:

$$\frac{\partial^{\alpha+\beta} A_{\mu,i}(\boldsymbol{x}_j)}{\partial x^\alpha \partial y^\beta} = 0, \quad \text{for each } i, j = 1, \ldots, N,\ \alpha, \beta \in \mathbb{N},\ \alpha + \beta \leq \begin{cases} [\mu] & \text{if } \mu > 1 \text{ is not an integer} \\ \mu - 1 & \text{if } \mu > 1 \text{ is an integer.} \end{cases}$$

Properties P1–P5 make CSBF suitable for interpolation of functional and derivative values. In fact, let us assume that at each node $\boldsymbol{x}_i$, $i = 1, \ldots, N$ a polynomial $P[f, \boldsymbol{x}_i]$ is given, capable of interpolating the value of a function f and/or some of its partial derivatives at the node $\boldsymbol{x}_i$; then the combined Shepard operator

$$S_P[f](\boldsymbol{x}) = \sum_{i=1}^{N} A_{\mu,i}(\boldsymbol{x})\, P[f, \boldsymbol{x}_i](\boldsymbol{x}) \tag{2}$$

satisfies

$$S_P[f](\boldsymbol{x}_j) = \sum_{i=1}^{N} A_{\mu,i}(\boldsymbol{x}_j)\, P[f, \boldsymbol{x}_i](\boldsymbol{x}_j) = P[f, \boldsymbol{x}_j](\boldsymbol{x}_j) \tag{3}$$

and

$$\frac{\partial^{\alpha+\beta} S_P[f](\boldsymbol{x}_j)}{\partial x^\alpha \partial y^\beta} = \sum_{i=1}^{N} \frac{\partial^{\alpha+\beta} A_{\mu,i}(\boldsymbol{x}_j)}{\partial x^\alpha \partial y^\beta} P[f, \boldsymbol{x}_i](\boldsymbol{x}_j) + \sum_{i=1}^{N} A_{\mu,i}(\boldsymbol{x}_j) \frac{\partial^{\alpha+\beta} P[f, \boldsymbol{x}_i](\boldsymbol{x}_j)}{\partial x^\alpha \partial y^\beta}$$
$$= \frac{\partial^{\alpha+\beta} P[f, \boldsymbol{x}_j](\boldsymbol{x}_j)}{\partial x^\alpha \partial y^\beta} \tag{4}$$

for each α, β as in P5.

The Local Shepard Basis Functions (LSBF)

$$\widetilde{\mathcal{W}}_{\mu,i}(\boldsymbol{x}) = \frac{\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_i|} - \frac{1}{R_{w_i}}\right)_+^\mu}{\sum_{k=1}^{N} \left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_k|} - \frac{1}{R_{w_k}}\right)_+^\mu} \tag{5}$$

are obtained from CSBF $A_{\mu,i}(\boldsymbol{x})$ by multiplying the euclidean distances $|\boldsymbol{x}-\boldsymbol{x}_j|^{-\mu}$ $j = 1, \ldots, N$ by Franke-Little weights

$$\left(1 - \frac{|\boldsymbol{x}-\boldsymbol{x}_j|}{R_{w_j}}\right)_+^\mu, \quad R_{w_j} > 0 \text{ for each } j = 1, \ldots, N$$

where

$$(x)_+ = \begin{cases} x, & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and R_{w_i} is the radius of influence about node $\boldsymbol{x}_i$ which, in practice, is taken to be just large enough to include N_w nodes; consequently LSBF are compactly supported functions which satisfy similar properties P1–P5. The modified Shepard operator

$$\widetilde{S}_P[f](\boldsymbol{x}) = \sum_{i=1}^{N} \widetilde{\mathcal{W}}_{\mu,i}(\boldsymbol{x}) P[f, \boldsymbol{x}_i](\boldsymbol{x}) \tag{6}$$

satisfies interpolation properties as the operator $S_P[f](\boldsymbol{x})$ and has the advantage of being a local interpolation operator, that is its value at a point $\boldsymbol{x} \in \Omega$ depends on the values of f at a small number of neighboring nodes only.

Another useful feature of combined (2) or modified (6) Shepard operators is the polynomial reproduction [11]. Let us denote by $\mathcal{P}_{\boldsymbol{x}}^d$ the space of polynomials p in the variable $\boldsymbol{x}$ up to degree $d \in \mathbb{N}$; we say that S_P reproduces polynomials up to degree d, if $S_P[p] = p$ for every $p \in \mathcal{P}_{\boldsymbol{x}}^d$. The number d is sometimes called the *algebraic degree of exactness* or *algebraic degree of precision* of the operator S_P [8,12]. The following property can be easily checked

P6 if for each $i = 1, \ldots, N$ there exists $d_i \in \mathbb{N}$ such that $P[p, \boldsymbol{x}_i] = p$ for all $p \in \mathcal{P}_{\boldsymbol{x}}^{d_i}$, then $S_P[p] = p$ for all $p \in \mathcal{P}_{\boldsymbol{x}}^{\min\{d_1,\ldots,d_N\}}$.

As well-known, the property of reproducing polynomials up to degree d is desirable for local linear operators of approximation, since it implies that the operator has approximation order $d+1$. That is, under certain conditions of regularity of function f and uniformity of the node set $\mathcal{X}$, $\widetilde{S}_P[f]$ tends to the function f, in the uniform norm, when the fill distance

$$h_{\mathcal{X},\Omega} = \sup_{\boldsymbol{x}\in\Omega} \min_{\boldsymbol{x}_i\in\mathcal{X}} |\boldsymbol{x}-\boldsymbol{x}_i| \tag{7}$$

tends to 0 and the rate of convergence is $O(h_{\mathcal{X},\Omega}^{d+1})$ [5, Cor. 4.8].

Remark 1. Local (polynomial) interpolation operator $P[\cdot, \boldsymbol{x}_i]$ reproduces polynomials of degree at most d_i if $P[f, \boldsymbol{x}_i]$ is the *unique solution* in $\mathcal{P}_{\boldsymbol{x}}^{d_i}$ of an interpolation problem with data taken from the continuous function f.

3. Complete Hermite–Birkhoff interpolation: an approach using Shepard operators

The goal of this section is to set the *complete Hermite–Birkhoff interpolation problem* on scattered data and to introduce a method to solve it, if possible. The method has polynomial reproduction property, as the Coman method, and it is as simple to implement as the Allasia and Bracco method.

We assume that for each $i = 1, 2, \ldots, N$ the set

$$\mathcal{I}_i(f) = \left\{f^{(p,q)}(\boldsymbol{x}_i), (p,q) \in I_i\right\}, \quad I_i \subset \mathbb{N}^2 \text{ finite set} \tag{8}$$

containing Hermite–Birkhoff data about a differentiable function $f \in C^r(\Omega)$, $r \geq 0$, satisfies the following condition:

C1 each datum in $\mathcal{I}_i(f)$ is a function value or a partial derivative value of order up to r_i, $0 \leq r_i \leq r$ of f at $\boldsymbol{x}_i$;
C2 if some partial derivative value of order s belongs to $\mathcal{I}_i(f)$, then all partial derivative values of order s belong to $\mathcal{I}_i(f)$.

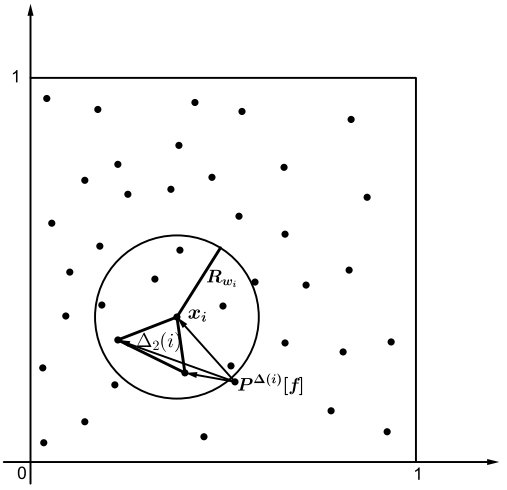


Fig. 1. The point-triangle association is realized in order to reduce the error of approximation of a polynomial $P^{\Delta(i)}[f](\boldsymbol{x})$, which is based on the vertices of the triangle $\Delta(i)$ and is *the unique solution* in a polynomial space $\mathcal{P}_{\boldsymbol{x}}^{d_i}$ of a three point Hermite–Birkhoff interpolation problem with data obtainable from $\mathcal{I}_i(f) \cup \mathcal{I}_j(f) \cup \mathcal{I}_k(f)$.

Definition 2. The dataset $\mathcal{I}(f) = \cup_{i=1}^{N} \mathcal{I}_i(f)$ is a set of *complete Hermite–Birkhoff data.*

Remark 3. Condition C2 implies that any directional derivative of order s of f at $\boldsymbol{x}_i$ can be obtained by the data in $\mathcal{I}_i(f)$.

When, for each $i = 1, 2, \ldots, N$, $r_i = r$ and the function value and all partial derivatives of order less than or equal to r belong to $\mathcal{I}_i(f)$, the dataset $\mathcal{I}(f)$ is also called a set of *uniform Hermite data*. According to Lorentz [13, Ch. 4.1] this is the most natural generalization of univariate Hermite data to the multivariate case.

Our approach to complete Hermite–Birkhoff interpolation is based on the following three steps (particular cases have been considered in [14–16]).

For each $i = 1, \ldots, N$,

S1 we associate the sample point $\boldsymbol{x}_i \in \mathcal{X}$ with a triangle $\Delta(i)$, with a vertex in $\boldsymbol{x}_i$ and other two vertices in certain interpolation nodes $\boldsymbol{x}_j, \boldsymbol{x}_k$ in the closed ball $B\left(\boldsymbol{x}_i, R_{w_i}\right) \subset \Omega$;

S2 we identify a polynomial space $\mathcal{P}_{\boldsymbol{x}}^{d_i}$, $d_i \in \mathbb{N}$, and a polynomial $P^{\Delta(i)}[f](\boldsymbol{x}) \in \mathcal{P}_{\boldsymbol{x}}^{d_i}$, based on the vertices of the triangle $\Delta(i)$, which is *the unique solution* of a Hermite–Birkhoff interpolation problem with interpolation dataset obtainable from $\mathcal{I}_i(f) \cup \mathcal{I}_j(f) \cup \mathcal{I}_k(f)$;

S3 we choose the pair $\left(\Delta(i), P^{\Delta(i)}[f](\boldsymbol{x})\right)$ such that the error of approximation of $P^{\Delta(i)}[f](\boldsymbol{x})$ is the smallest in $B\left(\boldsymbol{x}_i, R_{w_i}\right)$. See Fig. 1.

We call $\boldsymbol{x}_i$ the *referring vertex* of $\Delta(i)$ and we also denote it by $\boldsymbol{x}^0$; starting from $\boldsymbol{x}^0$ and moving counterclockwise we denote the vertices $\boldsymbol{x}_j, \boldsymbol{x}_k$ also by $\boldsymbol{x}^1, \boldsymbol{x}^2$; without loss of generality we assume $\boldsymbol{x}^1 = \boldsymbol{x}_j$. Thanks to condition **C2** we can assume that the interpolation dataset is constituted by data which are repeated applications of derivatives along the directed sides of the triangle $\Delta(i)$ to f at $\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2$. More precisely, we set $\boldsymbol{x}^l = \left(x^l, y^l\right)$, $l = 0, 1, 2$,

$$D_{rs} = \left(x^r - x^s\right) \frac{\partial}{\partial x} + \left(y^r - y^s\right) \frac{\partial}{\partial y}, \quad r, s = 0, 1, 2, \ r \neq s \tag{9}$$

and for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ in $\mathbb{N}^2$

$$D_0^{\boldsymbol{\alpha}} = D_{10}^{\alpha_1} D_{20}^{\alpha_2}, \qquad D_1^{\boldsymbol{\beta}} = D_{01}^{\beta_1} D_{21}^{\beta_2}, \qquad D_2^{\boldsymbol{\gamma}} = D_{02}^{\gamma_1} D_{12}^{\gamma_2}. \tag{10}$$

Then, by Remark 3, we refer the interpolation data of $\mathcal{I}_i(f)$, $\mathcal{I}_j(f)$, $\mathcal{I}_k(f)$ to the sides of the triangle $\Delta(i)$ with common vertex $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k$ respectively and we set

$$\mathcal{D}_{i,0}(f) = \left\{D_0^{\boldsymbol{\alpha}} f\left(\boldsymbol{x}^0\right) : \boldsymbol{\alpha} \in I_i\right\},$$
$$\mathcal{D}_{i,1}(f) = \left\{D_1^{\boldsymbol{\beta}} f\left(\boldsymbol{x}^1\right) : \boldsymbol{\beta} \in I_j\right\},$$
$$\mathcal{D}_{i,2}(f) = \left\{D_2^{\boldsymbol{\gamma}} f\left(\boldsymbol{x}^2\right) : \boldsymbol{\gamma} \in I_k\right\}.$$

A convenient representation of $P^{\Delta(i)}[f](\boldsymbol{x})$ uses a polynomial basis given in barycentric coordinates relative to the triangle $\Delta(i)$

$$\lambda_0(\boldsymbol{x}) = \frac{A(\boldsymbol{x}, \boldsymbol{x}^1, \boldsymbol{x}^2)}{A(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2)}, \qquad \lambda_1(\boldsymbol{x}) = \frac{A(\boldsymbol{x}^0, \boldsymbol{x}, \boldsymbol{x}^2)}{A(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2)}, \qquad \lambda_2(\boldsymbol{x}) = \frac{A(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x})}{A(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2)} \tag{11}$$

($A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ denotes the signed area of the triangle of vertices $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$) and coefficients $D_0^{\boldsymbol{\alpha}} f(\boldsymbol{x}^0) \in \mathcal{D}_{i,0}(f)$, $D_1^{\boldsymbol{\beta}} f(\boldsymbol{x}^1) \in \mathcal{D}_{i,1}(f)$, $D_2^{\boldsymbol{\gamma}} f(\boldsymbol{x}^2) \in \mathcal{D}_{i,2}(f)$. Therefore the polynomial $P^{\Delta(i)}[f](\boldsymbol{x})$ can be written as

$$P^{\Delta(i)}[f] = \sum_{\rho} Q_{0,\rho} D_0^{\boldsymbol{\alpha}_\rho} f(\boldsymbol{x}^0) + \sum_{\sigma} Q_{1,\sigma} D_1^{\boldsymbol{\beta}_\sigma} f(\boldsymbol{x}^1) + \sum_{\tau} Q_{2,\tau} D_2^{\boldsymbol{\gamma}_\tau} f(\boldsymbol{x}^2), \tag{12}$$

where $Q_{0,\rho} = Q_{0,\rho}(\lambda_0, \lambda_1, \lambda_2)$, $Q_{1,\sigma} = Q_{1,\sigma}(\lambda_0, \lambda_1, \lambda_2)$, $Q_{2,\tau} = Q_{2,\tau}(\lambda_0, \lambda_1, \lambda_2)$ are homogeneous polynomials in barycentric coordinates of degree d_i and

$$\{\boldsymbol{\alpha}_\rho\}_\rho = I_i, \qquad \{\boldsymbol{\beta}_\sigma\}_\sigma \subset I_j, \qquad \{\boldsymbol{\gamma}_\tau\}_\tau \subset I_k, \tag{13}$$

$$\sharp(\{\rho\}) + \sharp(\{\sigma\}) + \sharp(\{\tau\}) = \binom{d_i + 2}{d_i} = \dim\left(\mathcal{P}_{\boldsymbol{x}}^{d_i}\right). \tag{14}$$

Remark 4. From step S2 the polynomial $P^{\Delta(i)}[f]$ is required to interpolate, at the nodes $\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2$ data $\left\{D_0^{\boldsymbol{\alpha}_\rho} f(\boldsymbol{x}^0)\right\}_\rho$, $\left\{D_1^{\boldsymbol{\beta}_\sigma} f(\boldsymbol{x}^1)\right\}_\sigma$, $\left\{D_2^{\boldsymbol{\gamma}_\tau} f(\boldsymbol{x}^2)\right\}_\tau$ respectively. In particular, since our aim is to solve a complete Hermite–Birkhoff interpolation problem on scattered data, according to property (4) of combined Shepard operators, polynomial $P^{\Delta(i)}[f]$ is required to interpolate, at the node $\boldsymbol{x}_i$, all data from $\mathcal{D}_{i,0}(f)$; consequently $d_i \geq r_i$. However, polynomial $P^{\Delta(i)}[f]$ is not constrained to use and interpolate all the information in $\mathcal{D}_{i,1}(f) \cup \mathcal{D}_{i,2}(f)$, in line with two special cases provided by Coman in [8]. In the process of construction of $P^{\Delta(i)}[f]$ these are only auxiliary data, only some of which are needed to identify the polynomial. For this reason, in Eq. (13) one equality operator and two subset operators appear.

Remark 5. The use of barycentric coordinates and derivatives along the directed sides of the triangle, in the case of Hermite–Birkhoff interpolation on triangles, is suggested in [13, Ch. 8.1] as well as in [17], since both of them are affine invariants: in this case the polynomials in $\mathcal{P}_{\boldsymbol{x}}^{d_i}$ are expressed as polynomials in $\mathcal{P}_{\boldsymbol{\lambda}}^{d_i}$, the space of homogeneous polynomials of degree d_i in the indeterminate $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$. As a consequence, the non vanishing of the Vandermonde determinant of each particular interpolation problem does not depend on the position of the vertices of the triangle and can be checked once and for all. The unique solution of the related problem, if it exists, can be easily computed in barycentric coordinates by symbolic computation software like Mathematica and stored for later use. In the next section we shall provide a procedure that allows us to test whether the unique solution exists and, if so, to calculate it explicitly.

Remark 6. Dealing with Hermite–Birkhoff interpolation problem, it is possible that step S2 fails for any choice of the triangle $\Delta(i)$. This is the case, for example, of the dataset

$$\begin{aligned} &\mathcal{I}_1(f) = \{f(\boldsymbol{x}_1)\}, \\ &\mathcal{I}_k(f) = \left\{f^{(0,2)}(\boldsymbol{x}_k), f^{(1,1)}(\boldsymbol{x}_k), f^{(2,0)}(\boldsymbol{x}_k)\right\}, \quad k = 2, \ldots, N, \end{aligned}$$

for which the proposed method fails.

3.1. Hermite–Birkhoff interpolation on the vertices of a triangle

We refer to a fixed triangle $\Delta(i)$, $i = 1, \ldots, N$, and denote by $\mathcal{D}(i)$ the set

$$\mathcal{D}(i) = \left\{D_0^{\boldsymbol{\alpha}_\rho} f(\boldsymbol{x}^0)\right\}_\rho \cup \left\{D_1^{\boldsymbol{\beta}_\sigma} f(\boldsymbol{x}^1)\right\}_\sigma \cup \left\{D_2^{\boldsymbol{\gamma}_\tau} f(\boldsymbol{x}^2)\right\}_\tau$$

of functional and derivative data appearing, as coefficients, in the analytic expression (12) of $P^{\Delta(i)}[f]$; by (14) we get $\sharp(\mathcal{D}(i)) = \dim\left(\mathcal{P}_{\boldsymbol{x}}^{d_i}\right)$. We describe a procedure that allows us to test the existence and uniqueness of the polynomial $P^{\Delta(i)}[f]$ starting from the set $\mathcal{D}(i)$ and, if so, to calculate the analytic expression (12) of $P^{\Delta(i)}[f]$. To do this, we draw on the Lorentz notation introduced in [13], with minor adjustments to our specific case. To simplify notations, for the rest of this section, we shall omit index i and write Δ, $\mathcal{D}$, d, etc. instead of $\Delta(i)$, $\mathcal{D}(i)$, d_i, etc., where possible.

Following Lorentz, we pictorially represent the data given in $\mathcal{D}$ as follows. For each vertex $\boldsymbol{x}^k$, $k = 0, 1, 2$, we use a $(d+1) \times (d+1)$ lower triangular matrix E_k whose entrances are 1 or 0, depending on the interpolation conditions to be satisfied at that vertex.

Example 7. The Lagrange interpolation problem on the vertices of the triangle Δ, which corresponds to the set of interpolation data

$$\mathcal{D} = \left\{f\left(\boldsymbol{x}^0\right), f\left(\boldsymbol{x}^1\right), f\left(\boldsymbol{x}^2\right)\right\},$$

is depicted by

$$E_0 = \begin{array}{|ccc} 0 & & \\ 1 & 0 & \end{array}, \qquad E_1 = \begin{array}{|ccc} 0 & & \\ 1 & 0 & \end{array}, \qquad E_2 = \begin{array}{|ccc} 0 & & \\ 1 & 0 & \end{array},$$

where the 1s indicate that functional evaluations at the vertices are given data, while the 0s indicate that first order partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are not known.

Since we wish to express the polynomial $P^{\Delta}[f]$ in barycentric coordinates and in terms of derivatives along the directed sides of the triangle Δ, it is more convenient to relate the matrices E_k to the triangle Δ. We do this by marking the matrices E_k with the vertices of Δ as follows

$$E_0 = \begin{array}{|cccc} \boldsymbol{x}^2 & & & \\ \vdots & & & \\ e^0_{20} & \ddots & & \\ e^0_{10} & e^0_{11} & \ddots & \\ e^0_{00} & e^0_{01} & e^0_{02} & \dots \\ \boldsymbol{x}^0 & & & \boldsymbol{x}^1 \end{array} \qquad E_1 = \begin{array}{|cccc} \boldsymbol{x}^2 & & & \\ \vdots & & & \\ e^1_{20} & \ddots & & \\ e^1_{10} & e^1_{11} & \ddots & \\ e^1_{00} & e^1_{01} & e^1_{02} & \dots \\ \boldsymbol{x}^1 & & & \boldsymbol{x}^0 \end{array} \qquad E_2 = \begin{array}{|cccc} \boldsymbol{x}^1 & & & \\ \vdots & & & \\ e^2_{20} & \ddots & & \\ e^2_{10} & e^2_{11} & \ddots & \\ e^2_{00} & e^2_{01} & e^2_{02} & \dots \\ \boldsymbol{x}^2 & & & \boldsymbol{x}^0 \end{array} \tag{15}$$

and by assuming $e^k_{ij} = 1$ if, and only if, $D^{(i,j)}_k f\left(\boldsymbol{x}^k\right) \in \mathcal{D}$, $k = 0, 1, 2$. Therefore we can rewrite the polynomial (12) as follows

$$P^{\Delta}[f] = \sum_{\substack{(i,j) \\ e^0_{ij} \neq 0}} Q_{0,(i,j)} D^{(i,j)}_0 f\left(\boldsymbol{x}^0\right) + \sum_{\substack{(i,j) \\ e^1_{ij} \neq 0}} Q_{1,(i,j)} D^{(i,j)}_1 f\left(\boldsymbol{x}^1\right) + \sum_{\substack{(i,j) \\ e^2_{ij} \neq 0}} Q_{2,(i,j)} D^{(i,j)}_2 f\left(\boldsymbol{x}^2\right). \tag{16}$$

We can include the information provided by the three matrices E_k (15) into the $3 \times \sharp(\mathcal{D})$ Polya matrix [13]

$$E = \begin{pmatrix} e^0_{00} & e^0_{01} & e^0_{10} & e^0_{02} & e^0_{11} & e^0_{20} & \dots \\ e^1_{00} & e^1_{01} & e^1_{10} & e^1_{02} & e^1_{11} & e^1_{20} & \dots \\ e^2_{00} & e^2_{01} & e^2_{10} & e^2_{02} & e^2_{11} & e^2_{20} & \dots \end{pmatrix}. \tag{17}$$

Let us now introduce some other useful notations. We set α the list of non negative integer pairs

$$\alpha = \{(0,0)\,;\,(0,1)\,;\,(1,0)\,;\,(0,2)\,;\,(1,1)\,;\,(2,0)\,;\,\dots\} \tag{18}$$

and we associate, to list α, list η of non negative integer triplets

$$\eta = \{(d,0,0)\,;\,(d-1,0,1)\,;\,(d-1,1,0)\,;\,(d-2,0,2)\,;\,(d-2,1,1)\,;\,(d-2,2,0)\,;\,\dots\}. \tag{19}$$

We assume that α and η are two-column and three-column matrices, respectively, and set $\alpha(m,:) = (\alpha(m,1), \alpha(m,2))$, $\eta(m,:) = (\eta(m,1), \eta(m,2), \eta(m,3))$ for each $m = 1, 2, \dots, \sharp(\mathcal{D})$. Then the d-degree homogeneous polynomials $Q_{0,(i,j)}, Q_{1,(i,j)}, Q_{2,(i,j)}$ appearing in the analytic expression (16) of $P^{\Delta}[f]$ have the following representations

$$\begin{aligned} Q_{0,(i,j)} &= \sum_{m=1}^{\sharp(\mathcal{D})} a^{(i,j)}_{\eta(m,:)} \lambda_0^{\eta(m,1)} \lambda_1^{\eta(m,2)} \lambda_2^{\eta(m,3)}, \\ Q_{1,(i,j)} &= \sum_{m=1}^{\sharp(\mathcal{D})} b^{(i,j)}_{\eta(m,:)} \lambda_0^{\eta(m,1)} \lambda_1^{\eta(m,2)} \lambda_2^{\eta(m,3)}, \\ Q_{2,(i,j)} &= \sum_{m=1}^{\sharp(\mathcal{D})} c^{(i,j)}_{\eta(m,:)} \lambda_0^{\eta(m,1)} \lambda_1^{\eta(m,2)} \lambda_2^{\eta(m,3)}, \end{aligned}$$

where $a^{(i,j)}_{\eta(m,:)}, b^{(i,j)}_{\eta(m,:)}, c^{(i,j)}_{\eta(m,:)}$ are real coefficients to be determined. We determine them by imposing the interpolation conditions stored in the three Polya matrices (15):

$$\begin{aligned} &\text{for each } m,\ \mu = 1, 2, \dots, \sharp(\mathcal{D}), \quad k, \kappa = 0, 1, 2 \\ &\text{such that } e^k_{\alpha(m,:)} = 1 \quad \text{and} \quad e^{\kappa}_{\alpha(\mu,:)} = 1, \\ &D^{(\alpha(m,1),\alpha(m,2))}_k Q_{\kappa,\alpha(\mu,:)}\left(\boldsymbol{x}^k\right) = \begin{cases} 1 & \text{if } k = \kappa \text{ and } m = \mu \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{20}$$

The system of linear equations (20) can be rearranged in $\sharp(\mathcal{D})$ independent systems, each of which corresponds to a fixed pair (μ, κ) such that $e^{\kappa}_{\alpha(\mu,:)} = 1$:

$$D_k^{(\alpha(m,1),\alpha(m,2))} Q_{\kappa,\alpha(\mu,:)}\left(\boldsymbol{x}^k\right) = \begin{cases} e^k_{\alpha(m,:)} & \text{if } k = \kappa \text{ and } m = \mu \\ 0 & \text{otherwise} \end{cases},$$
$$k = 0, 1, 2, m = 1, 2, \ldots, \sharp(\mathcal{D}), \quad \text{such that } e^k_{\alpha(m,:)} = 1.$$

These systems vary only for the vector of known terms: in fact, each of them has $\sharp(\mathcal{D})$ linear equations whose coefficients are determined by the action of the linear functionals

$$\varphi \longrightarrow D_k^{(\alpha(m,1),\alpha(m,2))}\varphi,$$
$$m = 1, 2, \ldots, \sharp(\mathcal{D}), k = 0, 1, 2 \quad \text{such that } e^k_{\alpha(m,:)} = 1$$

on the polynomials in $\mathcal{P}^d_\lambda$ (See [9, Sec. 2] or [18, Ch. II]). Let M be the matrix of coefficients; the entries of M, as well as its determinant and, if it exists, the unique solution in $\mathcal{P}^d_{\boldsymbol{x}}$ of the interpolation problem can be easily computed by symbolic computation software like Mathematica.

Let us consider some examples.

Example 8. Let us assume that

$$E_0 = \begin{array}{c|cc} \boldsymbol{x}^2 & 0 & 0 \\ & 1 & 0 \\ \hline \boldsymbol{x}^0 & & \boldsymbol{x}^1 \end{array}, \qquad E_1 = \begin{array}{c|cc} \boldsymbol{x}^2 & 1 & 0 \\ & 0 & 0 \\ \hline \boldsymbol{x}^1 & & \boldsymbol{x}^0 \end{array}, \qquad E_2 = \begin{array}{c|cc} \boldsymbol{x}^1 & 0 & 0 \\ & 0 & 1 \\ \hline \boldsymbol{x}^2 & & \boldsymbol{x}^0 \end{array}. \tag{21}$$

Therefore $\sharp(\mathcal{D}) = 3$, Polya's matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and then $P^{\Delta}[f] \in \mathcal{P}^1_{\boldsymbol{x}}$. The vectors of multi-indices are

$$\alpha = \{(0,0); (0,1); (1,0)\},$$
$$\eta = \{(1,0,0); (0,0,1); (0,1,0)\}$$

consequently

$$Q_{0,(0,0)} = a^{(0,0)}_{(1,0,0)}\lambda_0 + a^{(0,0)}_{(0,0,1)}\lambda_1 + a^{(0,0)}_{(0,1,0)}\lambda_2,$$
$$Q_{1,(1,0)} = b^{(1,0)}_{(1,0,0)}\lambda_0 + b^{(1,0)}_{(0,0,1)}\lambda_1 + b^{(1,0)}_{(0,1,0)}\lambda_2,$$
$$Q_{2,(0,1)} = c^{(0,1)}_{(1,0,0)}\lambda_0 + c^{(0,1)}_{(0,0,1)}\lambda_1 + c^{(0,1)}_{(0,1,0)}\lambda_2.$$

By imposing the interpolation conditions in E we get

$$\begin{cases} Q_{0,(0,0)}\left(\boldsymbol{x}^0\right) = 1 \\ D_1^{(1,0)} Q_{0,(0,0)}\left(\boldsymbol{x}^1\right) = 0 \\ D_2^{(0,1)} Q_{0,(0,0)}\left(\boldsymbol{x}^2\right) = 0 \end{cases}, \quad \begin{cases} Q_{1,(1,0)}\left(\boldsymbol{x}^0\right) = 0 \\ D_1^{(1,0)} Q_{1,(1,0)}\left(\boldsymbol{x}^1\right) = 1 \\ D_2^{(0,1)} Q_{1,(1,0)}\left(\boldsymbol{x}^2\right) = 0 \end{cases}, \quad \begin{cases} Q_{2,(0,1)}\left(\boldsymbol{x}^0\right) = 0 \\ D_1^{(1,0)} Q_{2,(0,1)}\left(\boldsymbol{x}^1\right) = 0 \\ D_2^{(0,1)} Q_{2,(0,1)}\left(\boldsymbol{x}^2\right) = 1 \end{cases};$$

these systems have the same matrix of coefficients

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

whose determinant is not zero. The interpolation problem (21) has then the unique solution

$$P^{\Delta}[f] = f\left(\boldsymbol{x}^0\right) - (\lambda_1 + \lambda_2) D_1^{(1,0)} f\left(\boldsymbol{x}^1\right) - \lambda_2 D_2^{(0,1)} f\left(\boldsymbol{x}^2\right).$$

Example 9. Let us assume that

$$E_0 = \begin{array}{c|ccc} \boldsymbol{x}^2 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & 1 & 1 & 0 \\ \hline \boldsymbol{x}^0 & & & \boldsymbol{x}^1 \end{array}, \qquad E_1 = \begin{array}{c|ccc} \boldsymbol{x}^2 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ \hline \boldsymbol{x}^1 & & & \boldsymbol{x}^0 \end{array}, \qquad E_2 = \begin{array}{c|ccc} \boldsymbol{x}^1 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ \hline \boldsymbol{x}^2 & & & \boldsymbol{x}^0 \end{array}. \tag{22}$$

Therefore $\sharp(\mathcal{D}) = 6$, Polya's matrix is

$$E = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and then $P^{\Delta}[f] \in \mathcal{P}^2_{\boldsymbol{x}}$. The vectors of multi-indices are

$$\alpha = \{(0,0)\,;\,(0,1)\,;\,(1,0)\,;\,(0,2)\,;\,(1,1)\,;\,(2,0)\},$$
$$\eta = \{(2,0,0)\,;\,(1,0,1)\,;\,(1,1,0)\,;\,(0,0,2)\,;\,(0,1,1)\,;\,(0,2,0)\}$$

consequently

$$\begin{aligned}
Q_{0,(0,0)} &= a^{(0,0)}_{(2,0,0)}\lambda_0^2 + a^{(0,0)}_{(1,0,1)}\lambda_0\lambda_2 + a^{(0,0)}_{(1,1,0)}\lambda_0\lambda_1 + a^{(0,0)}_{(0,0,2)}\lambda_2^2 + a^{(0,0)}_{(0,1,1)}\lambda_1\lambda_2 + a^{(0,0)}_{(0,2,0)}\lambda_1^2,\\
Q_{0,(1,0)} &= a^{(1,0)}_{(2,0,0)}\lambda_0^2 + a^{(1,0)}_{(1,0,1)}\lambda_0\lambda_2 + a^{(1,0)}_{(1,1,0)}\lambda_0\lambda_1 + a^{(1,0)}_{(0,0,2)}\lambda_2^2 + a^{(1,0)}_{(0,1,1)}\lambda_1\lambda_2 + a^{(1,0)}_{(0,2,0)}\lambda_1^2,\\
Q_{0,(0,2)} &= a^{(0,2)}_{(2,0,0)}\lambda_0^2 + a^{(0,2)}_{(1,0,1)}\lambda_0\lambda_2 + a^{(0,2)}_{(1,1,0)}\lambda_0\lambda_1 + a^{(0,2)}_{(0,0,2)}\lambda_2^2 + a^{(0,2)}_{(0,1,1)}\lambda_1\lambda_2 + a^{(0,2)}_{(0,2,0)}\lambda_1^2,\\
Q_{1,(0,1)} &= b^{(0,1)}_{(2,0,0)}\lambda_0^2 + b^{(0,1)}_{(1,0,1)}\lambda_0\lambda_2 + b^{(0,1)}_{(1,1,0)}\lambda_0\lambda_1 + b^{(0,1)}_{(0,0,2)}\lambda_2^2 + b^{(0,1)}_{(0,1,1)}\lambda_1\lambda_2 + b^{(0,1)}_{(0,2,0)}\lambda_1^2,\\
Q_{1,(0,2)} &= b^{(0,2)}_{(2,0,0)}\lambda_0^2 + b^{(0,2)}_{(1,0,1)}\lambda_0\lambda_2 + b^{(0,2)}_{(1,1,0)}\lambda_0\lambda_1 + b^{(0,2)}_{(0,0,2)}\lambda_2^2 + b^{(0,2)}_{(0,1,1)}\lambda_1\lambda_2 + b^{(0,2)}_{(0,2,0)}\lambda_1^2,\\
Q_{2,(1,1)} &= c^{(1,1)}_{(2,0,0)}\lambda_0^2 + c^{(1,1)}_{(1,0,1)}\lambda_0\lambda_2 + c^{(1,1)}_{(1,1,0)}\lambda_0\lambda_1 + c^{(1,1)}_{(0,0,2)}\lambda_2^2 + c^{(1,1)}_{(0,1,1)}\lambda_1\lambda_2 + c^{(1,1)}_{(0,2,0)}\lambda_1^2.
\end{aligned}$$

By imposing the interpolation condition in E we get

$$\begin{cases} Q_{0,(0,0)}(\boldsymbol{x}^0) = 1 \\ D_0^{(1,0)}Q_{0,(0,0)}(\boldsymbol{x}^0) = 0 \\ D_0^{(0,2)}Q_{0,(0,0)}(\boldsymbol{x}^0) = 0 \\ D_1^{(0,1)}Q_{0,(0,0)}(\boldsymbol{x}^1) = 0 \\ D_1^{(0,2)}Q_{0,(0,0)}(\boldsymbol{x}^1) = 0 \\ D_2^{(1,1)}Q_{0,(0,0)}(\boldsymbol{x}^2) = 0 \end{cases}, \quad \begin{cases} Q_{0,(1,0)}(\boldsymbol{x}^0) = 0 \\ D_0^{(1,0)}Q_{0,(1,0)}(\boldsymbol{x}^0) = 1 \\ D_0^{(0,2)}Q_{0,(1,0)}(\boldsymbol{x}^0) = 0 \\ D_1^{(0,1)}Q_{0,(1,0)}(\boldsymbol{x}^1) = 0 \\ D_1^{(0,2)}Q_{0,(1,0)}(\boldsymbol{x}^1) = 0 \\ D_2^{(1,1)}Q_{0,(1,0)}(\boldsymbol{x}^2) = 0 \end{cases}, \quad \begin{cases} Q_{0,(0,2)}(\boldsymbol{x}^0) = 0 \\ D_0^{(1,0)}Q_{0,(0,2)}(\boldsymbol{x}^0) = 0 \\ D_0^{(0,2)}Q_{0,(0,2)}(\boldsymbol{x}^0) = 1 \\ D_1^{(0,1)}Q_{0,(0,2)}(\boldsymbol{x}^1) = 0 \\ D_1^{(0,2)}Q_{0,(0,2)}(\boldsymbol{x}^1) = 0 \\ D_2^{(1,1)}Q_{0,(0,2)}(\boldsymbol{x}^2) = 0 \end{cases},$$

$$\begin{cases} Q_{1,(0,1)}(\boldsymbol{x}^0) = 0 \\ D_0^{(1,0)}Q_{1,(0,1)}(\boldsymbol{x}^0) = 0 \\ D_0^{(0,2)}Q_{1,(0,1)}(\boldsymbol{x}^0) = 0 \\ D_1^{(0,1)}Q_{1,(0,1)}(\boldsymbol{x}^1) = 1 \\ D_1^{(0,2)}Q_{1,(0,1)}(\boldsymbol{x}^1) = 0 \\ D_2^{(1,1)}Q_{1,(0,1)}(\boldsymbol{x}^2) = 0 \end{cases}, \quad \begin{cases} Q_{1,(0,2)}(\boldsymbol{x}^0) = 0 \\ D_0^{(1,0)}Q_{1,(0,2)}(\boldsymbol{x}^0) = 0 \\ D_0^{(0,2)}Q_{1,(0,2)}(\boldsymbol{x}^0) = 0 \\ D_1^{(0,1)}Q_{1,(0,2)}(\boldsymbol{x}^1) = 0 \\ D_1^{(0,2)}Q_{1,(0,2)}(\boldsymbol{x}^1) = 1 \\ D_2^{(1,1)}Q_{1,(0,2)}(\boldsymbol{x}^2) = 0 \end{cases}, \quad \begin{cases} Q_{2,(1,1)}(\boldsymbol{x}^0) = 0 \\ D_0^{(1,0)}Q_{2,(1,1)}(\boldsymbol{x}^0) = 0 \\ D_0^{(0,2)}Q_{2,(1,1)}(\boldsymbol{x}^0) = 0 \\ D_1^{(0,1)}Q_{2,(1,1)}(\boldsymbol{x}^1) = 0 \\ D_1^{(0,2)}Q_{2,(1,1)}(\boldsymbol{x}^1) = 0 \\ D_2^{(1,1)}Q_{2,(1,1)}(\boldsymbol{x}^2) = 1 \end{cases},$$

whose matrix of coefficients is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 & -2 & 2 \\ 0 & -1 & 1 & 2 & -1 & 0 \end{pmatrix}$$

with non zero determinant. The unique solution of the interpolation problem (21) is

$$\begin{aligned} P^{\Delta}[f] = {} & f(\boldsymbol{x}^0) + (\lambda_1+\lambda_2)D_0^{(1,0)}f(\boldsymbol{x}^0) + \frac{1}{2}(\lambda_1+\lambda_2)^2 D_0^{(0,2)}f(\boldsymbol{x}^0) + \lambda_2 D_1^{(0,1)}f(\boldsymbol{x}^1) \\ & + \left(\frac{\lambda_1^2}{2} + \lambda_2\right) D_1^{(0,2)}f(\boldsymbol{x}^1) - \left(\lambda_1^2 + \lambda_2 + \lambda_1\lambda_2\right) D_2^{(1,1)}f(\boldsymbol{x}^2). \end{aligned}$$

3.2. The remainder term in the Hermite–Birkhoff interpolation on the vertices of a triangle

As above, $\Omega \subset \mathbb{R}^2$ is a compact, convex domain, $\Delta \subset B(\boldsymbol{x}_i, R_{w_i}) \subset \Omega$ is a triangle with referring vertex $\boldsymbol{x}^0 = \boldsymbol{x}_i$ and $P^{\Delta}[f] \in \mathcal{P}^d_{\boldsymbol{x}}$ is the unique solution (16) of a Hermite–Birkhoff interpolation problem on the vertices of the triangle Δ with

data taken from a function $f \in C^r(\Omega)$, $r \geq d$. We restrict to the class $C^{r,1}(\Omega) \subset C^r(\Omega)$ of functions φ with partial derivatives $\frac{\partial^r \varphi}{\partial x^{r-j} \partial y^j}$ Lipschitz-continuous in Ω for each $j = 0, \ldots, r$ and we set

$$|\varphi|_{d,1} = \sup \left\{ \frac{\left| \frac{\partial^d \varphi}{\partial x^{d-j} \partial y^j}(\boldsymbol{u}_1) - \frac{\partial^d \varphi}{\partial x^{d-j} \partial y^j}(\boldsymbol{u}_2) \right|}{|\boldsymbol{u}_1 - \boldsymbol{u}_2|}, \boldsymbol{u}_1 \neq \boldsymbol{u}_2 \text{ in } \Omega,\ j = 0, \ldots, d \right\}. \tag{23}$$

In order to get a bound for the remainder

$$R^{\Delta}[f](\boldsymbol{x}) = f(\boldsymbol{x}) - P^{\Delta}[f](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{24}$$

we make use of the result in the following theorem.

Theorem 10. *Let be $f \in C^{r,1}(\Omega)$. Then, for each $\boldsymbol{x} \in \Omega$, we have*

$$P^{\Delta}[f](\boldsymbol{x}) = T_d\left[f, \boldsymbol{x}^0\right](\boldsymbol{x}) + \delta_d^{\Delta}[f](\boldsymbol{x}),$$

where $T_d\left[f, \boldsymbol{x}^0\right](\boldsymbol{x})$ is the Taylor polynomial of order d for f at $\boldsymbol{x}^0$

$$T_d\left[f, \boldsymbol{x}^0\right](\boldsymbol{x}) = \sum_{\substack{(\alpha_1,\alpha_2)\in\mathbb{N}_0^2 \\ \alpha_1+\alpha_2 \leq d}} \frac{1}{\alpha_1!\alpha_2!} \frac{\partial^{\alpha_1+\alpha_2} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\left(\boldsymbol{x}^0\right)\left(x - x^0\right)^{\alpha_1}\left(y - y^0\right)^{\alpha_2} \tag{25}$$

and

$$\begin{aligned}\delta_d^{\Delta}[f](\boldsymbol{x}) = &\sum_{\substack{(i,j) \\ e_{ij}^1 \neq 0}} Q_{1,(i,j)}(\boldsymbol{x}) \int_0^1 \frac{D_{10}^{d-(i+j)+1} D_1^{(i,j)} f\left(\boldsymbol{x}^0 + t\left(\boldsymbol{x}^1 - \boldsymbol{x}^0\right)\right)}{(d-i-j)!}(1-t)^{d-(i+j)}\, dt \\ &+ \sum_{\substack{(i,j) \\ e_{ij}^2 \neq 0}} Q_{2,(i,j)}(\boldsymbol{x}) \int_0^1 \frac{D_{20}^{d-(i+j)+1} D_2^{(i,j)} f\left(\boldsymbol{x}^0 + t\left(\boldsymbol{x}^2 - \boldsymbol{x}^0\right)\right)}{(d-i-j)!}(1-t)^{d-(i+j)}\, dt.\end{aligned} \tag{26}$$

Proof. By applying the Taylor theorem with integral remainder [19, Ch. 7] to the derivatives $D_1^{(i,j)} f\left(\boldsymbol{x}^1\right)$ and $D_2^{(i,j)} f\left(\boldsymbol{x}^2\right)$ in the expression (16) of $P^{\Delta}[f](\boldsymbol{x})$ we get

$$P^{\Delta}[f](\boldsymbol{x}) = \widetilde{T}_d\left[f, \boldsymbol{x}^0\right](\boldsymbol{x}) + \delta_d^{\Delta}[f](\boldsymbol{x}),$$

where $\delta_d^{\Delta}[f](\boldsymbol{x})$ is as in (26) and

$$\begin{aligned}\widetilde{T}_d\left[f, \boldsymbol{x}^0\right](\boldsymbol{x}) = &\sum_{\substack{(i,j) \\ e_{ij}^0 \neq 0}} Q_{0,(i,j)}(\boldsymbol{x}) D_0^{(i,j)} f\left(\boldsymbol{x}^0\right) + \sum_{\substack{(i,j) \\ e_{ij}^1 \neq 0}} Q_{1,(i,j)}(\boldsymbol{x}) T_{d-(i+j)}\left[D_1^{(i,j)} f, \boldsymbol{x}^0\right]\left(\boldsymbol{x}^1\right) \\ &+ \sum_{\substack{(i,j) \\ e_{ij}^2 \neq 0}} Q_{2,(i,j)}(\boldsymbol{x}) T_{d-(i+j)}\left[D_2^{(i,j)} f, \boldsymbol{x}^0\right]\left(\boldsymbol{x}^2\right).\end{aligned}$$

We note that after computation and rearrangement, the polynomial $\widetilde{T}_d\left[f, \boldsymbol{x}^0\right]$ can be expressed only in terms of partial derivatives of f up to the order d at the point $\boldsymbol{x}^0$, with polynomial coefficients $p_{(\alpha_1,\alpha_2)}(\boldsymbol{x})$ of degree at most d, that is

$$\widetilde{T}_d\left[f, \boldsymbol{x}^0\right](\boldsymbol{x}) = \sum_{\substack{(\alpha_1,\alpha_2)\in\mathbb{N}_0^2 \\ \alpha_1+\alpha_2 \leq d}} \frac{\partial^{\alpha_1+\alpha_2} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\left(\boldsymbol{x}^0\right) p_{(\alpha_1,\alpha_2)}(\boldsymbol{x}). \tag{27}$$

The Taylor polynomial $T_d\left[f, \boldsymbol{x}^0\right]$ interpolates at $\boldsymbol{x}^0$ all partial derivatives of f up to the order d, that is

$$\frac{\partial^{\alpha_1+\alpha_2} T_d\left[f, \boldsymbol{x}^0\right]}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\left(\boldsymbol{x}^0\right) = \frac{\partial^{\alpha_1+\alpha_2} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\left(\boldsymbol{x}^0\right)$$

for each $(\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, such that $\alpha_1 + \alpha_2 \leq d$; from (27) it follows that

$$\widetilde{T}_d\left[T_d\left[f, \boldsymbol{x}^0\right], \boldsymbol{x}^0\right] = \widetilde{T}_d\left[f, \boldsymbol{x}^0\right]. \tag{28}$$

On the other hand, the polynomial operator $\widetilde{T}_d\left[\cdot, \boldsymbol{x}^0\right]$ reproduces polynomials up to degree d, since $\delta_d^{\Delta}[p] \equiv 0$ and $P^{\Delta}[p] = p$ for each $p \in \mathcal{P}_{\boldsymbol{x}}^d$. The Taylor polynomial $T_d\left[f, \boldsymbol{x}^0\right] \in \mathcal{P}_{\boldsymbol{x}}^d$ and therefore

$$\widetilde{T}_d\left[T_d\left[f, \boldsymbol{x}^0\right], \boldsymbol{x}^0\right] = T_d\left[f, \boldsymbol{x}^0\right]. \tag{29}$$

By equaling the right hand side terms of (28) and (29) we get

$$\widetilde{T}_d\left[f, \boldsymbol{x}^0\right] = T_d\left[f, \boldsymbol{x}^0\right]. \quad \blacksquare$$

Corollary 11. *In the hypothesis of Theorem 10 we have the following representation for the remainder term of* $P^{\Delta}[f](\boldsymbol{x})$

$$R^{\Delta}[f](\boldsymbol{x}) = R_d^T\left[f, \boldsymbol{x}^0\right](\boldsymbol{x}) - \delta_d^{\Delta}[f](\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{30}$$

where $R_d^T[f, \boldsymbol{x}_i](\boldsymbol{x})$ *is the remainder in the truncated Taylor expansion* [20, *Ch. 1*].

We are now able to give a bound for the remainder term (24). To this end we need some additional notations and results. We set

$$e_{\max} = \max\left\{\left|\boldsymbol{x}^0 - \boldsymbol{x}^1\right|, \left|\boldsymbol{x}^1 - \boldsymbol{x}^2\right|, \left|\boldsymbol{x}^2 - \boldsymbol{x}^0\right|\right\} \tag{31}$$

and

$$S = \frac{1}{\left|A\left(\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2\right)\right|}. \tag{32}$$

The barycentric coordinates $\lambda_1(\boldsymbol{x})$ and $\lambda_2(\boldsymbol{x})$ are bounded by

$$|\lambda_i(\boldsymbol{x})| \le (e_{\max} S)\left|\boldsymbol{x} - \boldsymbol{x}^0\right|, \quad i = 1, 2 \tag{33}$$

and

$$|\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})| \le (e_{\max} S)\left|\boldsymbol{x} - \boldsymbol{x}^0\right|. \tag{34}$$

By (9) and (10) we get

$$\begin{aligned} D_{10}^{d-(i+j)+1} D_1^{(i,j)} &= (-1)^{d-(i+j)+1} D_1^{(d+1-j,j)}, \\ D_{20}^{d-(i+j)+1} D_2^{(i,j)} &= (-1)^{d-(i+j)+1} D_2^{(d+1-j,j)}. \end{aligned} \tag{35}$$

Moreover,

Lemma 12. *Let* $f \in C^{r,1}(\Omega)$. *The* $(d+1)$*th order derivatives of* f *along the directed sides of the triangle* Δ *satisfy*

$$\left|D_k^{(\beta_1,\beta_2)} f(\boldsymbol{x})\right| \le 2^{d+1} e_{\max}^{d+1} |f|_{d,1}, \quad k = 0, 1, 2, \qquad \beta_1 + \beta_2 = d+1, \quad \boldsymbol{x} \in \Omega. \tag{36}$$

Proof. For $k = 0$, by (10) and (9) we have

$$D_0^{(\beta_1,\beta_2)} f(\boldsymbol{x}) = \sum_{i=0}^{\beta_1} \binom{\beta_1}{i} (x^1 - x^0)^{\beta_1 - i} \left(y^1 - y^0\right)^i \times \sum_{j=0}^{\beta_2} \binom{\beta_2}{j} (x^2 - x^0)^{\beta_2 - j} \left(y^2 - y^0\right)^j \frac{\partial^{d+1} f(\boldsymbol{x})}{\partial x^{d+1-i-j} \partial y^{i+j}}.$$

Since $d+1 \ge 1$, at least one between $d+1-i-j$ and $i+j$ is greater than or equal to 1, for each $i = 0, \ldots, \beta_1, j = 0, \ldots, \beta_2$. If $d+1-i-j \ge 1$

$$\left|\frac{\partial^{d+1} f(\boldsymbol{x})}{\partial x^{d+1-i-j} \partial y^{i+j}}\right| = \left|\lim_{h \to 0} \frac{\frac{\partial^d f(x+h,y)}{\partial x^{d-i-j} \partial y^{i+j}} - \frac{\partial^d f(x,y)}{\partial x^{d-i-j} \partial y^{i+j}}}{h}\right| \le |f|_{d,1}$$

otherwise

$$\left|\frac{\partial^{d+1} f(\boldsymbol{x})}{\partial x^{d+1-i-j} \partial y^{i+j}}\right| = \left|\lim_{h \to 0} \frac{\frac{\partial^d f(x,y+h)}{\partial x^{d+1-i-j} \partial y^{i+j-1}} - \frac{\partial^d f(x,y)}{\partial x^{d+1-i-j} \partial y^{i+j-1}}}{h}\right| \le |f|_{d,1}$$

for each $i = 0, \ldots, \beta_1, j = 0, \ldots, \beta_2$. Therefore we have

$$\begin{aligned}\left|D_0^{(\beta_1,\beta_2)} f(\boldsymbol{x})\right| &= \left|\sum_{i=0}^{\beta_1}\binom{\beta_1}{i}(x^1-x^0)^{\beta_1-i}\left(y^1-y^0\right)^i\sum_{j=0}^{\beta_2}\binom{\beta_2}{j}(x^2-x^0)^{\beta_2-j}\left(y^2-y^0\right)^j\frac{\partial^{d+1}f(\boldsymbol{x})}{\partial x^{d+1-i-j}\partial y^{i+j}}\right| \\ &\le \sum_{i=0}^{\beta_1}\binom{\beta_1}{i}\left|x^1-x^0\right|^{\beta_1-i}\left|y^1-y^0\right|^i\sum_{j=0}^{\beta_2}\binom{\beta_2}{j}\left|x^2-x^0\right|^{\beta_2-j}\left|y^2-y^0\right|^j\left|\frac{\partial^{d+1}f(\boldsymbol{x})}{\partial x^{d+1-i-j}\partial y^{i+j}}\right| \\ &\le 2^{d+1}\left|\boldsymbol{x}^1-\boldsymbol{x}^0\right|^{\beta_1-i}\left|\boldsymbol{x}^2-\boldsymbol{x}^0\right|^{\beta_2}|f|_{d,1} \\ &\le 2^{d+1}e_{\max}^{d+1}|f|_{d,1}.\end{aligned}$$

The cases $j = 1, 2$ are analogous. ■

By using relations (35) and bounds (33), (34) and (36) we obtain the bound

$$\left|\delta_d^{\Delta}[f](\boldsymbol{x})\right| \le 2^{d+1}|f|_{d,1}\sum_{k=0}^{d}C_k e_{\max}^{d+1-k}\left(e_{\max}^2 S\right)^k\left|\boldsymbol{x}-\boldsymbol{x}^0\right|^k, \quad \boldsymbol{x}\in\Omega, \tag{37}$$

where C_k are non-negative constants which depend on the interpolation conditions in $\mathcal{D}$. By settings (31) and (32) the quantity $e_{\max}^2 S$ depends only on the shape of the triangle Δ. Therefore, by taking the modulus of both sides of (30) and by bounding the Taylor remainder in standard way we finally obtain

$$\left|R^{\Delta}[f](\boldsymbol{x})\right| \le 2^{d+1}|f|_{d,1}\left(\frac{\left|\boldsymbol{x}-\boldsymbol{x}^0\right|^{d+1}}{d!}+\sum_{k=0}^{d}C_k e_{\max}^{d+1-k}\left(e_{\max}^2 S\right)^k\left|\boldsymbol{x}-\boldsymbol{x}^0\right|^k\right). \tag{38}$$

3.3. The combined Shepard Hermite–Birkhoff operator

With the aim of obtaining Shepard Hermite–Birkhoff operators with as great as possible algebraic precision, depending on the data in $\mathcal{D} = \cup_{i=1}^{N}\mathcal{D}(i)$, we set d the maximum value obtainable for the algebraic degree of precision. For example, in the case of bivariate Lagrange interpolation we set $d = 1$; we set $d = 2$ if the gradient of the function at each node is given, in addition to the function evaluation. The procedure of association to point $\boldsymbol{x}_i$ of the triangle $\Delta(i)$, as specified in S1–S3, can be automated as follows:

1. we enumerate the N_i nodes in the closed ball $B\left(\boldsymbol{x}_i, R_{w_i}\right)$ according to the increasing distance from $\boldsymbol{x}_i$;
2. we list the $\frac{(N_i-1)(N_i-2)}{2}$ triangles with referring vertex $\boldsymbol{x}_i$ in the ball $B\left(\boldsymbol{x}_i, R_{w_i}\right)$ according to the increasing values of $e_{\max}(i)\left(e_{\max}^2(i)S(i)\right)$;
3. we set $\Delta(i)$ the first triangle of the list for which $P^{\Delta(i)}[f](\boldsymbol{x})$ exists and has algebraic degree of precision d; if such a polynomial does not exist, then we set $\Delta(i)$ the first triangle for which $P^{\Delta(i)}[f](\boldsymbol{x})$ exists and has algebraic degree of precision $d-1$ and so on, until we find a useful triangle. The radius of the ball R_{w_i} can be increased in order to ensure the success of this procedure. We denote by d_i the degree of exactness of $P^{\Delta(i)}[f](\boldsymbol{x})$.

Definition 13. The combined Shepard Hermite–Birkhoff operator is defined by

$$S_{HB}[f](\boldsymbol{x}) = \sum_{i=1}^{N}\widetilde{W}_{\mu,i}(\boldsymbol{x})P^{\Delta(i)}[f](\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega \tag{39}$$

with remainder term

$$\begin{aligned}R_{HB}[f](\boldsymbol{x}) &= f(\boldsymbol{x}) - S_{HB}[f](\boldsymbol{x}) \\ &= \sum_{i=1}^{N}\widetilde{W}_{\mu,i}(\boldsymbol{x})\left(f(\boldsymbol{x}) - P^{\Delta(i)}[f](\boldsymbol{x})\right), \quad \boldsymbol{x}\in\Omega.\end{aligned} \tag{40}$$

As remarked in Section 2 the algebraic degree of exactness of the operator $S_{HB}[f]$ is $d_S = \min\{d_1, \ldots, d_N\}$ and therefore its approximation order is $d_S + 1$. In this respect we explicitly bound the remainder term (40) by setting

1. $\mathcal{I}_{\boldsymbol{x}} = \left\{i\in\{1,\ldots,N\} : |\boldsymbol{x}-\boldsymbol{x}_i| \le R_{w_i}\right\}, \ \boldsymbol{x}\in\Omega$;
2. $M = \sup_{\boldsymbol{x}\in\Omega}\{\sharp(\mathcal{I}_{\boldsymbol{x}})\}$, where $\sharp$ is the cardinality operator.
3. $R = \max_{i=1,\ldots,N} R_{w_i}$.

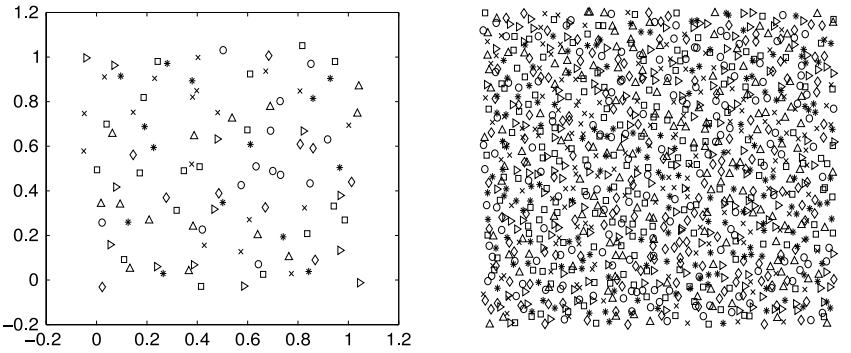


Fig. 2. The Franke dataset (left) and the set of 1089 regularly distributed interpolation nodes in the unit square $[0, 1] \times [0, 1]$ (right) with specified interpolation data. The association symbol-data is as reported in Table 1.

Table 1
The association symbol-data: the dot "•" indicates the presence of the data.

	$f(\boldsymbol{x}_i)$	$\frac{\partial f}{\partial x}(\boldsymbol{x}_i)$	$\frac{\partial f}{\partial y}(\boldsymbol{x}_i)$	$\frac{\partial^2 f}{\partial x^2}(\boldsymbol{x}_i)$	$\frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{x}_i)$	$\frac{\partial^2 f}{\partial y^2}(\boldsymbol{x}_i)$
▷	•					
◇	•	•	•			
○	•			•	•	•
×		•	•	•	•	•
□		•	•			
△				•	•	•
∗	•	•	•	•	•	•

Calculating from this we find

$$
\begin{aligned}
|R_{HB}[f](\boldsymbol{x})| &\leq \sum_{i \in \mathcal{I}_{\boldsymbol{x}}} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \left| f(\boldsymbol{x}) - P^{\Delta(i)}[f](\boldsymbol{x}) \right| \\
&\leq \sum_{i \in \mathcal{I}_{\boldsymbol{x}}} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \, 2^{d_i+1} \, |f|_{d_i,1} \left(\frac{|\boldsymbol{x} - \boldsymbol{x}_i|^{d_i+1}}{d_i!} + \sum_{k=0}^{d_i} C_k(i) \, e_{\max}^{d_i+1-k}(i) \left(e_{\max}^2(i) \, S(i) \right)^k |\boldsymbol{x} - \boldsymbol{x}_i|^k \right) \\
&\leq \sum_{i \in \mathcal{I}_{\boldsymbol{x}}} \widetilde{W}_{\mu,i}(\boldsymbol{x}) \, 2^{d_i+1} \, |f|_{d_i,1} \, R_{w_i}^{d_i+1} \left(\frac{1}{d_i!} + \sum_{k=0}^{d_i} C_k(i) \, 2^{d_i+1-k} \left(e_{\max}^2(i) \, S(i) \right)^k \right). \\
&\leq M \left(\max_{i=1,\ldots,N} |f|_{d_i,1} \right) \left(\max_{i=1,\ldots,N} K(i) \right) R^{d_S+1}
\end{aligned}
$$

for each $\boldsymbol{x} \in \Omega$. Constants

$$
K(i) = 2^{d_i+1} \left(\frac{1}{d_i!} + \sum_{k=0}^{d_i} C_k(i) \, 2^{d_i+1-k} \left(e_{\max}^2(i) \, S(i) \right)^k \right), \quad i = 1, \ldots, N
$$

depend on the polynomial $P^{\Delta(i)}[f]$ only.

4. Numerical results

To test the accuracy of approximation of the Shepard Hermite–Birkhoff operators in the bivariate interpolation of large sets of scattered data, we compare the reproduction quality of the operator $S_{HB}[f](\boldsymbol{x})$ (39) with those of the operator

$$
F_B(\boldsymbol{x}; f, \mathcal{X}) = \sum_{i=1}^{N} \sum_{(\alpha_1,\alpha_2) \in I_i} \frac{\partial^{\alpha_1+\alpha_2} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(\boldsymbol{x}_i) \, \frac{(x - x_i)^{\alpha_1} (y - y_i)^{\alpha_2}}{\alpha_1! \alpha_2!} \, \widetilde{W}_{\mu,i}(\boldsymbol{x})
$$

introduced in [9] (see Figs. 3 and 4). The numerical results are obtained by using the Franke dataset and a set of 1089 regularly distributed interpolation nodes in the unit square $[0, 1] \times [0, 1]$. The interpolation conditions, displayed in Fig. 2, are randomly chosen and the association symbol-data is as reported in Table 1.

In the case of the Franke dataset the operator $S_{HB}[f]$ has degree of exactness $d = 3$, while the operator $F_B(\boldsymbol{x}; f, \mathcal{X})$ does not have the polynomial reproduction property P6 since at some points the functional evaluations are not given; in particular, the interpolation data vary in number, as in Table 2.

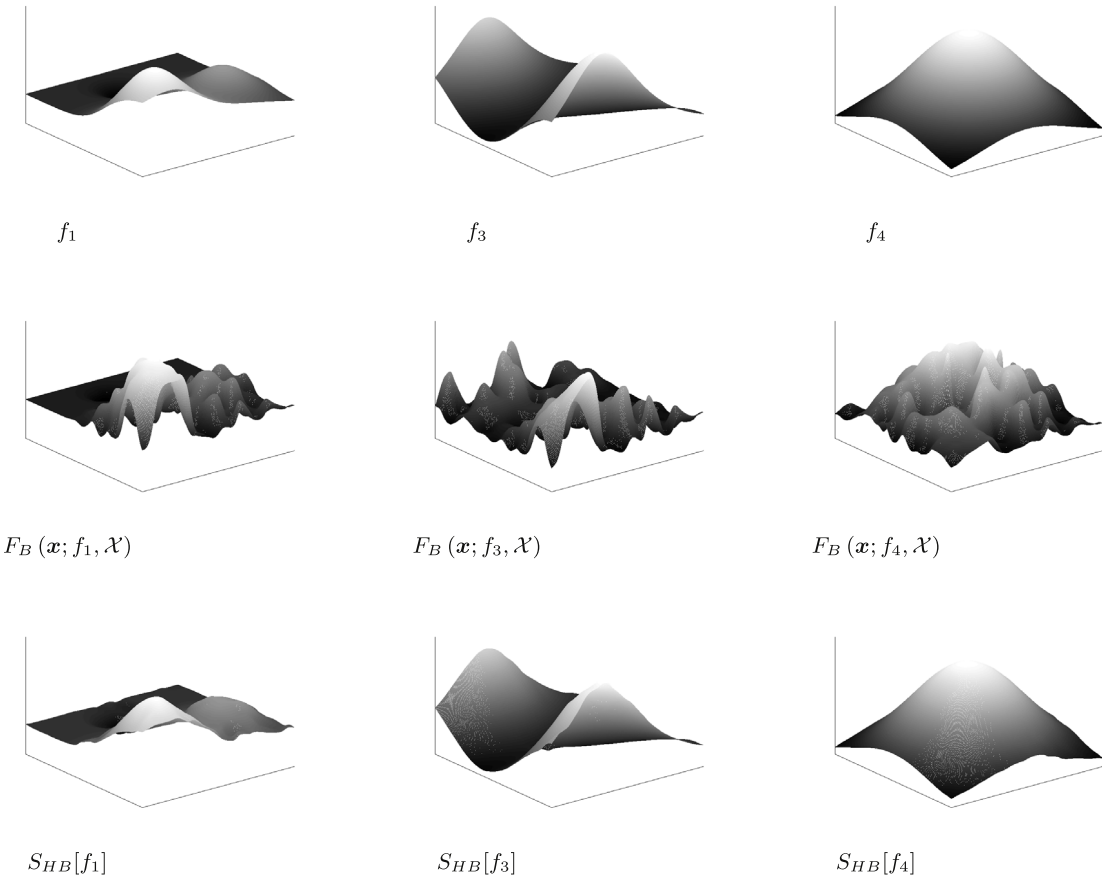


Fig. 3. Comparison between the reproduction quality of the operators $F_B(\boldsymbol{x}; f, \mathcal{X})$ and $S_{HB}[f]$ by using the Franke dataset with complete interpolation data as in Fig. 2.

Table 2
Variation, in number, of randomly chosen complete data for the Franke dataset.

Interpolation data	▷	◇	○	×	□	△	∗
Number of nodes	13	10	13	18	18	14	14

Table 3
Variation, in number, of randomly chosen complete data for the 1089 nodes.

Interpolation data	▷	◇	○	×	□	△	∗
Number of nodes	151	152	154	154	167	154	157

In the case of 1089 nodes in the unit square $[0, 1]\times[0, 1]$ the operator $S_{HB}[f]$ has degree of exactness $d = 2$ (7 polynomials $P^{\Delta(i)}[f]$ are quadratic), while the operator $F_B(\boldsymbol{x}; f, \mathcal{X})$ does not have the polynomial reproduction property; in particular, the interpolation data vary in number, as in Table 3.

Remark 14. The algorithm that we employ to obtain numerical tests works as follows:

1. it fixes the referring vertex $\boldsymbol{x}_0$ and the first N_w nearest nodes. The computational cost of such procedure is $N \log N$, since it is a sorting algorithm on all nodes;
2. it detects the points to construct the local interpolants, as described in steps S1–S3, at the following cost: with $O\left(\frac{N_w^2}{2}\log\left(\frac{N_w^2}{2}\right)\right)$ operations it lists the $\frac{(N_w-1)(N_w-2)}{2}$ triangles with referring vertex $\boldsymbol{x}_0$ in the ball $B\left(\boldsymbol{x}_0, R_{w_0}\right)$ according to the increasing values of $e_{\max}(i)\left(e^2_{\max}(i)\, S(i)\right)$. The enlargement of the radius of influence to a ball containing $N_w + 1$ nodes implies to fix $2\frac{(N_w-1)(N_w-2)}{2}$ new triangles, namely 2 triangles for each previously considered triangle, and therefore this procedure requires an extra cost of $O\left(2\frac{N_w^2}{2}\log\left(2\frac{N_w^2}{2}\right)\right)$. By induction, one can see that the enlargement

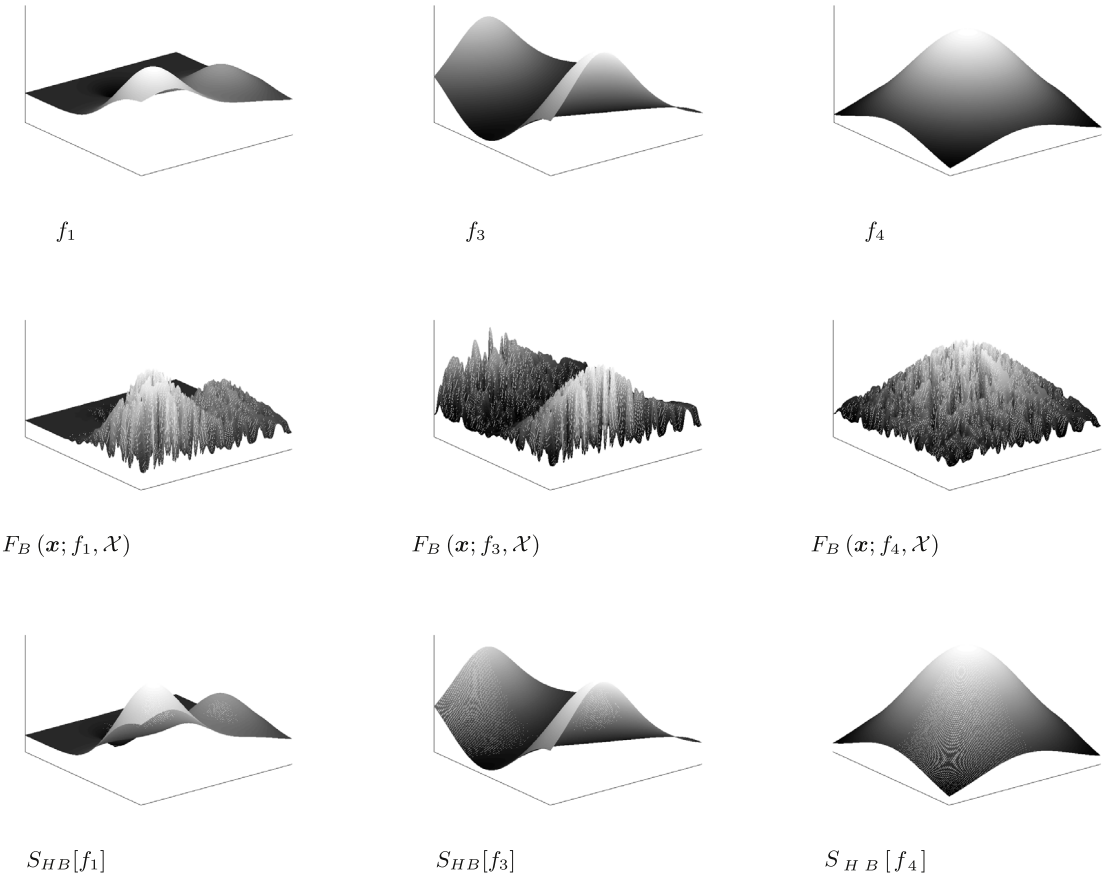


Fig. 4. Comparison between the reproduction quality of the operators $F_B(\boldsymbol{x}; f, \mathcal{X})$ and $S_{HB}[f]$ by using the 1089 nodes with complete interpolation data as in Fig. 2.

Table 4
The data are uniformly distributed. The radius of influence does not increase. The interpolant $S_{HB}[f]$ has degree of exactness 3.

Interpolation data	▷	◇	○	×	□	△	∗
Number of nodes	10	15	15	15	15	15	15

Table 5
$\mathcal{I}(f)$ contains few functional evaluations. The radius of influence increases once from 0.2367 to 0.2595. The interpolant $S_{HB}[f]$ has degree of exactness 3.

Interpolation data	▷	◇	○	×	□	△	∗
Number of nodes	40	5	5	15	15	15	5

of the radius of influence to a ball containing $N_w + k$ nodes implies to fix $\sum_{j=0}^{k} 2^j \frac{(N_w-1)(N_w-2)}{2}$ triangles and a cost of $\sum_{j=0}^{k} O\left(2^j \frac{N_w^2}{2} \log\left(2^j \frac{N_w^2}{2}\right)\right)$.

In order to show how the radius of influence could increase in practice, the method has been tested in a number of cases, by using the Franke dataset and varying the data in $\mathcal{I}(f)$. Numerical results are shown in Tables 4–6. The association symbol-data is as reported in Table 1. It results that a low number of available functional evaluations can cause an increase of the radius of influence.

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Table 6
$\mathcal{I}(f)$ contains few functional evaluations and no data of type $*$. The radius of influence increases 4 times: (a) from 0.2656 to 0.2804; (b) from 0.2512 to 0.2656; (c) from 0.2480 to 0.3594; (d) from 0.2213 to 0.8932. The interpolant $S_{HB}[f]$ has degree of exactness 2.

Interpolation data	$\triangleright$	$\diamond$	$\circ$	$\times$	$\square$	$\triangle$	$*$
Number of nodes	25	5	5	15	35	15	0

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.cam.2015.12.016.

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