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A unified numerical scheme for the multi-term time fractional diffusion and diffusion–wave equations with variable coefficients

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Abstract

We consider the numerical solutions of the multi-term time fractional diffusion and diffusion–wave equations with variable coefficients in a bounded domain. The time fractional derivatives are described in the Caputo sense. A unified numerical scheme based on finite difference method in time and Legendre spectral method in space is proposed. Detailed error analysis is given for the fully discrete scheme. The convergence rate of the proposed scheme in L^2 norm is $\mathrm{O}(\tau^2 + N^{1-m})$, where τ, N, and m are the time-step size, polynomial degree, and regularity in the space variable of the exact solution, respectively. Numerical examples are presented to illustrate the theoretical results.

Keywords: Fractional diffusion equation, Fractional diffusion–wave equation, Spectral method, Stability, Convergence
2010 MSC: Primary 65M12, 65M06, 65M70, 35R11.

1. Introduction

Fractional calculus involves investigating the properties and applications of the derivatives and integrals with non-integer orders. One can refer to [1] for an extensive list of recent applications and mathematical developments of the fractional calculus. Fractional differential equations are the equations

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involving the fractional derivatives of the unknown functions. There are many kinds of definitions for the fractional derivatives, such as Riemann–Liouville derivative, Caputo derivative, Grünwald–Letinikov derivative, etc.

The time fractional diffusion and diffusion–wave equations are the usual diffusion and wave equations with their first-order time derivative and second-order time derivative replaced by fractional derivatives of order $0 < \alpha < 1$, $1 < \alpha < 2$, respectively [2]. Both analytical and numerical investigations of them have been studied by many authors. For the solution theory of the time fractional diffusion and diffusion–wave equations, one can refer to [3–7]. For the numerical approximation of the time fractional diffusion and diffusion–wave equations, see [8–14], etc.

In this paper we consider the following multi-term time fractional diffusion and diffusion–wave equations with variable coefficients:

$$ {}_0^C D_t^{\gamma} u(x,t) + \sum_{i=1}^{s} b_i \, {}_0^C D_t^{\gamma_i} u(x,t) = \mathcal{L} u(x,t) + g(x,t), \quad -1 < x < 1,\ 0 < t \le T, \tag{1.1} $$

where

$$ \mathcal{L} u = \frac{\partial}{\partial x}\left(p(x) \frac{\partial u}{\partial x} \right) - q(x) u, $$

$$ p \in C^1[-1,1],\ q \in C[-1,1],\ p(x) > 0,\ q(x) \ge 0,\ x \in [-1,1], $$

$$ 0 < \gamma_s < \cdots < \gamma_1 < \gamma < 2, \quad b_i \ge 0,\ i = 1, \ldots, s,\ s \in \mathbb{N}_0, $$

and ${}_0^C D_t^{\gamma} u(x,t)$ is the Caputo fractional derivative of order γ with respect to t, its exact definition will be given in next section.

We endow the equations (1.1) with the following boundary conditions:

$$ u(-1,t) = 0, \qquad u(1,t) = 0, \qquad 0 \le t \le T, \tag{1.2} $$

and initial conditions

$$ u(x,0) = u_0(x), \qquad x \in (-1,1), \tag{1.3} $$

$$ u_t(x,0) = \psi(x), \qquad x \in (-1,1) \ \text{ for } 1 < \gamma < 2. \tag{1.4} $$

In this paper, in the case $0 < \gamma_s < \cdots < \gamma_1 < \gamma < 1$, (1.1) is called the multi-term time fractional diffusion equation. When $1 < \gamma_s < \cdots < \gamma_1 < \gamma < 2$, (1.1) is called the multi-term time fractional diffusion–wave equation. When

$0 < \gamma_s < \cdots < \gamma_i < 1 < \gamma_{i-1} < \cdots < \gamma_1 < \gamma < 2$, (1.1) is called the multi-term time fractional mixed diffusion and diffusion–wave equation. Luchko [15] considered the initial-boundary value problems for the generalized multi-term time-fractional diffusion equation, and showed some existence and uniqueness results. Jiang et al. [16] derived the analytical solutions for the multi-term time-space Caputo–Riesz fractional advection–diffusion equations on a finite domain. Li et al. [17] presented the well-posedness and the long-time asymptotic behavior of the initial-boundary value problems for the multi-term time-fractional diffusion equations. Ding and Nieto [18] used Laplace transform and Fourier transform methods to obtain the analytical solutions of the multi-term time-space fractional reaction–diffusion equations on the whole line, and presented the results in a compact and elegant form in terms of Mittag–Leffler functions.

Liu et al. [19] proposed some computationally effective numerical methods for simulating the multi-term time fractional wave–diffusion equations. Jin et al. [20] used a Galerkin finite element method to approximate the multi-term time fractional diffusion equation on a bounded convex polyhedral domain, and analysed the stability and error estimate for the semi-discrete and fully discrete schemes. Ren and Sun [21] presented a compact difference method for the multi-term time fractional diffusion–wave equation on one-dimensional and two-dimensional bounded domains.

However, the temporal accuracy of the previous methods is depending on the order of the fractional derivatives, and is usually less than two. There are also some papers in which second order discretization was proposed for the time discretization of the fractional derivative operators, see [22–25]. But the high order approximations for single fractional operator either cannot be directly applied to multi-term fractional operators, or the error analysis of them is hard to analyse. Most importantly, they are not workable for solving both the time fractional diffusion and diffusion–wave equations. Huang and Yang [26] proposed a unified difference–spectral method for the single term time-space fractional diffusion equations, but its extension to the multi-term cases is not clear. Recently, Tian et al. [27] proposed a class of second order approximations, called weighted and shifted Grünwald difference (WSGD) operators, for the Riemann–Liouville fractional derivatives. In this paper, we propose a unified numerical scheme which has second order accuracy in time and spectral accuracy in space for the problem (1.1)–(1.4) . The proposed scheme is based on finite difference method in the temporal direction and Legendre spectral method in the spatial direction. More precisely, for the multi-term time fractional diffusion and diffusion–wave equations, we first transform them into equivalent forms with the Riemman–Liouville fractional derivative operator and Riemman-Liouville fractional integral operator, respectively. Then we use weighted and shifted Grünwald difference (WSGD) operators to approximate the fractional operators, and based on a Crank–Nicolson technique, the convergence rate of the fully discrete scheme in L^2 norm is $\mathrm{O}(\tau^2 + N^{1-m})$, where τ, N, and m are the time-step size, polynomial degree, and regularity in the space variable of the exact solution, respectively. The stability and convergence of the fully discrete scheme are rigorously established.

The rest of the paper is organized as follows. In Section 2, some preliminaries and notations are shown. In Section 3, we construct a unified numerical scheme for the multi-term time fractional diffusion and diffusion–wave equations. In Section 4, the stability and convergence of the fully discrete scheme are analysed. We do some numerical experiments in Section 5. Finally, the summary and discussion are presented in Section 6.

2. Preliminaries and Notations

Let $\Lambda = (-1, 1)$. Throughout this paper, we use the usual Sobolev spaces $W^{r,p}(\Lambda)$ with norm $\|\cdot\|_{r,p}$. When $p = 2$, we denote $W^{r,2}(\Lambda)$ and its inner product, semi-norm, and norm by $H^r(\Lambda)$, $(\cdot,\cdot)_r$, $|\cdot|_r$, and $\|\cdot\|_r$, respectively. In particular, $(\cdot,\cdot) = (\cdot,\cdot)_0$, $\|\cdot\| = \|\cdot\|_0$. Furthermore, we denote

$$H_0^1(\Lambda) = \left\{ v \in H^1(\Lambda), v(\pm 1) = 0 \right\}.$$

Denote $L^2_w(\Lambda)$ as a weighted L^2 space with a weight function $w(x)$, and its inner product and norm are defined as:

$$(u, v)_w = \int_\Lambda uvw\mathrm{d}x, \quad \|v\|_w = \left(\int_\Lambda v^2 w\mathrm{d}x \right)^{\frac{1}{2}}.$$

We denote by $L^\infty(0, T; H^m(\Lambda))$ the space of the measurable functions $v : (0, T) \to H^m(\Lambda)$, such that

$$\|v\|_{L^\infty(H^m)} = ess \sup_{0 \le t \le T} \|v(t)\|_m < +\infty,$$

$L^2(0, T; H^m)$ the space of the measurable functions $v : (0, T) \to H^m(\Lambda)$, such that

$$\|v\|_{L^2(H^m)} = \left(\int_0^T \|v\|_m^2 \mathrm{d}t \right)^{\frac{1}{2}} < +\infty.$$

For simplicity, we denote $\partial_x^k v(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k} v(x)$. Throughout the paper, c denotes a generic positive constant.

Let N be a positive integer, we denote by $\mathbb{P}_N(\Lambda)$ the space of all polynomials of degree less than or equal to N. $\mathbb{P}_N^0 := \{\phi \in \mathbb{P}_N(\Lambda) : \phi(\pm 1) = 0\}$. Next we introduce some projection approximation results.

Let $\pi_N^{1,0}$ be the H_0^1-orthogonal projection operator from $H_0^1(\Lambda)$ into $\mathbb{P}_N^0$, such that for all $u \in H_0^1(\Lambda)$,

$$\left(\partial_x \pi_N^{1,0} u, \partial_x v_N\right) = \left(\partial_x u, \partial_x v_N\right), \quad \forall v_N \in \mathbb{P}_N^0. \tag{2.1}$$

For the projection operator $\pi_N^{1,0}$, one has the following approximation result:

Lemma 2.1 ([28]). *For all $u \in H_0^1(\Lambda) \cap H^m(\Lambda)$, we have*

$$\|u - \pi_N^{1,0} u\|_k \le C N^{k-m} \|u\|_m, \quad k = 0, 1, \, m \ge 1,$$

where C is a positive constant independent of N.

In this paper, we need a modified projection operator $\Pi_N^{1,0} : H_0^1(\Lambda) \to \mathbb{P}_N^0$, defined as following:

$$\left(p(x)\partial_x(u - \Pi_N^{1,0} u), \partial_x v_N\right) + \left(q(x)(u - \Pi_N^{1,0} u), v_N\right) = 0, \quad \forall v_N \in \mathbb{P}_N^0. \tag{2.2}$$

Then one has the following lemma:

Lemma 2.2. *For all $u \in H_0^1(\Lambda) \cap H^m(\Lambda)$, we have*

$$\|\partial_x(u - \Pi_N^{1,0} u)\|_{p(x)}^2 + \|u - \Pi_N^{1,0} u\|_{q(x)}^2 \le c N^{2-2m} \|u\|_m^2, \quad m \ge 1,$$

where c is a positive constant independent of N.

Proof. According to the definition of the operator $\Pi_N^{1,0}$, we have

$$\begin{aligned}
&\|\partial_x(u - \Pi_N^{1,0} u)\|_{p(x)}^2 + \|u - \Pi_N^{1,0} u\|_{q(x)}^2 \\
=& \left(p(x)\partial_x(u - \Pi_N^{1,0} u), \partial_x(u - \Pi_N^{1,0} u)\right) + \left(q(x)(u - \Pi_N^{1,0} u), u - \Pi_N^{1,0} u\right) \\
=& \left(p(x)\partial_x(u - \Pi_N^{1,0} u), \partial_x(u - \pi_N^{1,0} u)\right) + \left(q(x)(u - \Pi_N^{1,0} u), u - \pi_N^{1,0} u\right) \\
\le& \|\partial_x(u - \Pi_N^{1,0} u)\|_{p(x)} \|\partial_x(u - \pi_N^{1,0} u)\|_{p(x)} + \|u - \Pi_N^{1,0} u\|_{q(x)} \|u - \pi_N^{1,0} u\|_{q(x)} \\
\le& \left(\|\partial_x(u - \Pi_N^{1,0} u)\|_{p(x)}^2 + \|u - \Pi_N^{1,0} u\|_{q(x)}^2\right)^{\frac{1}{2}} \left(\|\partial_x(u - \pi_N^{1,0} u)\|_{p(x)}^2 + \|u - \pi_N^{1,0} u\|_{q(x)}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus we have

$$\|\partial_x(u - \Pi_N^{1,0} u)\|_{p(x)}^2 + \|u - \Pi_N^{1,0} u\|_{q(x)}^2 \le \|\partial_x(u - \pi_N^{1,0} u)\|_{p(x)}^2 + \|u - \pi_N^{1,0} u\|_{q(x)}^2.$$

Then according to the boundedness of $p(x)$, $q(x)$, and Lemma 2.1, the desired result is obtained. $\square$

The following Poincaré inequality is useful.

Lemma 2.3. *For $u(x) \in C^1[-1,1]$, with $u(-1) = u(1) = 0$, we have*

$$\|u\| \leq \frac{1}{\sqrt{2}}\|\partial_x u\|.$$

Proof. The inequality can be obtained by a simple computation. □

We first give a discrete Grönwall's inequality.

Lemma 2.4 ([29]). *Let k, B, and a_μ, b_μ, c_μ, γ_μ, for integers $\mu \geq 0$, be nonnegative numbers such that*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq k\sum_{\mu=0}^{n} \gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B, \quad n \geq 0.$$

Suppose that $k\gamma_\mu < 1$, for all μ, and set $\sigma_\mu \equiv (1 - k\gamma_\mu)^{-1}$. Then

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq \exp\left(k\sum_{\mu=0}^{n} \sigma_\mu\gamma_\mu\right)\left\{k\sum_{\mu=0}^{n} c_\mu + B\right\}, \quad n \geq 0.$$

Next, we give some definitions from fractional calculus. For simplicity, denote $\partial_t^k v(t) = \frac{\mathrm{d}^k v(t)}{\mathrm{d}t^k}$. Following [30], for a given function $f(t)$, $\alpha > 0$, we denote by ${}_0I_t^\alpha f(t)$ the left-sided Riemann–Liouville fractional integral of order α, defined as

$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > 0. \tag{2.3}$$

For $n - 1 < \alpha < n$, we denote by ${}^{RL}_{\ 0}D_t^\alpha f(t)$ the left-sided Riemann–Liouville fractional derivative of order α, defined as ${}^{RL}_{\ 0}D_t^\alpha f(t) = \partial_t^n\, {}_0I_t^{n-\alpha} f(t)$, that is

$${}^{RL}_{\ 0}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\partial_t^n \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s, \quad t > 0. \tag{2.4}$$

For $n - 1 < \alpha < n$, we denote by ${}^{C}_{0}D_t^\alpha f(t)$ the left-sided Caputo fractional derivative of order α, defined as ${}^{C}_{0}D_t^\alpha = {}_0I_t^{n-\alpha}\partial_t^n f(t)$, that is

$${}^{C}_{0}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\partial_s^n f(s)\mathrm{d}s, \quad t > 0. \tag{2.5}$$

Then according to Theorem 3.8 in [31], we have the following formula

$$ {}_0I_t^{\alpha}\, {}_0^CD_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k. \tag{2.6} $$

We recall some useful properties about the fractional derivatives and integrals. For $\alpha > 0$, $\beta > 0$, we have

$$ {}_0I_t^{\alpha}\, {}_0I_t^{\beta} f(t) = {}_0I_t^{\alpha+\beta} f(t) = {}_0I_t^{\beta}\, {}_0I_t^{\alpha} f(t), \tag{2.7} $$

and

$$ {}_0^{RL}D_t^{\alpha}\, {}_0I_t^{\alpha} f(t) = f(t). \tag{2.8} $$

For $0 < \alpha < 1$, $0 < \alpha_1, \alpha_2 < 1$, if $f(0) = 0$, then we have

$$ {}_0^CD_t^{\alpha} f(t) = {}_0^{RL}D_t^{\alpha} f(t), \tag{2.9} $$

and

$$ {}_0^{RL}D_t^{\alpha_1}\, {}_0^{RL}D_t^{\alpha_2} f(t) = {}_0^{RL}D_t^{\alpha_1+\alpha_2} f(t) = {}_0^{RL}D_t^{\alpha_2}\, {}_0^{RL}D_t^{\alpha_1} f(t). \tag{2.10} $$

3. Construction of the unified numerical scheme

In the following parts of this paper, we assume $u(x, 0) \equiv 0$, otherwise, we consider $\widetilde{u} = u - u_0$.

For a positive integer M, let $t_k = k\tau$, $k = 0, 1, \ldots, M$, where $\tau = T/M$ is the time-step size. Given a grid function $w = \{w^k | 0 \le k \le M\}$, we define

$$ w^{k+\frac{1}{2}} = \frac{1}{2}\big(w^{k+1} + w^k\big), \quad \delta_t w^{k+\frac{1}{2}} = \frac{1}{\tau}\big(w^{k+1} - w^k\big). $$

First we have the following lemma:

Lemma 3.1. *We denote ${}_0I_t^{\beta} f(t)$ as ${}_0^{RL}D_t^{-\beta} f(t)$ for $\beta > 0$. Then the equation (1.1) is equivalent to the following form*

$$ \partial_t u + \sum_{i=1}^{s} b_i\, {}_0^{RL}D_t^{\alpha_i} u(x, t) = {}_0^{RL}D_t^{\alpha} \mathcal{L} u(x, t) + f(x, t), \tag{3.1} $$

where $\alpha = 1 - \gamma$, $\alpha_i = 1 + \gamma_i - \gamma$; $f(x, t) = {}_0^{RL}D_t^{1-\gamma} g(x, t)$ for the case $0 < \gamma_s < \cdots < \gamma_1 < \gamma < 1$ and $f(x, t) = {}_0I_t^{\gamma-1} g(x, t) + \psi + \sum_{i=1}^{s'} b_i\, {}_0I_t^{\gamma-\gamma_i} \psi$ for the case $0 < \gamma_s < \cdots < \gamma_{s'+1} \le 1 < \gamma_{s'} < \cdots < \gamma_1 < \gamma < 2$.

Proof. (1) In the case $0 < \gamma_s < \cdots < \gamma_1 < \gamma < 1$, as $u(x,0) = 0$, according to (2.9), the equation (1.1) is equivalent to

$$ {}_0^C D_t^{\gamma} u(x,t) + \sum_{i=1}^{s} b_i \, {}_0^{RL} D_t^{\gamma_i} u(x,t) = \mathcal{L}u(x,t) + g(x,t). \tag{3.2} $$

As ${}_0^C D_t^{\gamma} u(x,t) = {}_0 I_t^{1-\gamma} \partial_t u(x,t)$, then according to (2.8) and (2.10), we transform (3.2) into

$$ \partial_t u + \sum_{i=1}^{s} b_i \, {}_0^{RL} D_t^{1+\gamma_i-\gamma} u(x,t) = {}_0^{RL} D_t^{1-\gamma} \mathcal{L}u(x,t) + f(x,t), $$

where $f(x,t) = {}_0^{RL} D_t^{1-\gamma} g(x,t)$.

(2) We next consider the case $1 < \gamma_s < \cdots < \gamma_1 < \gamma < 2$.

Since

$$ \begin{aligned} {}_0^C D_t^{\gamma} u(x,t) &= {}_0 I_t^{2-\gamma} \partial_t^2 u(x,t) \\ &= {}_0 I_t^{1-(\gamma-1)} \partial_t (\partial_t u(x,t)) \\ &= {}_0^C D_t^{\gamma-1} \partial_t u(x,t), \end{aligned} $$

then according to (2.6), (2.7), and noticing that $u_t(x,0) = \psi$, we have

$$ {}_0 I_t^{\gamma-1} \, {}_0^C D_t^{\gamma} u(x,t) = {}_0 I_t^{\gamma-1} \, {}_0^C D_t^{\gamma-1} \partial_t u(x,t) = \partial_t u(x,t) - \psi, $$

and

$$ \begin{aligned} {}_0 I_t^{\gamma-1} \, {}_0^C D_t^{\gamma_i} u(x,t) &= {}_0 I_t^{\gamma-1} \, {}_0^C D_t^{\gamma_i-1} \partial_t u(x,t) \\ &= {}_0 I_t^{\gamma-\gamma_i} \, {}_0 I_t^{\gamma_i-1} \, {}_0^C D_t^{\gamma_i-1} \partial_t u(x,t) \\ &= {}_0 I_t^{\gamma-\gamma_i} (\partial_t u(x,t) - \psi) \\ &= {}_0^C D_t^{1+\gamma_i-\gamma} u(x,t) + {}_0 I_t^{\gamma-\gamma_i} \psi. \end{aligned} $$

Thus we transform the initial equation (1.1) into its equivalent form

$$ \partial_t u(x,t) + \sum_{i=1}^{s} b_i \, {}_0^C D_t^{1+\gamma_i-\gamma} u(x,t) = {}_0 I_t^{\gamma-1} \mathcal{L}u + f(x,t), \tag{3.3} $$

where $f(x,t) = {}_0 I_t^{\gamma-1} g(x,t) + \psi + \sum_{i=1}^{s} b_i \, {}_0 I_t^{\gamma-\gamma_i} \psi$.

Noticing that $0 < 1 + \gamma_i - \gamma = 1 - (\gamma - \gamma_i) < 1$, and $\gamma - 1 > 0$, according to (2.9), the equation (3.3) is equivalent to

$$\partial_t u(x,t) + \sum_{i=1}^{s} b_i \, {}^{RL}_{0}D_t^{1+\gamma_i-\gamma} u(x,t) = {}^{RL}_{0}D_t^{1-\gamma} \mathcal{L} u + f(x,t).$$

(3) Now we turn to the case $0 < \gamma_s < \cdots < \gamma_{s'+1} \le 1 < \gamma_{s'} < \cdots < \gamma_1 < \gamma < 2$. At present, we need just tackle the case $0 < \gamma_i < 1$.

In the case $1 < \gamma - \gamma_i < 2$, according to (2.6) and (2.7), we have

$$ {}_0I_t^{\gamma-1} \, {}^{C}_{0}D_t^{\gamma_i} u = {}_0I_t^{\gamma-\gamma_i-1} \, {}_0I_t^{\gamma_i} \, {}^{C}_{0}D_t^{\gamma_i} u = {}_0I_t^{\gamma-\gamma_i-1} u = {}^{RL}_{0}D_t^{1+\gamma_i-\gamma} u. \tag{3.4}$$

In the case $0 < \gamma - \gamma_i < 1$, according to (2.7) and (2.9), we have

$$ {}_0I_t^{\gamma-1} \, {}^{C}_{0}D_t^{\gamma_i} u = {}_0I_t^{\gamma-1} \, {}_0I_t^{1-\gamma_i} \partial_t u = {}_0I_t^{\gamma-\gamma_i} \partial_t u = {}^{C}_{0}D_t^{1+\gamma_i-\gamma} u = {}^{RL}_{0}D_t^{1+\gamma_i-\gamma} u. \tag{3.5}$$

In the case $\gamma - \gamma_i = 1$, ${}_0I_t^{\gamma-1} \, {}^{C}_{0}D_t^{\gamma_i} u = {}_0I_t^{\gamma_i} \, {}^{C}_{0}D_t^{\gamma_i} u = u$, which can be incorporated in either the case (3.4) or (3.5).

Therefore, the equation (1.1) is transformed into

$$\partial_t u(x,t) + \sum_{i=1}^{s} b_i \, {}^{RL}_{0}D_t^{1+\gamma_i-\gamma} u(x,t) = {}^{RL}_{0}D_t^{1-\gamma} \mathcal{L} u + f(x,t),$$

where $f(x,t) = {}_0I_t^{\gamma-1} g(x,t) + \psi + \sum_{i=1}^{s'} b_i \, {}_0I_t^{\gamma-\gamma_i} \psi$. □

For the approximation of the Riemann–Liouville fractional derivative ${}^{RL}_{0}D_t^{\alpha} u$, as $u(x,0) = 0$, one can continuously extend the solution $u(x,t)$ to be zero for $t < 0$. Thus we use the weighted and shifted Grünwald difference (WSGD) operator as in [14], that is, for $u(\cdot,t) \in L^1(\mathbb{R})$, ${}^{RL}_{-\infty}D_t^{\alpha+2} u(\cdot,t)$ and and its Fourier transform belong to $L^1(\mathbb{R})$, we have

$$ {}^{RL}_{0}D_t^{\alpha} u(x,t_{k+1}) = \tau^{-\alpha} \sum_{j=0}^{k+1} \lambda_j^{(\alpha)} u(x,t_{k+1-j}) + \mathrm{O}(\tau^2), \; 0 \le \alpha \le 1, \tag{3.6}$$

where

$$\lambda_0^{(\alpha)} = \frac{2+\alpha}{2} g_0^{(\alpha)}, \quad \lambda_j^{(\alpha)} = \frac{2+\alpha}{2} g_j^{(\alpha)} - \frac{\alpha}{2} g_{j-1}^{(\alpha)}, \; j \ge 1, \tag{3.7}$$

and $g_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$ for $j \geq 0$.

For the discretization of integral operator ${}_0I_t^{\alpha}$, we use the weighted and shifted Grünwald difference operator as in [14], that is, for $u(\cdot,t) \in L^1(\mathbb{R})$, ${}_{-\infty}I_t^{\alpha}u(\cdot,t)$ and $(i\omega)^{2-\alpha}\mathcal{F}[f](\omega)$ belong to $L^1(\mathbb{R})$, we have

$$ {}_0I_t^{\alpha}u(\cdot,t_{k+1}) = \tau^{\alpha}\sum_{j=0}^{k+1}\mu_j^{(\alpha)}u(\cdot,t_{k+1-j}) + \mathrm{O}(\tau^2), \tag{3.8} $$

where

$$ \mu_0^{(\alpha)} = \left(1-\frac{\alpha}{2}\right)\omega_0^{(\alpha)}, \qquad \mu_j^{(\alpha)} = \left(1-\frac{\alpha}{2}\right)\omega_j^{(\alpha)} + \frac{\alpha}{2}\omega_{j-1}^{(\alpha)},\ j \geq 1, \tag{3.9} $$

and $\omega_j^{(\alpha)} = (-1)^j \binom{-\alpha}{j}$ for $j \geq 0$.

Remark 3.1. *For more details about the second order weighted and shifted Grünwald difference (WSGD) operator, one can refer to [27]. And* (3.6) *is the case* $(p,q) = (0,-1)$ *in paper [27].*

If we denote ${}_0I_t^{\alpha}f(t)$ by ${}^{RL}_{\ 0}D_t^{-\alpha}f(t)$, and notice that (3.6) and (3.7) with α replaced by $-\alpha$ are exactly the same as (3.8) and (3.9). Thus we can extend the (3.6) and (3.7) to cover both the cases $-1 \leq \alpha \leq 0$ and $0 \leq \alpha \leq 1$.

For convenience, denote $u^{k+1}(x) = u(x,t_{k+1})$, and

$$ \mathcal{D}_\tau^{\alpha}u^{k+1} = \tau^{-\alpha}\sum_{j=0}^{k+1}\lambda_j^{(\alpha)}u^{k+1-j}, -1 \leq \alpha \leq 1. $$

Thus ${}^{RL}_{\ 0}D_t^{\alpha}u(x,t_{k+1}) = \mathcal{D}_\tau^{\alpha}u^{k+1} + \mathrm{O}(\tau^2)$, by virtue of (3.6).

We discretize the space using Legendre spectral method. Therefore, based on a Crank–Nicolson technique, the fully discrete scheme for (3.1) is as follows: find $u_N^{k+1} \in \mathbb{P}_N^0$, with $u_N^0 = 0$, such that

$$ \begin{aligned} & (\delta_t u_N^{k+\frac{1}{2}}, v_N) + \frac{1}{2}\sum_{i=1}^{s} b_i(\mathcal{D}_\tau^{\alpha_i}u_N^{k+1} + \mathcal{D}_\tau^{\alpha_i}u_N^{k}, v_N) \\ = & -\frac{1}{2}\left(p(\mathcal{D}_\tau^{\alpha}\partial_x u_N^{k+1} + \mathcal{D}_\tau^{\alpha}\partial_x u_N^{k}), \partial_x v_N\right) \\ & -\frac{1}{2}\left(q(\mathcal{D}_\tau^{\alpha}u_N^{k+1} + \mathcal{D}_\tau^{\alpha}u_N^{k}), v_N\right) + (f^{k+\frac{1}{2}}, v_N), \quad \forall v_N \in \mathbb{P}_N^0, \end{aligned} \tag{3.10} $$

where $k = 0, 1, \ldots, M-1$.

For $\{u_N^j\}_{j=0}^k$ given, the well-posedness of the problem (3.10) is guaranteed by the well-known Lax–Milgram lemma, as we can rewritten (3.10) as

$$a(u_N^{k+1}, v_N) = F(v_N), \quad \forall v_N \in \mathbb{P}_N^0,$$

where

$$\begin{aligned} a(u_N^{k+1}, v_N) =& (u_N^{k+1}, v_N) + \frac{1}{2}\sum_{i=1}^{s} b_i \tau^{1-\alpha_i} \lambda_0^{(\alpha_i)} (u_N^{k+1}, v_N) \\ &+ \frac{1}{2}\tau^{1-\alpha}\lambda_0^{(\alpha)}(p\partial_x u_N^{k+1}, \partial_x v_N) + \frac{1}{2}\tau^{1-\alpha}\lambda_0^{(\alpha)}(q u_N^{k+1}, v_N) \end{aligned}$$

is a continuous and coercive bilinear form on $\mathbb{P}_N^0 \times \mathbb{P}_N^0$, $F(v_N)$ is a linear functional independent of u_N^{k+1}.

In the next section, we will give the stability and convergence of the scheme (3.10).

4. Stability and convergence of the fully discrete scheme

We first need a lemma about the coefficients $\{\lambda_j^{(\alpha)}\}_{j=0}^\infty$.

Lemma 4.1 ([14]). *Let $\{\lambda_j^{(\alpha)}\}_{j=0}^\infty$ be defined as in (3.7), then for any positive integer k and real vector $(v_1, v_2, \ldots, v_k) \in \mathbb{R}^k$, it holds that*

$$\sum_{n=0}^{k-1}\left(\sum_{j=0}^{n}\lambda_j^{(\alpha)} v_{n+1-j}\right) v_{n+1} \geq 0.$$

By virtue of this lemma, one immediately get the following lemma:

Lemma 4.2. *Let $\{\lambda_j^{(\alpha)}\}_{j=0}^\infty$ be defined as in (3.7), $r(x)$ be a nonnegative continuous function, then for any positive integer k and real-valued continuous functions $v_1(x), v_2(x), \ldots, v_k(x)$, we have*

$$\sum_{n=0}^{k-1}\left(r(x)\sum_{j=0}^{n}\lambda_j^{(\alpha)} v_{n+1-j}(x), v_{n+1}(x)\right) \geq 0.$$

Here, one should notice the notation $(\cdot,\cdot)$ denotes the inner product on Λ.

Proof. For each $x \in \Lambda$, by virtue of Lemma 4.1, we have

$$r(x) \sum_{n=0}^{k-1} \left(\sum_{j=0}^{n} \lambda_j^{(\alpha)} v_{n+1-j}(x) \right) v_{n+1}(x) \geq 0.$$

Then it follows that

$$\int_\Lambda \sum_{n=0}^{k-1} r(x) \left(\sum_{j=0}^{n} \lambda_j^{(\alpha)} v_{n+1-j}(x) \right) v_{n+1}(x) \mathrm{d}x \geq 0.$$

Finally, using the linearity of the definite integral, we get

$$\sum_{n=0}^{k-1} \int_\Lambda r(x) \left(\sum_{j=0}^{n} \lambda_j^{(\alpha)} v_{n+1-j}(x) \right) v_{n+1}(x) \mathrm{d}x \geq 0.$$

□

For the stability of the fully discrete scheme (3.10) we have the following theorem.

Theorem 4.1. *Suppose $\tau < 1$, the fully discrete scheme* (3.10) *is stable in the sense that for $1 \leq n \leq M$, it holds*

$$\|u_N^n\|^2 \leq \exp\left(\frac{T}{1-\tau} \right) \left(\tau \sum_{k=0}^{n-1} \|f^{k+\frac{1}{2}}\|^2 \right).$$

Proof. Taking $v_N = u_N^{k+1} + u_N^k$ in (3.10) gives

$$\begin{aligned} &\frac{1}{\tau}\left(\|u_N^{k+1}\|^2 - \|u_N^k\|^2\right) + \frac{1}{2} \sum_{i=1}^{s} b_i \tau^{-\alpha_i} \left(\sum_{j=0}^{k} \lambda_j^{(\alpha_i)} (u_N^{k+1-j} + u_N^{k-j}), u_N^{k+1} + u_N^k \right) \\ &= -\frac{\tau^{-\alpha}}{2} \left(p(x) \sum_{j=0}^{k} \lambda_j^{(\alpha)} (\partial_x u_N^{k+1-j} + \partial_x u_N^{k-j}), \partial_x u_N^{k+1} + \partial_x u_N^k \right) \\ &\quad - \frac{\tau^{-\alpha}}{2} \left(q(x) \sum_{j=0}^{k} \lambda_j^{(\alpha)} (u_N^{k+1-j} + u_N^{k-j}), u_N^{k+1} + u_N^k \right) + (f^{k+\frac{1}{2}}, u_N^{k+1} + u_N^k). \end{aligned} \tag{4.1}$$

Summing up for k in (4.1) from 0 to $n-1$, and noticing that

$$(f^{k+\frac{1}{2}}, u_N^{k+1} + u_N^k) \le \|f^{k+\frac{1}{2}}\|^2 + \frac{1}{2}\|u_N^{k+1}\|^2 + \frac{1}{2}\|u_N^k\|^2,$$

we get, by Lemma 4.2, that

$$\begin{aligned}
\|u_N^n\|^2 \le & -\frac{1}{2}\sum_{i=1}^{s} b_i \tau^{1-\alpha_i} \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}\lambda_j^{(\alpha_i)}(u_N^{k+1-j}+u_N^{k-j}), u_N^{k+1}+u_N^k\right)\\
& -\frac{\tau^{1-\alpha}}{2}\sum_{k=0}^{n-1}\left(p(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(\partial_x u_N^{k+1-j}+\partial_x u_N^{k-j}), \partial_x u_N^{k+1}+\partial_x u_N^k\right)\\
& -\frac{\tau^{1-\alpha}}{2}\sum_{k=0}^{n-1}\left(q(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(u_N^{k+1-j}+u_N^{k-j}), u_N^{k+1}+u_N^k\right)\\
& +\frac{\tau}{2}\sum_{k=0}^{n-1}\|u_N^{k+1}\|^2 + \frac{\tau}{2}\sum_{k=0}^{n-1}\|u_N^k\|^2 + \tau\sum_{k=0}^{n-1}\|f^{k+\frac{1}{2}}\|^2\\
\le & \ \tau\sum_{k=1}^{n}\|u_N^k\|^2 + \tau\sum_{k=0}^{n-1}\|f^{k+\frac{1}{2}}\|^2.
\end{aligned}$$

Then according to the discrete Grönwall's inequality in Lemma 2.4, we get that

$$\|u_N^n\|^2 \le \exp\left(\frac{T}{1-\tau}\right)\left(\tau\sum_{k=0}^{n-1}\|f^{k+\frac{1}{2}}\|^2\right).$$

□

For the convergence of the fully discrete scheme (3.10), we have

Theorem 4.2. *Let u be the exact solution of (3.1), $\{u_N^k\}_{k=0}^M$ be the solution of the problem (3.10). Suppose $u, {}^{RL}_{0}D_t^{\alpha_i}u \in L^\infty(0,T;H^m(\Lambda))$, $\partial_t u \in L^2(0,T;H^m(\Lambda))$, $m \ge 1$, and u satisfies the conditions above (3.6) and (3.8). Then we have, for $\tau < 1/2$,*

$$\begin{aligned}
\|u(t_k) - u_N^k\|^2 \le & \exp\left(\frac{2T}{1-2\tau}\right)\big(cN^{2-2m}\|\partial_t u\|^2_{L^2(H^m)} + c_u\tau^4\\
& + cN^{2-2m}\sum_{i=1}^{s} b_i^2 \|{}^{RL}_{0}D_t^{\alpha_i}u\|^2_{L^\infty(H^m)}\big) + cN^{2-2m}\|u\|^2_{L^\infty(H^m)},
\end{aligned}$$

where c is a positive constant independent of N, c_u is a constant depending on u.

Proof. Let $e_N^j = u^j - u_N^j$, $\widetilde{e}_N^j = \Pi_N^{1,0}u^j - u_N^j$, $\widehat{e}_N^j = u^j - \Pi_N^{1,0}u^j$, thus we have $e_N^j = \widetilde{e}_N^j + \widehat{e}_N^j$. From the equation (3.1) and the fully discrete scheme (3.10), we have the following error equation,

$$\begin{aligned}
&(\delta_t e_N^{k+\frac{1}{2}}, v_N) + \frac{1}{2}\sum_{i=1}^{s} b_i\big(\mathcal{D}_\tau^{\alpha_i} e_N^{k+1} + \mathcal{D}_\tau^{\alpha_i} e_N^k, v_N\big)\\
=& -\frac{\tau^{-\alpha}}{2}\left(p(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(\partial_x e_N^{k+1-j} + \partial_x e_N^{k-j}), \partial_x v_N\right)\\
&-\frac{\tau^{-\alpha}}{2}\left(q(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(e_N^{k+1-j} + e_N^{k-j}), v_N\right) + (R_\tau^{k+1}, v_N), \quad \forall v_N \in \mathbb{P}_N^0,
\end{aligned}$$

where $|R_\tau^{k+1}| \le c_u\tau^2$.

Then as $e_N^j = \widetilde{e}_N^j + \widehat{e}_N^j$, and according to the definition of the projection operator $\Pi_{\mathcal{N}}^{1,0}$, we have

$$\begin{aligned}
&(\delta_t \widetilde{e}_N^{k+\frac{1}{2}}, v_N) + \frac{1}{2}\sum_{i=1}^{s} b_i(\mathcal{D}_\tau^{\alpha_i} \widetilde{e}_N^{k+1} + \mathcal{D}_\tau^{\alpha_i} \widetilde{e}_N^k, v_N)\\
=& -\frac{\tau^{-\alpha}}{2}\left(p(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(\partial_x \widetilde{e}_N^{k+1-j} + \partial_x \widetilde{e}_N^{k-j}), \partial_x v_N\right) + (R_\tau^{k+1}, v_N)\\
&-\frac{\tau^{-\alpha}}{2}\left(q(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(\widetilde{e}_N^{k+1-j} + \widetilde{e}_N^{k-j}), v_N\right) - (\delta_t \widehat{e}_N^{k+\frac{1}{2}}, v_N)\\
&-\frac{1}{2}\sum_{i=1}^{s} b_i(\mathcal{D}_\tau^{\alpha_i} \widehat{e}_N^{k+1} + \mathcal{D}_\tau^{\alpha_i} \widehat{e}_N^k, v_N), \quad \forall v_N \in \mathbb{P}_N^0. \qquad (4.2)
\end{aligned}$$

Taking $v_N = \widetilde{e}_N^{k+1} + \widetilde{e}_N^k$ in (4.2), gives

$$
\begin{aligned}
&\frac{1}{\tau}\left(\|\widetilde{e}_N^{k+1}\|^2 - \|\widetilde{e}_N^k\|^2\right) + \sum_{i=1}^{s} b_i \frac{\tau^{-\alpha_i}}{2}\left(\sum_{j=0}^{k}\lambda_j^{(\alpha_i)}(\widetilde{e}_N^{k+1-j} + \widetilde{e}_N^{k-j}), \widetilde{e}_N^{k+1} + \widetilde{e}_N^k\right)\\
=& -\frac{\tau^{-\alpha}}{2}\left(p(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(\partial_x\widetilde{e}_N^{k+1-j} + \partial_x\widetilde{e}_N^{k-j}), \partial_x\widetilde{e}_N^{k+1} + \partial_x\widetilde{e}_N^k\right)\\
&-\frac{\tau^{-\alpha}}{2}\left(q(x)\sum_{j=0}^{k}\lambda_j^{(\alpha)}(\widetilde{e}_N^{k+1-j} + \widetilde{e}_N^{k-j}), \widetilde{e}_N^{k+1} + \widetilde{e}_N^k\right) + (R_\tau^{k+1}, \widetilde{e}_N^{k+1} + \widetilde{e}_N^k)\\
&-(\delta_t\widehat{e}_N^{k+\frac{1}{2}}, \widetilde{e}_N^{k+1} + \widetilde{e}_N^k) - \frac{1}{2}\sum_{i=1}^{s} b_i(\mathcal{D}_\tau^{\alpha_i}\widehat{e}_N^{k+1} + \mathcal{D}_\tau^{\alpha_i}\widehat{e}_N^k, \widetilde{e}_N^{k+1} + \widetilde{e}_N^k). \qquad (4.3)
\end{aligned}
$$

As $\mathcal{D}_\tau^{\alpha_i}\widehat{e}_N^k = {}_0^{RL}D_t^{\alpha_i}\widehat{e}_N^k + \mathrm{O}(\tau^2)$, and according to the Lemmas 2.2 and 2.3, we have

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{s} b_i(\mathcal{D}_\tau^{\alpha_i}\widehat{e}_N^{k+1} + \mathcal{D}_\tau^{\alpha_i}\widehat{e}_N^k, \widetilde{e}_N^{k+1} + \widetilde{e}_N^k)\\
\le& \frac{1}{2}\sum_{i=1}^{s} s b_i^2\|\mathcal{D}_\tau^{\alpha_i}\widehat{e}_N^{k+1} + \mathcal{D}_\tau^{\alpha_i}\widehat{e}_N^k\|^2 + \frac{1}{4}\|\widetilde{e}_N^{k+1}\|^2 + \frac{1}{4}\|\widetilde{e}_N^k\|^2\\
\le& cN^{2-2m}\sum_{i=1}^{s} b_i^2\|{}_0^{RL}D_t^{\alpha_i}u\|_{L^\infty(H^m)}^2 + c_u\tau^4 + \frac{1}{4}\|\widetilde{e}_N^{k+1}\|^2 + \frac{1}{4}\|\widetilde{e}_N^k\|^2.
\end{aligned}
$$

Summing up (4.3) for $0 \le k \le n-1$, and noticing that

$$
\begin{aligned}
-(\delta_t\widehat{e}_N^{k+\frac{1}{2}}, \widetilde{e}_N^k + \widetilde{e}_N^k) =& -\left(\frac{1}{\tau}\int_{t_k}^{t_{k+1}}\partial_t\widehat{e}_N \mathrm{d}t, \widetilde{e}_N^{k+1} + \widetilde{e}_N^k\right)\\
\le& \frac{1}{\tau}\int_{t_k}^{t_{k+1}}\|\partial_t\widehat{e}_N\|^2\mathrm{d}t + \frac{1}{2}\|\widetilde{e}_N^{k+1}\|^2 + \frac{1}{2}\|\widetilde{e}_N^k\|^2,
\end{aligned}
$$

and

$$
(R_\tau^{k+1}, \widetilde{e}_N^{k+1} + \widetilde{e}_N^k) \le 2\|R_\tau^{k+1}\|^2 + \frac{1}{4}\|\widetilde{e}_N^{k+1}\|^2 + \frac{1}{4}\|\widetilde{e}_N^k\|^2,
$$

we get, by Lemmas 4.2, 2.2 and 2.3, that

$$\begin{aligned}
\|\widetilde{e}_N^n\|^2 \leq & -\sum_{i=1}^{s} b_i \frac{\tau^{1-\alpha_i}}{2} \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \lambda_j^{(\alpha_i)} (\widetilde{e}_N^{k+1-j} + \widetilde{e}_N^{k-j}), \widetilde{e}_N^{k+1} + \widetilde{e}_N^k \right) \\
& - \frac{\tau^{1-\alpha}}{2} \sum_{k=0}^{n-1} \left(p(x) \sum_{j=0}^{k} \lambda_j^{(\alpha)} (\partial_x \widetilde{e}_N^{k+1-j} + \partial_x \widetilde{e}_N^{k-j}), \partial_x \widetilde{e}_N^{k+1} + \partial_x \widetilde{e}_N^k \right) \\
& - \frac{\tau^{1-\alpha}}{2} \sum_{k=0}^{n-1} \left(q(x) \sum_{j=0}^{k} \lambda_j^{(\alpha)} (\widetilde{e}_N^{k+1-j} + \widetilde{e}_N^{k-j}), \widetilde{e}_N^{k+1} + \widetilde{e}_N^k \right) \\
& + \tau \sum_{k=0}^{n-1} \|\widetilde{e}_N^{k+1}\|^2 + \tau \sum_{k=0}^{n-1} \|\widetilde{e}_N^k\|^2 + 2\tau \sum_{k=0}^{n-1} \|R_\tau^{k+1}\|^2 + \int_0^{t_n} \|\partial_t \widehat{e}_N\|^2 \mathrm{d}t \\
& + \tau \sum_{k=0}^{n-1} cN^{2-2m} \sum_{i=1}^{s} b_i^2 \| {}_0^{RL}D_t^{\alpha_i} u\|_{L^\infty(H^m)}^2 + \tau \sum_{k=0}^{n-1} c_u \tau^4 \\
\leq & 2\tau \sum_{k=1}^{n} \|\widetilde{e}_N^k\|^2 + cN^{2-2m} \int_0^T \|\partial_t u\|_m^2 \mathrm{d}t + c_u \tau^4 \\
& + cN^{2-2m} \sum_{i=1}^{s} b_i^2 \| {}_0^{RL}D_t^{\alpha_i} u\|_{L^\infty(H^m)}^2 .
\end{aligned}$$

Then according to the discrete Grönwall's inequality in Lemma 2.4, we get

$$\begin{aligned}
& \|\widetilde{e}_N^n\|^2 \\
\leq & \exp\left(\frac{2T}{1-2\tau}\right) (cN^{2-2m} \|\partial_t u\|_{L^2(H^m)}^2 + cN^{2-2m} \sum_{i=1}^{s} b_i^2 \| {}_0^{RL}D_t^{\alpha_i} u\|_{L^\infty(H^m)}^2 + c_u \tau^4).
\end{aligned}$$

Finally, using the triangular inequality $\|e_N^n\| \leq \|\widetilde{e}_N^n\| + \|\widehat{e}_N^n\|$, and Lemmas 2.2 and 2.3, we get the desired result.

$\square$

5. Numerical experiment

5.1. Implementation

Let $\beta_0 = \tau^{1-\alpha}/2$, $\beta_i = \tau^{1-\alpha_i}/2$, we rewrite the equation (3.10) in the form

$$\begin{aligned}(u_N^{k+1}, v_N) &+ \sum_{i=1}^{s} b_i\beta_i\lambda_0^{(\alpha_i)}(u_N^{k+1}, v_N) + \beta_0\lambda_0^{(\alpha)}\left(p(u_N^{k+1})_x, (v_N)_x\right) \\ &+ \beta_0\lambda_0^{(\alpha)}\left(qu_N^{k+1}, v_N\right) = F_N^{k+1}(v_N),\ \forall v_N \in \mathbb{P}_N^0, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned}F_N^{k+1}(v_N) =&(u_N^k, v_N) - \sum_{i=1}^{s} b_i\beta_i\left(\sum_{j=0}^{k}(\lambda_j^{(\alpha_i)} + \lambda_{j+1}^{(\alpha_i)})u_N^{k-j}, v_N\right) \\ &- \beta_0\left(p\sum_{j=0}^{k}(\lambda_j^{(\alpha)} + \lambda_{j+1}^{(\alpha)})\partial_x u_N^{k-j}, \partial_x v_N\right) \\ &- \beta_0\left(q\sum_{j=0}^{k}(\lambda_j^{(\alpha)} + \lambda_{j+1}^{(\alpha)})u_N^{k-j}, v_N\right) + \tau(f^{k+\frac{1}{2}}, v_N).\end{aligned}$$

Let $L_n(x)$ denotes Legendre polynomials with degree n. We choose the basis functions as

$$\phi_j(x) = L_j(x) - L_{j+2}(x), \quad j \geq 0.$$

One can verify that $\phi_j(\pm 1) = 0$. Thus $\mathbb{P}_N^0 = \text{span}\{\phi_j : j = 0, 1, \ldots, N-2\}$.

We express the function u_N^{k+1} in terms of the basis functions $\{\phi_j(x)\}_{j=0}^{N-2}$

$$u_N^{k+1}(x) = \sum_{j=0}^{N-2} \widetilde{u}_j^{k+1}\phi_j(x),$$

where $\{\widetilde{u}_j^{k+1}\}_{j=0}^{N-2}$ are the frequency coefficients that we want to solve.

Choosing each test function v_N to be $\phi_l(x), l = 0, 1, \ldots, N-2$, we obtain

$$\begin{aligned}&\sum_{j=0}^{N-2}(\phi_j, \phi_l)\,\widetilde{u}_j^{k+1} + \sum_{i=1}^{s} b_i\beta_i\lambda_0^{(\alpha_i)}\sum_{j=1}^{N-1}(\phi_j, \phi_l)\,\widetilde{u}_j^{k+1} \\ &+\beta_0\lambda_0^{(\alpha)}\sum_{j=0}^{N-2}\left(p\partial_x\phi_j, \partial_x\phi_l\right)\widetilde{u}_j^{k+1} + \beta_0\lambda_0^{(\alpha)}\sum_{j=0}^{N-2}\left(q\phi_j, \phi_l\right)\widetilde{u}_j^{k+1} \\ =\ & F_N^{k+1}(\phi_l).\end{aligned}$$

Let

$$\mathbf{U}^{k+1} = [\widetilde{u}_0^{k+1}, \widetilde{u}_1^{k+1}, \dots, \widetilde{u}_{N-2}^{k+1}]^{\mathrm{T}},$$
$$\mathbf{F}^{k+1} = [F_N^{k+1}(\phi_0), F_N^{k+1}(\phi_1), \dots, F_N^{k+1}(\phi_{N-2})]^{\mathrm{T}}.$$

We denote the matrices

$$\mathbf{A} = \big((\phi_j, \phi_i)\big)_{i,j=0}^{N-2}, \quad \mathbf{B} = \Big(\big(p\partial_x\phi_j, \partial_x\phi_i\big)\Big)_{i,j=0}^{N-2},$$
$$\mathbf{C} = \Big(\big(q\phi_j, \phi_i\big)\Big)_{i,j=0}^{N-2}.$$

Then we arrive at the following matrix statement of the problem (5.1):

$$\Big((1+\sum_{i=1}^{s} b_i\beta_i\lambda_0^{(\alpha_i)})\mathbf{A} + \beta_0\lambda_0^{(\alpha)}\mathbf{B} + \beta_0\lambda_0^{(\alpha)}\mathbf{C}\Big)\mathbf{U}^{k+1} = \mathbf{F}^{k+1}.$$

The components of the matrices can be efficiently computed by Legendre-Gauss quadrature formula [32].

5.2. Numerical results

We carry out some numerical experiments and present some results to confirm our theoretical statements.

Firstly, we consider the problem (1.1)–(1.4) with exact solutions.

Example 5.1. *We consider the problem* (1.1)–(1.4) *in the case* $s = 1$, $b_1 = 1$, *with an exact analytical solution:*

$$u(x,t) = (t^{2+\gamma} + t^2)\sin(\pi x),$$

and $p(x) = 2 - \sin(x)$, $q(x) = 1 - \cos(x)$. *The corresponding forcing terms are*

$$g(x,t) = \left(\frac{\Gamma(3+\gamma)t^2}{2} + \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} + \frac{\Gamma(3+\gamma)t^{2+\gamma-\gamma_1}}{\Gamma(3+\gamma-\gamma_1)} + \frac{2t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right)\sin(\pi x)$$
$$+ (t^{2+\gamma}+t^2)\sin(\pi x)(1-\cos(x)+\pi^2(2-\sin(x))) + \pi\cos(x)(t^{2+\gamma}+t^2)\cos(\pi x),$$

$$f(x,t) = \left((2+\gamma)t^{1+\gamma} + 2t + \frac{\Gamma(3+\gamma)t^{1+2\gamma-\gamma_1}}{\Gamma(2+2\gamma-\gamma_1)} + \frac{2t^{1+\gamma-\gamma_1}}{\Gamma(2+\gamma-\gamma_1)}\right)\sin(\pi x)$$
$$+ (\frac{\Gamma(3+\gamma)t^{1+2\gamma}}{\Gamma(2+2\gamma)} + \frac{2t^{1+\gamma}}{\Gamma(2+\gamma)})[\sin(\pi x)(1-\cos(x)+\pi^2(2-\sin(x))) + \pi\cos(x)\cos(\pi x)].$$

Example 5.2. *We consider the problem* (1.1)–(1.4) *in the case* $s = 1$, $b_1 = 1$, *with an exact solution which has limited regularity :*

$$u(x,t) = t^3(1-x^2)x^{\frac{16}{3}},$$

(one can verify $u \in H^5(\Lambda)$, *but* $\notin H^6(\Lambda)$*), and* $p(x) = 2 - \sin(x)$, $q(x) = 1 - \cos(x)$. *The corresponding forcing terms are*

$$g(x,t) = \left(\frac{6t^{3-\gamma}}{\Gamma(4-\gamma)} + \frac{6t^{3-\gamma_1}}{\Gamma(4-\gamma_1)}\right)(1-x^2)x^{\frac{16}{3}} + t^3(2-\sin(x))(\frac{418}{9}x^{\frac{16}{3}} - \frac{208}{9}x^{\frac{10}{3}})$$
$$+ t^3\cos(x)(\frac{16}{3}x^{\frac{13}{3}} - \frac{22}{3}x^{\frac{19}{3}}) + t^3(1-\cos(x))(1-x^2)x^{\frac{16}{3}},$$

$$f(x,t) = \left(3t^2 + \frac{6t^{2+\gamma-\gamma_1}}{\Gamma(3+\gamma-\gamma_1)}\right)(1-x^2)x^{\frac{16}{3}} + \frac{6t^{2+\gamma}}{\Gamma(3+\gamma)}(2-\sin(x))(\frac{418}{9}x^{\frac{16}{3}} - \frac{208}{9}x^{\frac{10}{3}})$$
$$+\frac{6t^{2+\gamma}}{\Gamma(3+\gamma)}\cos(x)(\frac{16}{3}x^{\frac{13}{3}} - \frac{22}{3}x^{\frac{19}{3}}) + \frac{6t^{2+\gamma}}{\Gamma(3+\gamma)}(1-\cos(x))(1-x^2)x^{\frac{16}{3}}.$$

Example 5.3. *We consider the problem* (1.1)–(1.4) *in the case* $s = 4$, $b_1 = b_2 = b_3 = b_4 = 1$, *with an exact analytical solution:*

$$u(x,t) = t^3\sin(\pi x),$$

and $p(x) = 2 - \sin(x)$, $q(x) = 1 - \cos(x)$. *The corresponding forcing terms are*

$$g(x,t) = \left(\frac{6t^{3-\gamma}}{\Gamma(4-\gamma)} + \sum_{i=1}^{4}\frac{6t^{3-\gamma_i}}{\Gamma(4-\gamma_i)}\right)\sin(\pi x) + \pi t^3\cos(x)\cos(\pi x)$$
$$+ t^3\sin(\pi x)\left(1-\cos(x)+\pi^2(2-\sin(x))\right),$$

$$f(x,t) = \left(3t^2 + \sum_{i=1}^{4}\frac{6t^{2+\gamma-\gamma_i}}{\Gamma(3+\gamma-\gamma_i)}\right)\sin(\pi x) + \pi\frac{6t^{2+\gamma}}{\Gamma(3+\gamma)}\cos(x)\cos(\pi x)$$
$$+ \frac{6t^{2+\gamma}}{\Gamma(3+\gamma)}\sin(\pi x)\left(1-\cos(x)+\pi^2(2-\sin(x))\right).$$

To confirm the temporal accuracy, we choose N big enough to eliminate the error caused by spacial discretization. For Examples 5.1 and 5.3 we take $N = 15$, while for Example 5.2 we take $N = 100$.

Tables 1–7 show the errors $\|u(T) - u_N^M\|$ $(T = 1)$ and the corresponding temporal convergence rates. From which, we can see the temporal accuracy is second-order, which is consistent with our theoretical analysis. The convergence rate is given by the formula: Rate= $\log_{\frac{\tau_1}{\tau_2}} \frac{e_1}{e_2}$ (e_i is the error corresponding to τ_i). All the calculations are performed in MATLAB.

Table 1: L^2 errors and convergence rates in the case $0 < \gamma_1 < \gamma < 1$ for Example 5.1.

τ	$\gamma = 0.1,\ \gamma_1 = 0.01$		$\gamma = 0.5,\ \gamma_1 = 0.4$		$\gamma = 0.9,\ \gamma_1 = 0.75$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	1.0392e-02	1.9810	7.7551e-03	1.9849	2.5941e-03	1.9854
1/20	2.6324e-03	1.9840	1.9592e-03	1.9900	6.5510e-04	1.9924
1/40	6.6545e-04	1.9842	4.9321e-04	1.9942	1.6465e-04	1.9960
1/80	1.6819e-04	1.9855	1.2380e-04	1.9966	4.1274e-05	1.9979
1/160	4.2470e-05	1.9872	3.1023e-05	1.9979	1.0334e-05	1.9988
1/320	1.0712e-05	*	7.7673e-06	*	2.5855e-06	*

Table 2: L^2 errors and convergence rates in the case $1 < \gamma_1 < \gamma < 2$ for Example 5.1.

τ	$\gamma = 1.1,\ \gamma_1 = 1.01$		$\gamma = 1.5,\ \gamma_1 = 1.25$		$\gamma = 1.9,\ \gamma_1 = 1.65$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	6.0235e-04	1.9880	4.8377e-03	1.9788	5.5096e-03	1.9853
1/20	1.5184e-04	1.9931	1.2273e-03	1.9890	1.3915e-03	1.9935
1/40	3.8144e-05	1.9961	3.0918e-04	1.9942	3.4944e-04	1.9960
1/80	9.5620e-06	1.9977	7.7605e-05	1.9969	8.7604e-05	1.9969
1/160	2.3943e-06	1.9986	1.9442e-05	1.9983	2.1948e-05	1.9975
1/320	5.9915e-07	*	4.8662e-06	*	5.4964e-06	*

Table 3: L^2 errors and convergence rates in the case $0 < \gamma_1 < 1 < \gamma < 2$ for Example 5.1.

τ	$\gamma = 1.1,\ \gamma_1 = 0.35$		$\gamma = 1.5,\ \gamma_1 = 0.5$		$\gamma = 1.9,\ \gamma_1 = 0.75$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	9.4197e-04	1.9695	6.2393e-03	1.9765	8.8740e-03	1.9900
1/20	2.4053e-04	1.9849	1.5854e-03	1.9886	2.2340e-03	1.9957
1/40	6.0765e-05	1.9925	3.9949e-04	1.9944	5.6016e-04	1.9980
1/80	1.5271e-05	1.9962	1.0026e-04	1.9972	1.4023e-04	1.9991
1/160	3.8276e-06	1.9981	2.5113e-05	1.9986	3.5080e-05	1.9995
1/320	9.5815e-07	*	6.2843e-06	*	8.7727e-06	*

Table 4: L^2 errors and convergence rates in the case $0 < \gamma_1 < \gamma < 1$ for Example 5.2.

τ	$\gamma = 0.1,\ \gamma_1 = 0.01$		$\gamma = 0.5,\ \gamma_1 = 0.4$		$\gamma = 0.9,\ \gamma_1 = 0.75$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	1.2608e-03	1.9461	6.2878e-04	1.9569	1.7042e-04	1.9731
1/20	3.2721e-04	1.9738	1.6197e-04	1.9791	4.3405e-05	1.9880
1/40	8.3301e-05	1.9873	4.1083e-05	1.9901	1.0942e-05	1.9973
1/80	2.1009e-05	1.9946	1.0341e-05	1.9970	2.7405e-06	2.0114
1/160	5.2719e-06	2.0008	2.5907e-06	2.0056	6.7975e-07	2.0547
1/320	1.3172e-06	*	6.4516e-07	*	1.6361e-07	*

Table 5: L^2 errors and convergence rates in the case $1 < \gamma_1 < \gamma < 2$ for Example 5.2.

τ	$\gamma = 1.1,\ \gamma_1 = 1.01$		$\gamma = 1.5,\ \gamma_1 = 1.25$		$\gamma = 1.9,\ \gamma_1 = 1.65$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	1.1482e-04	1.9862	3.0294e-04	1.9844	4.2068e-04	1.9946
1/20	2.8981e-05	1.9948	7.6559e-05	1.9920	1.0557e-04	1.9983
1/40	7.2717e-06	2.0033	1.9246e-05	1.9966	2.6423e-05	1.9998
1/80	1.8138e-06	2.0252	4.8228e-06	2.0006	6.6066e-06	2.0015
1/160	4.4559e-07	2.1094	1.2052e-06	2.0090	1.6500e-06	2.0066
1/320	1.0326e-07	*	2.9944e-07	*	4.1059e-07	*

Table 6: L^2 errors and convergence rates in the case $0 < \gamma_1 < 1 < \gamma < 2$ for Example 5.2.

τ	$\gamma = 1.1,\ \gamma_1 = 0.35$		$\gamma = 1.5,\ \gamma_1 = 0.5$		$\gamma = 1.9,\ \gamma_1 = 0.75$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	7.5546e-05	1.9743	3.1467e-04	1.9831	3.8329e-04	2.0136
1/20	1.9226e-05	1.9878	7.9598e-05	1.9917	9.4924e-05	2.0049
1/40	4.8471e-06	1.9980	2.0015e-05	1.9958	2.3651e-05	2.0012
1/80	1.2134e-06	2.0141	5.0181e-06	1.9977	5.9079e-06	2.0004
1/160	3.0040e-07	2.0467	1.2565e-06	1.9974	1.4766e-06	2.0009
1/320	7.2710e-08	*	3.1469e-07	*	3.6892e-07	*

Table 7: L^2 errors, convergence rates, and CPU time for Example 5.3.

$\gamma, \gamma_1, \gamma_2, \gamma_3, \gamma_4$	τ	$Error$	$Rate$	$CPU(s)$
$\gamma = 0.9$	1/10	3.5222e-003	1.9737	0.0460
$\gamma_1 = 0.8$	1/20	8.9675e-004	1.9873	0.0830
$\gamma_2 = 0.7$	1/40	2.2617e-004	1.9937	0.2058
$\gamma_3 = 0.5$	1/80	5.6789e-005	1.9969	0.5133
$\gamma_4 = 0.3$	1/160	1.4228e-005	1.9984	1.4476
	1/320	3.5608e-006	*	5.0817
$\gamma = 1.9$	1/10	1.6412e-003	1.9209	0.0393
$\gamma_1 = 1.7$	1/20	4.3342e-004	1.9647	0.0703
$\gamma_2 = 1.5$	1/40	1.1104e-004	1.9836	0.1670
$\gamma_3 = 1.3$	1/80	2.8076e-005	1.9921	0.4676
$\gamma_4 = 1.1$	1/160	7.0573e-006	1.9962	1.5730
	1/320	1.7690e-006	*	4.3373
$\gamma = 1.6$	1/10	2.8967e-003	1.9705	0.0445
$\gamma_1 = 1.3$	1/20	7.3912e-004	1.9865	0.0786
$\gamma_2 = 0.9$	1/40	1.8652e-004	1.9934	0.1405
$\gamma_3 = 0.6$	1/80	4.6843e-005	1.9968	0.3027
$\gamma_4 = 0.3$	1/160	1.1737e-005	1.9984	1.3381
	1/320	2.9376e-006	*	5.6394

Next we check the spatial accuracy with respect to the polynomial degree N. By fixing the time step small enough to avoid the contamination of the temporal error. We choose $\tau = 0.001$. We take the cases $\gamma = 1.5$, $\gamma_1 = 1.25$; $\gamma = 0.5$, $\gamma_1 = 0.4$; $\gamma = 1.5$, $\gamma_1 = 0.5$ to illustrate.

Figs. 1–3 present the L^2 errors with respect to N in semi-log scale for Example 5.1. From which, we can see the errors decay exponentially, that is the so-called spectral accuracy.

Figs. 4–6 show the the L^2 errors with respect to N in log-log scale for Example 5.2. Since its solution belongs to $H^5(\Lambda)$, but $\notin H^6(\Lambda)$, we can see the convergence rates are between N^{-4} and N^{-5}, which conforms with our theoretical analysis.

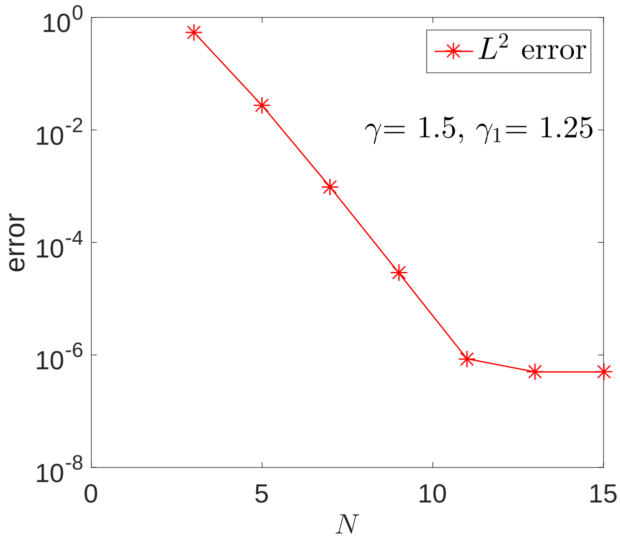

Fig. 1: $\gamma = 1.5$, $\gamma_1 = 1.25$ for Example 5.1

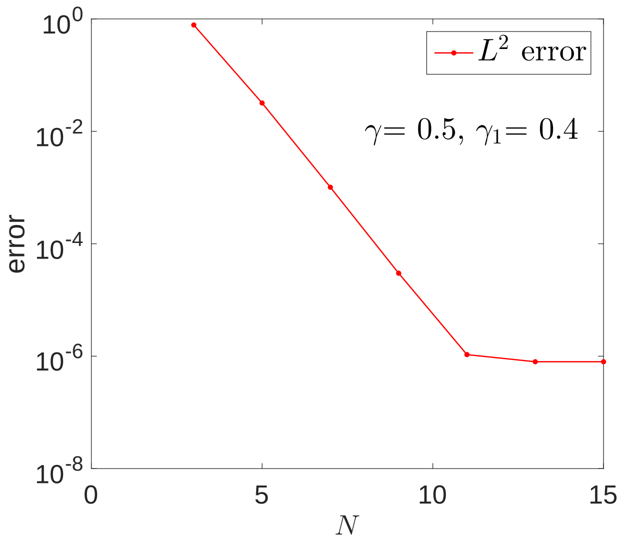

Fig. 2: $\gamma = 0.5$, $\gamma_1 = 0.4$ for Example 5.1

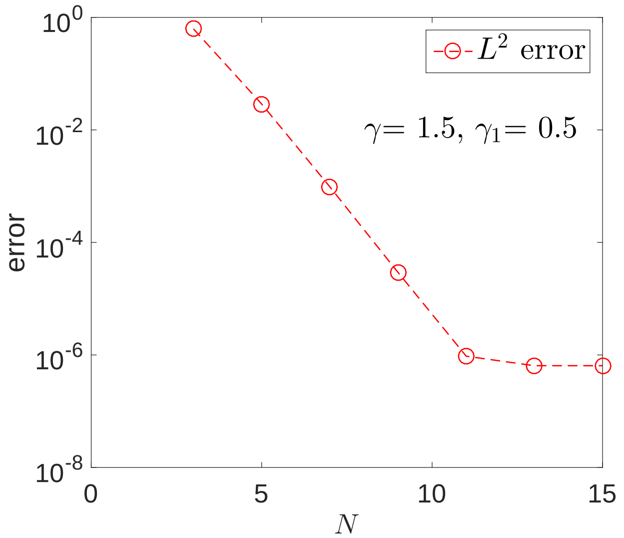

Fig. 3: $\gamma = 1.5$, $\gamma_1 = 0.5$ for Example 5.1

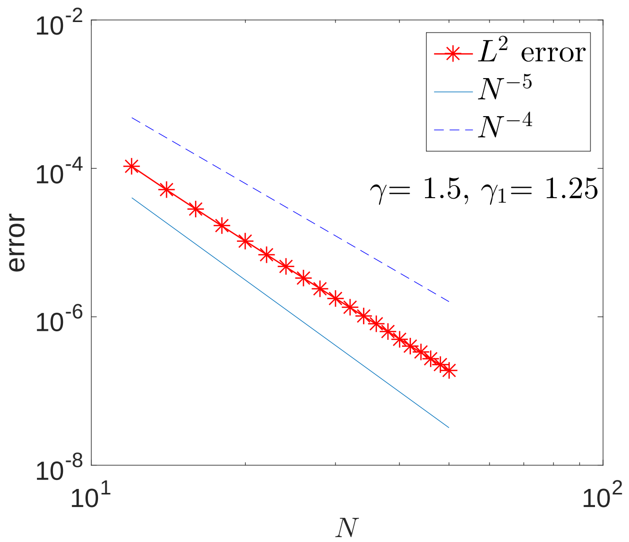

Fig. 4: $\gamma = 1.5$, $\gamma_1 = 1.25$ for Example 5.2

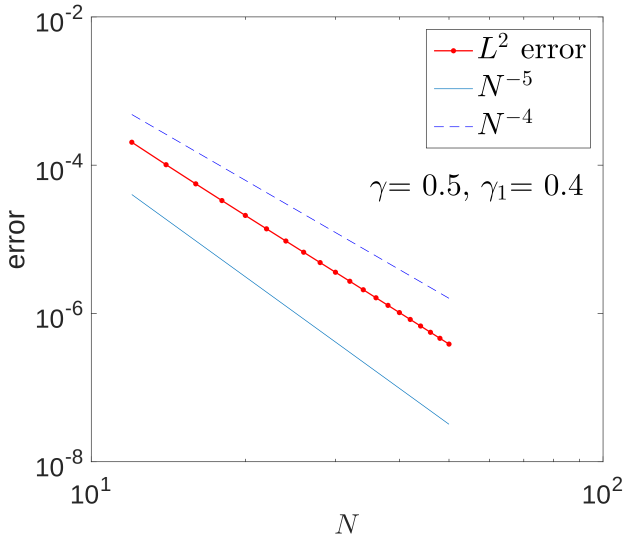

Fig. 5: $\gamma = 0.5,\ \gamma_1 = 0.4$ for Example 5.2

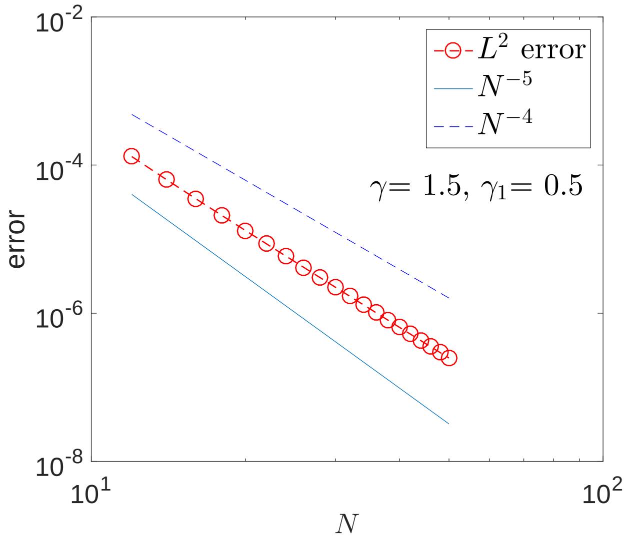

Fig. 6: $\gamma = 1.5,\ \gamma_1 = 0.5$ for Example 5.2

Next, we consider the problem (1.1)–(1.4) with general problem data.

Example 5.4. *We consider the problem* (1.1)–(1.4) *in the case* $s = 1$, $b_1 = 1$, *with* $u_0 = 0$, $\psi = \cos(x)$, *the forcing term* $g(x,t) = t^2 \sin(\pi x)$, *and coefficients* $p(x) = 2 - \sin(x)$, $q(x) = 1 - \cos(x)$. *The corresponding forcing term in equation* (3.1) *is*

$$f(x,t) = \begin{cases} \frac{2t^{1+\gamma}}{\Gamma(2+\gamma)} \sin(\pi x), & \text{if } 0 < \gamma_1 < \gamma < 1, \\ \frac{2t^{1+\gamma}}{\Gamma(2+\gamma)} \sin(\pi x) + \cos(x), & \text{if } 0 < \gamma_1 < 1 < \gamma < 2, \\ \frac{2t^{1+\gamma}}{\Gamma(2+\gamma)} \sin(\pi x) + \cos(x) + \frac{t^{\gamma-\gamma_1}}{\Gamma(1+\gamma-\gamma_1)} \cos(x), & \text{if } 1 < \gamma_1 < \gamma < 2. \end{cases}$$

As the exact solution u is unknown, we use the reference solution U which is computed on a much finer mesh in stead of u. We choose $\tau = 0.001$ and $N = 15$ to compute the reference solution. Tables 8–10 show the errors $\|U - u_N^M\|$ $(T = 1)$ and the corresponding temporal convergence rates.

In both $0 < \gamma_1 < \gamma < 1$ and $0 < \gamma_1 < 1 < \gamma < 2$ cases, we can see the temporal accuracy is second-order, which is consistent with our analysis. However, in the case $1 < \gamma_1 < \gamma < 2$, the temporal accuracy is less than second-order (but greater than one), probably depends on the order of the fractional derivatives. It is the consequence of the forcing term $f(x,t)$ with a term $\frac{t^{\gamma-\gamma_1}}{\Gamma(1+\gamma-\gamma_1)} \cos(x)$, so the exact solution is not regular enough to guarantee the accuracy of our scheme.

Table 8: L^2 errors and convergence rates in the case $0 < \gamma_1 < \gamma < 1$ for Example 5.4.

τ	$\gamma = 0.1$, $\gamma_1 = 0.01$		$\gamma = 0.5$, $\gamma_1 = 0.4$		$\gamma = 0.9$, $\gamma_1 = 0.75$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	2.3406e-04	1.9857	1.4556e-04	1.9925	4.0535e-05	2.0135
1/20	5.9096e-05	1.9869	3.6580e-05	1.9933	1.0039e-05	2.0079
1/40	1.4909e-05	1.9908	9.1875e-06	2.0011	2.4961e-06	2.0099
1/80	3.7511e-06	2.0129	2.2952e-06	2.0241	6.1977e-07	2.0296
1/160	9.2945e-07	2.1053	5.6427e-07	2.1158	1.5180e-07	2.1191
1/320	2.1601e-07	*	1.3019e-07	*	3.4942e-08	*

Table 9: L^2 errors and convergence rates in the case $1 < \gamma_1 < \gamma < 2$ for Example 5.4.

τ	$\gamma = 1.1,\ \gamma_1 = 1.01$		$\gamma = 1.5,\ \gamma_1 = 1.25$		$\gamma = 1.9,\ \gamma_1 = 1.65$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	7.1981e-04	0.9362	1.9723e-03	1.2382	4.3859e-03	1.4711
1/20	3.7618e-04	0.9694	8.3607e-04	1.3151	1.5820e-03	1.4858
1/40	1.9213e-04	1.0595	3.3601e-04	1.3346	5.6487e-04	1.4255
1/80	9.2183e-05	1.1616	1.3323e-04	1.3752	2.1029e-04	1.4009
1/160	4.1207e-05	1.3482	5.1359e-05	1.5149	7.9633e-05	1.5172
1/320	1.6185e-05	*	1.7972e-05	*	2.7821e-05	*

Table 10: L^2 errors and convergence rates in the case $0 < \gamma_1 < 1 < \gamma < 2$ for Example 5.4.

τ	$\gamma = 1.1,\ \gamma_1 = 0.35$		$\gamma = 1.5,\ \gamma_1 = 0.5$		$\gamma = 1.9,\ \gamma_1 = 0.75$	
	Error	*Rate*	*Error*	*Rate*	*Error*	*Rate*
1/10	1.9741e-04	2.1507	1.3835e-03	2.0369	6.2398e-03	1.7311
1/20	4.4458e-05	2.1804	3.3716e-04	2.0141	1.8796e-03	1.8693
1/40	9.8082e-06	2.0397	8.3471e-05	2.0118	5.1446e-04	1.9209
1/80	2.3854e-06	2.0285	2.0697e-05	2.0303	1.3586e-04	2.0207
1/160	5.8468e-07	2.1187	5.0669e-06	2.1194	3.3482e-05	2.1182
1/320	1.3463e-07	*	1.1661e-06	*	7.7118e-06	*

6. Summary and discussion

We have presented and analysed a unified numerical scheme for the multi-term time fractional diffusion and diffusion–wave equations with variable coefficients in a bounded domain. The scheme employs the Legendre spectral method in space and the weighted and shifted Grünwald difference operators for the discretization of the time fractional operators. The stability and convergence of the fully discrete scheme have been rigorously established. We have carried out some numerical experiments to confirm the theoretical results.

In our assumptions, $u_0(x) \equiv 0$, if $u_0 \neq 0$, we consider $\widetilde{u} = u - u_0$. Since ${}_0^C D_t^\gamma u_0(x) \equiv 0$, after transformation, the new forcing term $\widetilde{g}(x,t) = g(x,t) + \mathcal{L}u_0$, with the other terms unchanged in the new equation. It should be pointed out that in our analysis we assume the solution u satisfies some good regularity, for the solutions which do not satisfy our assumption, it needs further investigation about the convergence of our method.

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