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On a three dimensional Cauchy problem for inhomogeneous Helmholtz equation associated with perturbed wave number

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Abstract

In recent years, the Cauchy problem for inhomogenous Helmholtz equation (CPHE) is often considered as the wave number k is a constant number. At present, we investigate a three dimensional CPHE with inhomogenous Cauchy conditions given at $z = 0$ while wave number k is perturbed. The problem is well-known to be ill-posed in the sense of Hadamard. Therefore, we regularize the problem by applying the truncation method and possess error estimate between the exact solution and the regularized solution. A numerical experiment is given for the purpose of illustrating our method.

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1 Introduction

In this paper, we consider a three dimensional Cauchy problem for inhomogeneous Helmholtz equation (HE) associated with perturbed wave number as follows

$$\begin{aligned} \Delta u(x,y,z) + k^2 u(x,y,z) &= S(x,y,z), (x,y) \in \mathbb{R}^2, z \in (0;1), && (1.1) \\ u_z(x,y,0) &= f(x,y), (x,y) \in \mathbb{R}^2, && (1.2) \\ u(x,y,0) &= g(x,y), (x,y) \in \mathbb{R}^2, && (1.3) \end{aligned}$$

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where Δ denotes the Laplace operator, and $(f, g, k) \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times \mathbb{R}^+$, $S \in L^2(\mathbb{R}^2 \times (0,1))$ are given data.

The Cauchy problem for the Helmholtz equation frequently occurs in many areas such as acoustics, electromagnetics and elasticity and etc. Therefore, CPHE plays an important role in many engineering applications and is researched in such areas of physics by many authors. For example, we can list some studies which are related to vibration of structure [1], acoustic [2], electromagnetic scattering [5], wave propagation and scattering [9] and etc. On the other hand, a different version of Helmholtz equation which is called the modified Helmholtz equation (MHE) is considered by some authors in [3, 7, 12]. The Cauchy problem for HE or MHE is ill-posed in the sense that the solution, if it exists, does not depend continuously on the given data. It means that a small perturbation in given data may cause large error in the solution of the problem. Hence, a regularization is necessary to get the stable solution of the CPHE.

In fact, several approximation methods have been proposed to deal with the CPHE. For instance, L. Marin et al. [9, 10, 11] used the Landweber method with the boundary element method (BEM) and the conjugate gradient method with BEM to solve the problem. In [17], R. Shi et al. applied the method in order to cut off high frequencies and gave logarithmical type of the error estimate between the exact solution and the regularized solution. By using the fast multiple-accelerated integral equation method, W. Cheng et al. [3] solved the MHE. Thereout, the error estimate of logarithmical type was presented by T. H. Nguyen et al. [12] in which their regularization method was the quasi-reversibility one. Until now, there are many researchs which concerned with CPHE so that we can not list out all of papers.

Very recently, in [7], P. T. Hieu et al. investigated the Cauchy problem for MHE with inhomogeneous Cauchy data given at $y = 0$,

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) - k^2 u(x,y) &= f(x,y), x \in \mathbb{R}, y \in (0;1), && (1.4)\\ u_y(x,0) &= \varphi(x), x \in \mathbb{R}, && (1.5)\\ u(x,0) &= \psi(x), x \in \mathbb{R}, && (1.6)\end{aligned}$$

in which $\varphi(\cdot), \psi(x\cdot)$, $f(\cdot,\cdot)$ are given functions which belong to $L^2(\mathbb{R})$, $L^2(\mathbb{R})$, $L^2(\mathbb{R} \times (0,1))$, respectively. By using the Fourier transform and the truncation method, the authors have regularized the problem (1.4) - (1.6) and obtained the logarithmical type of the error estimate between the exact solution and the regularized solution. Morever, under some strong conditions of the exact solution of the problem (1.4) - (1.6), the Hölder type of error estimate was obtained in [7].

In 2014, Tuan N. H. et al. [18] studied the problem of determining the electric field of a 3D Helmholtz equation as follows

$$\begin{aligned} \Delta u(\xi, z) + k^2 u(\xi, d) &= -f, \quad \xi \in \Omega,\ z \in [0,d], && (1.7)\\ u(\xi, d) &= g(\xi),\ \xi \in \Omega, && (1.8)\\ \frac{\partial}{\partial z} u(\xi, d) &= h(\xi),\ \xi \in \Omega, && (1.9)\\ u(\cdot, d) &\in L^2(\mathbb{R}^2), && (1.10)\end{aligned}$$

where $g, h \in L^2(\mathbb{R}^2)$ and $\Omega \subset \mathbb{R}^2$ is a nonempty open set. The authors have aslo regularized the problem (1.7) - (1.10) by applying the Fourier transform and the truncated method. In addition, the authors gave the Hölder type of the error estimate which was proposed in [18].

Aside from that, Viet T. Q. et al. [19] researched the following three dimensional Cauchy problem for inhomogeneous Helmholtz-type equations

$$\begin{aligned} \Delta u + k^2 u &= f(x,y,z), (x,y) \in \Omega,\ z \in [0,c], \quad (1.11)\\ u_z(x,y,0) &= g(x,y), (x,y) \in \Omega, \quad (1.12)\\ u(x,y,0) &= \varphi(x,y), (x,y) \in \Omega, \quad (1.13)\\ ßu(x,y,z) &= 0, (x,y) \in \partial\Omega,\ z \in [0,c], \quad (1.14) \end{aligned}$$

in which ß in (1.14) is the homogeneous Dirichlet boundary condition or the homogeneous Neumann boundary condition and $g, \varphi \in L^2(\mathbb{R}^2)$, $\Omega = (0,a) \times (0,b)$, $a, b, c > 0$. By employing a general filter regularization method, the authors solved the problem (1.11) - (1.14) and the authors regularized the problem by the combining quasi-boundary method and the truncated method. Then, the logarithmic type of the error estimate was given in [19].

In fact, in many practical applications, we cannot get exactly the wavenumber k because of the heterogeneity of the environment. Therefore, it is natural to consider the case of perturbation of the wavenumber k. Moreover, although there are many papers related to the modified Helmholtz equations associated with constant wavenumber, but the results for the perturbed wavenumber are still limited. Because of these above reasons, we consider the problem (1.1) - (1.3), which is the original case of the Helmholtz equation associated with a perturbed wavenumber k in an infinite rectangular parallelepiped $\{(x,y,z) \,|(x,y) \in \mathbb{R}^2, 0 < z < 1\}$. Then, we can obtain the error estimate of Hölder type for the case $z \in [0;1)$.

The rest of the paper is organized as follows. In Section 2, we give some basis knowledge which support for the proof of our mains results. Besides that, the Hölder type of the error estimate is also introduced by using the truncated method. Then, the numerical example is given to test the effectiveness of our method in Section 3. Finally, we have a conclusion in Section 4.

2 Main results

Definition 2.1 *Let $f(\cdot,\cdot) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we have the Fourier transform of the function f as follows*

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} f(x,y)\, e^{-i(\xi_1 x + \xi_2 y)} dx dy,\ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Next, we get some following lemmas which is useful to prove our main results.

Lemma 2.1 *For $\alpha \in (0,1)\,, b > 1$ and $0 < x \leq \frac{1}{b}\ln\frac{4}{\sqrt{\alpha}}$,the following inequalities hold*

$$\begin{aligned} i)\ 0 &< \left(1-\frac{\alpha}{4}e^{bx}\right)\cosh zx \leq 4^{\frac{z-b}{b}}\delta\left(b,\alpha,z\right)\alpha^{-\frac{z}{b}}, \\ ii)\ 0 &< \left(1-\frac{\alpha}{4}e^{bx}\right)\frac{\sinh zx}{x} \leq 4^{\frac{z-b}{b}}\delta\left(b,\alpha,z\right)\alpha^{-\frac{z}{b}}, \end{aligned}$$

where $\delta\left(b,\alpha,z\right) = (4-\alpha)^{\frac{b-z}{b}}\,, \forall z \in [0;1]\,.$

Proof. For $0 < x \leq \frac{4}{\alpha}$, we have $1-\frac{\alpha}{4}x \geq 0$. By applying the following inequality $uv \leq \left(\frac{u+v}{2}\right)^2, \forall u,v \geq 0,$ we have

$$\frac{\alpha}{4}x\left(1-\frac{\alpha}{4}x\right) \leq \left(\frac{\frac{\alpha}{4}x+1-\frac{\alpha}{4}x}{2}\right)^2,$$

then we obtain

$$x\left(1-\frac{\alpha}{4}x\right) \leq \alpha^{-1}, 0 < x \leq \frac{4}{\alpha}. \tag{2.1}$$

For $0 < x \leq \frac{1}{b}\ln\frac{4}{\sqrt{\alpha}}$ and $\alpha \in (0,1)\,,$ we get $1-\frac{\alpha}{4}e^{bx} > 0.$

By applying two inequalities $\cosh zx \leq e^{zx}; \frac{\sinh zx}{x} \leq e^{zx}$, we can get $\left(1-\frac{\alpha}{4}e^{bx}\right)\cosh zx \leq \left(1-\frac{\alpha}{4}e^{bx}\right)e^{zx}$ and $\left(1-\frac{\alpha}{4}e^{bx}\right)\frac{\sinh zx}{x} \leq \left(1-\frac{\alpha}{4}e^{bx}\right)e^{zx}.$

From (2.1), we get

$$\begin{aligned} \left(1-\frac{\alpha}{4}e^{bx}\right)\cosh zx &\leq \left(1-\frac{\alpha}{4}e^{bx}\right)e^{zx} \\ &= \left(1-\frac{\alpha}{4}e^{bx}\right)^{\frac{b-z}{b}}\left[e^{bx}\left(1-\frac{\alpha}{4}e^{bx}\right)\right]^{\frac{z}{b}} \\ &\leq 4^{\frac{z-b}{b}}\left(4-\alpha\right)^{\frac{b-z}{b}}\ \alpha^{-\frac{z}{b}}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \left(1-\frac{\alpha}{4}e^{bx}\right)\cosh zx &\leq 4^{\frac{z-b}{b}}\delta\left(b,\alpha,z\right)\alpha^{-\frac{z}{b}}, \\ \left(1-\frac{\alpha}{4}e^{bx}\right)\frac{\sinh zx}{x} &\leq 4^{\frac{z-b}{b}}\delta\left(b,\alpha,z\right)\alpha^{-\frac{z}{b}}, \end{aligned}$$

where $\delta\left(b,\alpha,z\right) = (4-\alpha)^{\frac{b-z}{b}}\,, \forall z \in [0;1]\,.$

The proof of Lemma 2.1 is completed. ∎

Lemma 2.2 *Let $b > a > 0$, $\varphi_1(x) = \cosh zx$ and $\psi_1(x) = \frac{\sinh zx}{x}$, $z \in [0;1]$. Then, we have the following inequalities*

$$\begin{aligned} \left|\cosh(zb)-\cosh(za)\right| &\leq (b-a)\cosh(zc_1), \\ \left|\frac{\sinh(zb)}{b}-\frac{\sinh(za)}{a}\right| &\leq (b-a)\frac{\sinh(zc_2)}{c_2}, \end{aligned}$$

in which $c_1, c_2 \in [a,b]$.

Proof. It is easy to see that φ_1 is continuous and derivative on $[a,b]$. By using the Lagrange's theorem, there exists $c_1 \in [a,b]$ satisfying

$$\begin{aligned} |\varphi_1(b) - \varphi_1(a)| &= z\,(b-a)\sinh(zc_1) \\ &\leq (b-a)\sinh(zc_1) \\ &\leq (b-a)\cosh(zc_1). \end{aligned}$$

Similarly, we also see that ψ_1 is continuous and derivative on $[a,b]$. By applying Lagrange's theorem again, we obtain $c_2 \in [a,b]$ such that

$$|\psi_1(b) - \psi_1(a)| = (b-a)\left(\frac{zc_2\cosh(zc_2) - \sinh(zc_2)}{c_2^2}\right). \tag{2.2}$$

Morever, let $h(x) = e^{2zx} - 2zx - 1, x \geq 0, z \in (0,1]$.
By simple calculation, we can check that h is an increase function in $(0,\infty)$.
From that, we have

$$2zc_2 + 1 \leq e^{2zc_2}.$$

Hence

$$zc_2 e^{-zc_2} \leq \frac{e^{zc_2} - e^{-zc_2}}{2}.$$

So that, we infer that

$$zc_2\left(\cosh(zc_2) - \sinh(zc_2)\right) \leq \sinh(zc_2).$$

Therefore

$$\frac{zc_2\cosh(zc_2) - \sinh(zc_2)}{c_2^2} \leq \frac{z\sinh(zc_2)}{c_2}. \tag{2.3}$$

From (2.2) and (2.3), we deduce that

$$|\psi(b) - \psi(a)| \leq (b-a)\frac{\sinh(zc_2)}{c_2}. \tag{2.4}$$

The proof of Lemma 2.1 is completed. ■

Lemma 2.3 *Let* $b > a > 0$, $\varphi_2(x) = \cos zx$ *and* $\psi_2(x) = \dfrac{\sin zx}{x}$, $z \in (0,1]$, *the following inequalities hold*

$$\begin{aligned} |\varphi_2(b) - \varphi_2(a)| &\leq (b-a)(z+1)\,b, \\ |\psi_2(b) - \psi_2(a)| &\leq (b-a)(z+1)\,b. \end{aligned}$$

Proof. It is obvious that φ_2 is continuous and derivative on $[a,b]$. Thus, we have $c_3 \in [a,b]$ such that

$$\begin{aligned} |\varphi_2(b) - \varphi_2(a)| &= (b-a)\left|-z\sin(zc_3)\right| \\ &\leq (b-a)\,z^2c_3 \\ &\leq (b-a)(z+1)b. \end{aligned}$$

It is similar to the function ψ_2, then we obtain $c_4 \in [a, b]$ satisfying

$$|\psi_2(b) - \psi_2(a)| = (b-a)\left|\frac{zc_4\cos(zc_4) - \sin(zc_4)}{c_4^2}\right|. \tag{2.5}$$

For $c_4 \geq 1$, we have

$$\begin{aligned} -(z+1)c_4 &\leq \frac{-zc_4^3 - c_4^3}{c_4^2} \\ &\leq \frac{zc_4\cos(zc_4) - \sin(zc_4)}{c_4^2} \\ &\leq \frac{zc_4^3 + c_4^3}{c_4^2} \\ &\leq (z+1)b. \end{aligned}$$

Thus, we get

$$\left|\frac{zc_4\cos(zc_4) - \sin(zc_4)}{c_4^2}\right| \leq (z+1)b. \tag{2.6}$$

For $c_4 < 1$, we obtain

$$\sin(zc_4) > 0, \cos(zc_4) > 0.$$

Let $q(x) = \sin(zx) - zx\cos(zx)$, $z \in [0;1]$, $0 < x < 1$. It is easy to prove that q is an increase function in $(0, \infty)$. So that, we obtain

$$\left|\frac{zc_4\cos(zc_4) - \sin(zc_4)}{c_4^2}\right| = \frac{\sin(zc_4) - zc_4\cos(zc_4)}{c_4^2}. \tag{2.7}$$

It can be shown that

$$\begin{aligned} &\frac{\sin(zc_4) - zc_4\cos(zc_4)}{c_4^2} \\ \leq\ &\frac{zc_4(1-\cos(zc_4))}{c_4^2} \\ =\ &\frac{z2\sin^2\left(\frac{zc_4}{2}\right)}{c_4} \\ \leq\ &(z+1)b. \end{aligned} \tag{2.8}$$

From (2.5), (2.6), (2.7) and (2.8), we have

$$|\psi_2(b) - \psi_2(a)| \leq (b-a)(z+1)b. \tag{2.9}$$

The proof of Lemma 2.3 is completed. ■

Without loss of generality, we can assume that $k_\varepsilon \geq k_{ex}$. For short notation, we let $\eta_\varepsilon = \sqrt{|\xi|^2 - k_\varepsilon^2}$, $\eta_{ex} = \sqrt{|\xi|^2 - k_{ex}^2}$, $\theta_\varepsilon = \sqrt{k_\varepsilon^2 - |\xi|^2}$ and $\theta_{ex} = \sqrt{k_{ex}^2 - |\xi|^2}$. By using the Fourier transform, we can clearly find out the form of the solution of (1.1) - (1.3) corresponding to (f_{ex}, g_{ex}, k_{ex}) and $(f_{ex}, g_{ex}, k_\varepsilon)$, respectively, in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times \mathbb{R}^+$ as follows

$$\begin{aligned}
&\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\xi, z)\\
= &\left[\widehat{g_{ex}}(\xi)\cosh(z\eta_{ex}) + \widehat{f_{ex}}(\xi)\frac{\sinh(z\eta_{ex})}{\eta_{ex}} + \int_0^z \widehat{S}(\xi,t)\frac{\sinh((z-t)\eta_{ex})}{\eta_{ex}}dt\right] \times \chi_{\mathbb{R}^2\backslash B_{k_{ex}}(O)}(\xi)\\
&+\left[\widehat{g_{ex}}(\xi)\cos(z\theta_{ex}) + \widehat{f_{ex}}(\xi)\frac{\sin(z\theta_{ex})}{\theta_{ex}} + \int_0^z \widehat{S}(\xi,t)\frac{\sin((z-t)\theta_{ex})}{\theta_{ex}}dt\right] \times \chi_{B_{k_{ex}}(O)}(\xi),
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\xi, z)\\
= &\left[\widehat{g_{ex}}(\xi)\cosh(z\eta_\varepsilon) + \widehat{f_{ex}}(\xi)\frac{\sinh(z\eta_\varepsilon)}{\eta_\varepsilon} + \int_0^z \widehat{S}(\xi,t)\frac{\sinh((z-t)\eta_\varepsilon)}{\eta_\varepsilon}dt\right] \times \chi_{\mathbb{R}^2\backslash B_{k_\varepsilon}(O)}(\xi)\\
&+\left[\widehat{g_{ex}}(\xi)\cos(z\theta_\varepsilon) + \widehat{f_{ex}}(\xi)\frac{\sin(z\theta_\varepsilon)}{\theta_\varepsilon} + \int_0^z \widehat{S}(\xi,t)\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}dt\right] \times \chi_{B_{k_\varepsilon}(O)}(\xi),
\end{aligned} \tag{2.11}$$

where $|\xi| = \sqrt{\xi_1^2+\xi_2^2}$, $B_{k_{ex}}(O) = \{\xi \in \mathbb{R}^2 |\, |\xi| < k_{ex}\}$ and $B_{k_\varepsilon}(O) = \{\xi \in \mathbb{R}^2 |\, |\xi| < k_\varepsilon\}$.

Let us remark that the terms $\cosh(z\eta_{ex})$ and $\sinh(z\eta_{ex})$ in (2.10) are unbounded in $\mathbb{R}^2$, i.e., these terms are the unstable causes. To approximate u^{ex}, we have to replace these terms by some better terms. In particular, the regularized solutions corresponding to $(f_\varepsilon, g_\varepsilon, k_\varepsilon)$ and $(f_{ex}, g_{ex}, k_\varepsilon)$ in $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times \mathbb{R}^+$ for (2.10) are given by applying the truncation method and replacing $\cosh(z\eta_{ex})$ and $\sinh(z\eta_{ex})$ by $\left(1 - \frac{\alpha(\varepsilon)}{4}e^{b\eta_\varepsilon}\right)\cosh(z\eta_{ex})$ and $\left(1 - \frac{\alpha(\varepsilon)}{4}e^{b\eta_\varepsilon}\right)\sinh(z\eta_{ex})$, respectively. From that, we have the regularized solutions as follows

$$\begin{aligned}
&\widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\xi, z)\\
= &\left[\widehat{g}_\varepsilon(\xi)\cos(z\theta_\varepsilon) + \widehat{f}_\varepsilon(\xi)\frac{\sin(z\theta_\varepsilon)}{\theta_\varepsilon} + \int_0^z \widehat{S}(\xi,t)\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}dt\right] \times \chi_{B_{k_\varepsilon}(O)}(\xi)\\
&+\left(1 - \frac{\alpha(\varepsilon)}{4}e^{b\eta_\varepsilon}\right)\left[\widehat{g}_\varepsilon(\xi)\cosh(z\eta_\varepsilon) + \widehat{f}_\varepsilon(\xi)\frac{\sinh(z\eta_\varepsilon)}{\eta_\varepsilon}\right] \times \chi_{V^{\beta_\varepsilon(\varepsilon)}_{k_\varepsilon}}(\xi)\\
&+\left(1 - \frac{\alpha(\varepsilon)}{4}e^{b\eta_\varepsilon}\right)\left[\int_0^z \widehat{S}(\xi,t)\frac{\sinh((z-t)\eta_\varepsilon)}{\eta_\varepsilon}dt\right] \times \chi_{V^{\beta_\varepsilon(\varepsilon)}_{k_\varepsilon}}(\xi),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\xi,z)\\
=\ &\left[\widehat{g_{ex}}(\xi)\cos(z\theta_\varepsilon)+\widehat{f_{ex}}(\xi)\frac{\sin(z\theta_\varepsilon)}{\theta_\varepsilon}+\int_0^z\widehat{S}(\xi,t)\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}dt\right]\times\chi_{B_{k_\varepsilon}(O)}(\xi)\\
&+\left(1-\frac{\alpha(\varepsilon)}{4}e^{b\eta_\varepsilon}\right)\left[\widehat{g_{ex}}(\xi)\cosh(z\eta_\varepsilon)+\widehat{f_{ex}}(\xi)\frac{\sinh(z\eta_\varepsilon)}{\eta_\varepsilon}\right]\times\chi_{V_{k_\varepsilon}^{\beta_\varepsilon(\varepsilon)}}(\xi)\\
&+\left(1-\frac{\alpha(\varepsilon)}{4}e^{b\eta_\varepsilon}\right)\left[\int_0^z\widehat{S}(\xi,t)\frac{\sinh((z-t)\eta_\varepsilon)}{\eta_\varepsilon}dt\right]\times\chi_{V_{k_\varepsilon}^{\beta_\varepsilon(\varepsilon)}}(\xi),
\end{aligned} \tag{2.13}$$

where $\alpha(\varepsilon),\beta(\varepsilon)$ are two regularized parameters which shall be chosen later and $V_{k_\varepsilon}^{\beta_\varepsilon(\varepsilon)}=\{\xi\in\mathbb{R}^2|k_\varepsilon<|\xi|\le\beta_\varepsilon(\varepsilon)\}$. For convenience, we denote $\alpha(\varepsilon)$ by α, $\beta_\varepsilon(\varepsilon)$ by β_ε, and $\|\cdot\|_{L^2(\mathbb{R}^2)}$ is the norm in $L^2(\mathbb{R}^2)$.

In this paper, we require some assumptions on the exact data (f_{ex},g_{ex},k_{ex}) as follows

(H_1) : Assume that there exists five positive numbers A,B,C,D,E such that

$$A\le k_{ex}\le B,\ \sup_{|\xi|\in[-B-1;B+1]}|\widehat{g_{ex}}(\xi)|\le C,\ \sup_{|\xi|\in[-B-1;B+1]}\left|\widehat{f_{ex}}(\xi_1,\xi_2)\right|\le D,$$

$$\sup_{|\xi|\in[-B-1;B+1]}\left(\int_0^1\left|\widehat{S}(\xi,t)\right|^2dt\right)\le E^2.$$

(H_2) : Assume that $e^{|\xi|}\widehat{g_{ex}}(\xi)$, $e^{|\xi|}\widehat{f_{ex}}(\xi)$, $e^{|\xi|}\int_0^1\left|\widehat{S}(\xi,t)\right|dt\in L^2(\mathbb{R}^2)$, i.e., there exists a positive number $F>0$ such that

$$\int_{\mathbb{R}^2}e^{2|\xi|}\left[|\widehat{g_{ex}}(\xi)|^2+\left|\widehat{f_{ex}}(\xi)\right|^2+\left|\int_0^1\left|\widehat{S}(\xi,t)\right|dt\right|^2\right]d\xi\le F^2.$$

In Theorem 2.1, we let (f_{ex},g_{ex},k_{ex}), $(f_\varepsilon,g_\varepsilon,k_\varepsilon)\in L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times\mathbb{R}^+$ be the exact data and the measured data, respectively such that

$$\|f_\varepsilon-f_{ex}\|_{L^2(\mathbb{R}^2)}\le\varepsilon,\|g_\varepsilon-g_{ex}\|_{L^2(\mathbb{R}^2)}\le\varepsilon,|k_\varepsilon-k_{ex}|\le\varepsilon.$$

Theorem 2.1 *Let* $\varepsilon\in(0,1),\alpha=\varepsilon,\ \beta_\varepsilon=\sqrt{\frac{1}{b^2}\ln^2\frac{4}{\sqrt{\alpha(\varepsilon)}}+k_\varepsilon^2},\ b>1$ *and* (f_{ex},g_{ex},k_{ex}) *satisfy the condition* (H_1) *and* (H_2). *Assume that* $u^{ex}_{(f_{ex},g_{ex},k_{ex})}$, $u^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$ *are* *defined by (2.10) and (2.12), respectively, then we obtain the following error estimate*

$$\left\| u^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z) \right\|_{L^2(\mathbb{R}^2)} \le \varepsilon^{\frac{1-z}{2b}} N, \tag{2.14}$$

where $z \in [0,1)$ *and*

$$\begin{aligned} N &= 8\sqrt{2} + M + 2\sqrt{3}F, \\ M &= \sqrt{3}F + \sqrt{3\,(2B+1)}\Big[F + \left(1+e^{B+1}\right)(\,C+D+E)\sqrt{\pi} \\ &\quad +2\,(B+1)\left(\|g_{ex}\|_{L^2(\mathbb{R}^2)} + \|f_{ex}\|_{L^2(\mathbb{R}^2)} + EB\pi\right)\Big]. \end{aligned}$$

Proof. By applying the triangle inequality, we have

$$\begin{aligned} & \left\| u^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z) \right\|_{L^2(\mathbb{R}^2)} \\ = & \left\| \widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z) \right\|_{L^2(\mathbb{R}^2)} \\ \le & \left\| \widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - \widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) \right\|_{L^2(\mathbb{R}^2)} + \left\| \widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) \right\|_{L^2(\mathbb{R}^2)} \\ & + \left\| \widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z) \right\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{2.15}$$

Firstly, from (2.12) and (2.13), we get

$$\begin{aligned} & \left\| \widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - \widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) \right\|^2_{L^2(\mathbb{R}^2)} \\ = & \int\limits_{V^{\beta_\varepsilon}_{k_\varepsilon}} \left| \left(1 - \frac{\alpha}{4} e^{b\eta_\varepsilon}\right) \left[\cosh(z\eta_\varepsilon)\left(\widehat{g_\varepsilon}(\xi) - \widehat{g_{ex}}(\xi)\right) + \frac{\sinh(z\eta_\varepsilon)}{\eta_\varepsilon}\left(\widehat{f_\varepsilon}(\xi) - \widehat{f_{ex}}(\xi)\right)\right]\right|^2 d\xi \\ & + \int\limits_{B_{k_\varepsilon}(O)} \left| \left(\widehat{g_\varepsilon}(\xi) - \widehat{g_{ex}}(\xi)\right)\cos(z\theta_\varepsilon) + \frac{\sin(z\theta_\varepsilon)}{\theta_\varepsilon}\left(\widehat{f_\varepsilon}(\xi) - \widehat{f_{ex}}(\xi)\right)\right|^2 d\xi. \end{aligned}$$

By applying Lemma 2.1, we have

$$\begin{aligned} & \left\| \widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - \widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z) \right\|^2_{L^2(\mathbb{R}^2)} \\ \le & \; 4^{2\frac{z-b}{b}} \delta^2(b,\alpha,z) \alpha^{-2\frac{z}{b}} \int\limits_{V^{\beta_\varepsilon}_{k_\varepsilon}} \left[\left|\widehat{g_\varepsilon}(\xi) - \widehat{g_{ex}}(\xi)\right| + \left|\widehat{f_\varepsilon}(\xi) - \widehat{f_{ex}}(\xi)\right| \right]^2 d\xi \\ & + \int\limits_{B_{k_\varepsilon}(O)} \left[\left|\widehat{g_\varepsilon}(\xi) - \widehat{g_{ex}}(\xi)\right| + \left|\widehat{f_\varepsilon}(\xi) - \widehat{f_{ex}}(\xi)\right| \right]^2 d\xi \\ \le & \; 2\left(4^{2\frac{z-b}{b}} \delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}} + 1\right)\left(\|g_\varepsilon - g_{ex}\|^2_{L^2(\mathbb{R}^2)} + \|f_\varepsilon - f_{ex}\|^2_{L^2(\mathbb{R}^2)}\right) \\ = & \; 2\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}}\left(4^{2\frac{z-b}{b}} + \alpha^{2\frac{z}{b}}(4-\alpha)^{2\frac{z-b}{b}}\right)\left(\|g_\varepsilon - g_{ex}\|^2_{L^2(\mathbb{R}^2)} + \|f_\varepsilon - f_{ex}\|^2_{L^2(\mathbb{R}^2)}\right). \end{aligned}$$

Hence, we get

$$
\begin{aligned}
&\left\|\widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-\widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\\
\leq\ & 4\delta^2(b,\alpha,z)\alpha^{-2\frac{z}{b}}\left(\|g_\varepsilon-g_{ex}\|^2_{L^2(\mathbb{R}^2)}+\|f_\varepsilon-f_{ex}\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned}
$$

Then, we obtain

$$
\left\|\widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-\widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\right\|_{L^2(\mathbb{R}^2)}\leq 8\sqrt{2}\alpha^{-\frac{z}{b}}\varepsilon. \tag{2.16}
$$

From (2.11), (2.13), we deduce that

$$
\begin{aligned}
&\left\|\widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\\
=&\int\limits_{V^{\beta_\varepsilon}_{k_\varepsilon}}\left|\frac{\alpha}{4}e^{b\eta_\varepsilon}\left[\widehat{g_{ex}}(\xi)\cosh(z\eta_\varepsilon)+\frac{\sinh(z\eta_\varepsilon)}{\eta_\varepsilon}\widehat{f_{ex}}(\xi)+\int_0^z\widehat{S}(\xi,t)\frac{\sinh((z-t)\eta_\varepsilon)}{\eta_\varepsilon}dt\right]\right|^2d\xi\\
&+\int\limits_{\mathbb{R}^2\setminus B_{\beta_\varepsilon}(O)}\left|\widehat{g_{ex}}(\xi)\cosh(z\eta_\varepsilon)+\frac{\sinh(z\eta_\varepsilon)}{\eta_\varepsilon}\widehat{f_{ex}}(\xi)+\int_0^z\widehat{S}(\xi,t)\frac{\sinh((z-t)\eta_\varepsilon)}{\eta_\varepsilon}dt\right|^2d\xi.
\end{aligned}
$$

From $\beta_\varepsilon=\sqrt{\frac{1}{b^2}\ln^2\frac{4}{\sqrt{\alpha}}+k_\varepsilon^2}$, we get some inequalities as follows

$$
|\xi|^2-k_\varepsilon^2>\frac{1}{b^2}\ln^2\frac{4}{\sqrt{\alpha}},\forall|\xi|>\beta_\varepsilon, \tag{2.17}
$$

$$
\eta_\varepsilon=\sqrt{|\xi|^2-k_\varepsilon^2}<\frac{1}{b}\ln\frac{4}{\sqrt{\alpha}},\forall|\xi|<_{\beta_\varepsilon}. \tag{2.18}
$$

From (H_2), (2.17), (2.18) and by using some basic inequalities $\cosh zx\leq e^{zx}$, $\frac{\sinh zx}{x}\leq e^{zx}$, $\frac{\cosh ax}{\cosh bx}\leq 2e^{(a-b)x}$, $\frac{\sinh ax}{\sinh bx}\leq 2e^{(a-b)x}$, where $x>0$, $b>a>0$, $0\leq z<1$, we get

$$
\begin{aligned}
&\left\|\widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\\
\leq\ &\frac{\alpha^2}{16}e^{2b\frac{1}{b}\ln\frac{4}{\sqrt{\alpha}}}\int\limits_{V^{\beta_\varepsilon}_{k_\varepsilon}}e^{2|\xi|}\left[|\widehat{g_{ex}}(\xi)|+\left|\widehat{f_{ex}}(\xi)\right|+\int_0^1\left|\widehat{S}(\xi,t)\right|dt\right]^2d\xi\\
&+\int\limits_{\mathbb{R}^2\setminus B_{\beta_\varepsilon}(O)}4e^{2(z-1)\eta_\varepsilon}e^{2|\xi|}\left[|\widehat{g_{ex}}(\xi)|+\left|\widehat{f_{ex}}(\xi)\right|+\int_0^1\left|\widehat{S}(\xi,t)\right|dt\right]^2d\xi\\
\leq\ &3\alpha F^2+\ 12e^{2(z-1)\frac{1}{b}\ln\frac{4}{\sqrt{\alpha}}}F^2\\
\leq\ &3\alpha F^2+\ 12\alpha^{(1-z)/b}F^2.
\end{aligned}
$$

This implies that

$$\left\|\widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)\right\|_{L^2(\mathbb{R}^2)}\le\sqrt{3\alpha}F+2\sqrt{3}\sqrt{\alpha}^{(1-z)/b}F. \tag{2.19}$$

From (2.10) and (2.11), we estimate

$$\left\|\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}=I_1(z)+I_2(z)+I_3(z),$$

in which

$$\begin{aligned}
I_1(z) &= \int\limits_{\mathbb{R}^2\setminus B_{k_\varepsilon}(O)}\Bigg|\widehat{g_{ex}}(\xi)\left(\varphi_1(\eta_\varepsilon)-\varphi_1(\eta_{ex})\right)+\widehat{f_{ex}}(\xi)\left(\psi_1(\eta_\varepsilon)-\psi_1(\eta_{ex})\right)\\
&\quad+\int_0^z\widehat{S}(\xi,t)\left(\frac{\sinh((z-t)\eta_\varepsilon)}{\eta_\varepsilon}-\frac{\sinh((z-t)\eta_{ex})}{\eta_{ex}}\right)dt\Bigg|^2 d\xi,\\
I_2(z) &= \int\limits_{V^{k_\varepsilon}_{k_{ex}}}\Bigg|\widehat{g_{ex}}(\xi)\left(\varphi_2(\theta_\varepsilon)-\varphi_1(\eta_{ex})\right)+\widehat{f_{ex}}(\xi)\left(\psi_2(\theta_\varepsilon)-\psi_1(\eta_{ex})\right)\\
&\quad+\int_0^z\widehat{S}(\xi,t)\left(\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}-\frac{\sinh((z-t)\eta_{ex})}{\eta_{ex}}\right)dt\Bigg|^2 d\xi,\\
I_3(z) &= \int\limits_{B_{k_{ex}}(O)}\Bigg|\widehat{g_{ex}}(\xi)\left(\varphi_2(\theta_\varepsilon)-\varphi_2(\theta_{ex})\right)+\widehat{f_{ex}}(\xi)\left(\psi_2(\theta_\varepsilon)-\psi_2(\theta_{ex})\right)\\
&\quad+\int_0^z\widehat{S}(\xi,t)\left(\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}-\frac{\sin((z-t)\theta_{ex})}{\theta_{ex}}\right)dt\Bigg|^2 d\xi.
\end{aligned}$$

By using (H_2), the inequalities $\cosh zx\le e^{zx}$, $\dfrac{\sinh zx}{x}\le e^{zx}$ and $(a+b+c)^2\le 3(a^2+b^2+c^2)$ where $x>0$, $0\le z\le 1$ and Lemma 2.1, we obtain $k_1,k_2,k_3\in[k_{ex},k_\varepsilon]$ such that

$$\begin{aligned}
I_1(z) &\le (k_\varepsilon^2-k_{ex}^2)\int\limits_{\mathbb{R}^2\setminus B_{k_\varepsilon}(O)}\left[\left|\widehat{g_{ex}}(\xi)\right|\cosh\left(z\sqrt{|\xi|^2-k_1^2}\right)+\left|\widehat{f_{ex}}(\xi)\right|\frac{\sinh\left(z\sqrt{|\xi|^2-k_2^2}\right)}{\sqrt{|\xi|^2-k_2^2}}\right.\\
&\quad\left.+\int_0^z\left|\widehat{S}(\xi,t)\right|\frac{\sinh\left((z-t)\sqrt{|\xi|^2-k_3^2}\right)}{\sqrt{|\xi|^2-k_3^2}}dt\right]^2 d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq (k_\varepsilon^2 - k_{ex}^2) \int\limits_{\mathbb{R}^2 \setminus B_{k_\varepsilon}(O)} e^{2|\xi|} \left[|\widehat{g_{ex}}(\xi)| + \left| \widehat{f_{ex}}(\xi) \right| + \int_0^1 \left| \widehat{S}(\xi,t) \right| dt \right]^2 d\xi \\
&\leq 3\varepsilon (k_\varepsilon + k_{ex}) F^2 \\
&\leq 3\varepsilon (2B+1) F^2, \qquad (2.20)
\end{aligned}
$$

in which

$$
\int\limits_{\mathbb{R}^2} e^{2|\xi|} \left[|\widehat{g_{ex}}(\xi)|^2 + \left| \widehat{f_{ex}}(\xi) \right|^2 + \left| \int_0^1 \left| \widehat{S}(\xi,t) \right| dt \right|^2 \right] d\xi \leq F^2.
$$

For $|\xi| \in (k_{ex}, k_\varepsilon)$ and $z \in (0,1]$, we get

$$
\begin{aligned}
-1 - \cosh(z\eta_{ex}) &\leq \cos(z\theta_\varepsilon) - \cosh(z\eta_{ex}) \\
&\leq 1 + \cosh(z\eta_{ex}),
\end{aligned}
$$

then, we infer that $|\cos(z\theta_\varepsilon) - \cosh(z\eta_{ex})| \leq 1 + \cosh(z\eta_{ex})$.

This implies that

$$
|\cos(z\theta_\varepsilon) - \cosh(z\eta_{ex})| \leq 1 + e^{z\sqrt{|\xi|^2 - k_{ex}^2}} \leq 1 + e^{z\sqrt{k_\varepsilon^2 - k_{ex}^2}}. \qquad (2.21)
$$

On the other hand, we have

$$
\begin{aligned}
-1 - \frac{\sinh(z\eta_{ex})}{\eta_{ex}} &\leq \frac{\sin(z\theta_\varepsilon)}{\theta_\varepsilon} - \frac{\sinh(z\eta_{ex})}{\eta_{ex}} \\
&\leq 1 - \frac{\sinh(z\eta_{ex})}{\eta_{ex}} \\
&\leq 1 + \frac{\sinh(z\eta_{ex})}{\eta_{ex}}.
\end{aligned}
$$

Hence, we obtain

$$
\left| \frac{\sin(z\theta_\varepsilon)}{\theta_\varepsilon} - \frac{\sinh(z\eta_{ex})}{\eta_{ex}} \right| \leq 1 + e^{z\sqrt{|\xi|^2 - k_{ex}^2}} \leq 1 + e^{z\sqrt{k_\varepsilon^2 - k_{ex}^2}}. \qquad (2.22)
$$

Alternatively, we have

$$
\left| \frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon} - \frac{\sinh((z-t)\eta_{ex})}{\eta_{ex}} \right| \leq z - t + e^{(z-t)\eta_{ex}} \leq 1 + e^{z\eta_{ex}} \leq 1 + e^{z\sqrt{k_\varepsilon^2 - k_{ex}^2}}. \qquad (2.23)
$$

From (2.21), (2.22), (2.23) and the condition (H_1), we get

$$\begin{aligned}
I_2(z) &\leq 3\Bigg[\left(1+e^{z\sqrt{k_\varepsilon^2-k_{ex}^2}}\right)^2\left(\int\limits_{V_{k_{ex}}^{k_\varepsilon}}|\widehat{g_{ex}}(\xi)|^2 d\xi+\int\limits_{V_{k_{ex}}^{k_\varepsilon}}\left|\widehat{f_{ex}}(\xi)\right|^2 d\xi\right)\\
&\quad+\int\limits_{V_{k_{ex}}^{k_\varepsilon}}\left|\int\limits_0^z\widehat{S}(\xi,t)\left(\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}-\frac{\sinh((z-t)\eta_{ex})}{\eta_{ex}}\right)dt\right|^2 d\xi\Bigg]\\
&\leq 3\left(1+e^{z\sqrt{k_\varepsilon^2-k_{ex}^2}}\right)^2(k_\varepsilon^2-k_{ex}^2)\pi\left(C^2+D^2+E^2\right)\\
&\leq 3\left(1+e^{B+1}\right)^2\left(C^2+D^2+E^2\right)\varepsilon(2B+1)\pi. \qquad (2.24)
\end{aligned}$$

By using Lemma 2.3, we obtain

$$\begin{aligned}
I_3(z) &\leq 3\Bigg[(z+1)^2\left(k_\varepsilon^2-k_{ex}^2\right)k_\varepsilon^2\left(\|\widehat{g_{ex}}\|^2_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\quad+\int\limits_{B_{k_{ex}}(O)}\left|\int\limits_0^z\widehat{S}(\xi,t)\left(\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}-\frac{\sin((z-t)\theta_{ex})}{\theta_{ex}}\right)dt\right|^2 d\xi\Bigg]\\
&\leq 3\Bigg[(z+1)^2\left(k_\varepsilon^2-k_{ex}^2\right)k_\varepsilon^2\left(\|\widehat{g_{ex}}\|^2_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\quad+\int\limits_{B_{k_{ex}}(O)}\left(\int\limits_0^z\left|\widehat{S}(\xi,t)\right|^2\left|\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}-\frac{\sin((z-t)\theta_{ex})}{\theta_{ex}}\right|^2 dt\right)d\xi\Bigg].
\end{aligned}$$

Alternatively, we estimate

$$\left|\frac{\sin((z-t)\theta_\varepsilon)}{\theta_\varepsilon}-\frac{\sin((z-t)\theta_{ex})}{\theta_{ex}}\right|\leq\sqrt{k_\varepsilon^2-k_{ex}^2}\,(z-t+1)\,k_\varepsilon\leq\sqrt{k_\varepsilon^2-k_{ex}^2}\,(z+1)\,k_\varepsilon.$$

Hence, we obtain

$$\begin{aligned}
I_3(z) &\leq 3\Bigg[(z+1)^2\left(k_\varepsilon^2-k_{ex}^2\right)k_\varepsilon^2\left(\|\widehat{g_{ex}}\|^2_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\quad+(z+1)^2\left(k_\varepsilon^2-k_{ex}^2\right)k_\varepsilon^2E^2\int\limits_{\mathbb{R}^2}\chi_{B_{k_{ex}}(O)}(\xi)d\xi\Bigg]\\
&\leq 12\varepsilon(k_\varepsilon+k_{ex})(B+1)^2\left(\|\widehat{g_{ex}}\|^2_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|^2_{L^2(\mathbb{R}^2)}+E^2k_{ex}^2\pi\right)\\
&\leq 12\varepsilon(2B+1)(B+1)^2\left(\|\widehat{g_{ex}}\|^2_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|^2_{L^2(\mathbb{R}^2)}+E^2B^2\pi\right). \qquad (2.25)
\end{aligned}$$

From (2.20), (2.24) and (2.25), we deduce that

$$\begin{aligned}
&\left\|\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\\
\leq\;& 3\varepsilon(2B+1)F^2+3\left(1+e^{B+1}\right)^2\left(\;C^2+D^2+\;E^2\right)\varepsilon(2B+1)\pi\\
&+12\varepsilon(2B+1)\left(B+1\right)^2\left(\left\|\widehat{g_{ex}}\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|^2_{L^2(\mathbb{R}^2)}+E^2B^2\pi\right).
\end{aligned}$$

Thus, we infer

$$\begin{aligned}
&\left\|\widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,z)-\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\right\|_{L^2(\mathbb{R}^2)}\\
\leq\;& \sqrt{3\varepsilon\left(2B+1\right)}\left[F+\left(1+e^{B+1}\right)\left(\;C+D+E\right)\sqrt{\pi}\right.\\
&\left.+2\left(B+1\right)\left(\left\|\widehat{g_{ex}}\right\|_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|_{L^2(\mathbb{R}^2)}+EB\pi\right)\right].
\end{aligned}\tag{2.26}$$

From (2.16), (2.19) and (2.26), we have

$$\begin{aligned}
&\left\|u^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,\cdot,z)-u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,\cdot,z)\right\|_{L^2(\mathbb{R}^2)}\\
\leq\;& 8\sqrt{2}\alpha^{-\frac{z}{b}}\varepsilon+\sqrt{3\alpha}F+2\sqrt{3}\sqrt{\alpha}^{(1-z)/b}F\\
&+\sqrt{3\varepsilon\left(2B+1\right)}\left[F+\left(1+e^{B+1}\right)\left(\;C+D+E\right)\sqrt{\pi}\;.\right.\\
&\left.+2\left(B+1\right)\left(\left\|\widehat{g_{ex}}\right\|_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|_{L^2(\mathbb{R}^2)}+EB\pi\right)\right].
\end{aligned}$$

Choosing $\alpha=\varepsilon, z\in(0,1)$, we get

$$\left\|u^{\varepsilon,b}_{((f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z)-u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z)\right\|_{L^2(\mathbb{R}^2)}\leq 8\sqrt{2}\varepsilon^{\frac{b-z}{b}}+\sqrt{\varepsilon}M+\sqrt{\varepsilon}^{(1-z)/b}2\sqrt{3}F\leq\varepsilon^{\frac{1-z}{2b}}N,$$

where

$$\begin{aligned}
M\;&=\;\sqrt{3}F+\sqrt{3\left(2B+1\right)}\left[F+\left(1+e^{B+1}\right)\left(\;C+D+E\right)\sqrt{\pi}\right.\\
&\quad\left.+2\left(B+1\right)\left(\left\|\widehat{g_{ex}}\right\|_{L^2(\mathbb{R}^2)}+\left\|\widehat{f_{ex}}\right\|_{L^2(\mathbb{R}^2)}+EB\pi\right)\right],\\
N\;&=\;8\sqrt{2}+M+2\sqrt{3}F.
\end{aligned}$$

The proof of Lemma 2.1 is completed. ■

Remark 1 *From the proof of Theorem 2.1, we prove that for $z\in[0;1)$, the regularized solution $u^{\varepsilon,a}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$ converges to $u^{ex}_{(f_{ex},g_{ex},k_{ex})}$ when ε tends to zero. However, it is difficult to derive its approximation at $z=1$. So that, we need an adjustment in condition (H_2) in order to get the stable estimate at $z=1$. In particular, we require $e^{(1+\gamma)|\xi|}\widehat{g_{ex}}$, $e^{(1+\gamma)|\xi|}\widehat{f_{ex}}$, $e^{(1+\gamma)|\xi|}\int_0^1\left|\widehat{S}(\xi,t)\right|dt\in L^2(\mathbb{R}^2)$, i.e.,*

$$\int_{\mathbb{R}^2\setminus B_{\beta_\varepsilon}(O)}e^{2(1+\gamma)|\xi|}\left[\left|\widehat{g_{ex}}(\xi)\right|+\left|\widehat{f_{ex}}(\xi)\right|+\int_0^1\left|\widehat{S}(\xi,t)\right|dt\right]d\xi\leq K^2,$$

where γ is chosen such that $0 < \gamma < \min\{2b-2; b\}$.

Under these above assumptions, we have the following estimate at $z = 1$

$$\begin{aligned}
&\left\|\widehat{u}^{\varepsilon,b}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,1) - \widehat{u}^{ex}_{(f_{ex},g_{ex},k_\varepsilon)}(\cdot,1)\right\|^2_{L^2(\mathbb{R}^2)} \\
\leq\ & 3\alpha K^2 + \int\limits_{\mathbb{R}^2\setminus B_{\beta_\varepsilon}(O)} 4e^{-2\gamma\eta_\varepsilon} e^{2(1+\gamma)|\xi|} \left[|\widehat{g_{ex}}(\xi)| + \left|\widehat{f_{ex}}(\xi)\right| + \int\limits_0^1 \left|\widehat{S}(\xi,t)\right| dt \right]^2 d\xi \\
\leq\ & 3K^2\left(4\alpha^{\gamma/b} + \alpha\right).
\end{aligned}$$

Chosing $\alpha = \varepsilon$, we get

$$\begin{aligned}
&\left\|u^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,1) - u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,1)\right\|_{L^2(\mathbb{R}^2)} \\
\leq\ & 8\sqrt{2}\varepsilon^{\frac{b-1}{b}} + \sqrt{\varepsilon}M' + 2\sqrt{3}K\sqrt{\varepsilon}^{\gamma/b} \\
\leq\ & N'\sqrt{\varepsilon}^{\gamma/b},
\end{aligned}$$

where

$$\begin{aligned}
N' &= 8\sqrt{2} + M' + 2\sqrt{3}K, \\
M' &= \sqrt{3}K + \sqrt{3(2B+1)}\left[K + \left(1+e^{B+1}\right)(C+D+E)\sqrt{\pi}\right. \\
&\quad \left. +2(B+1)\left(\|\widehat{g_{ex}}\|_{L^2(\mathbb{R}^2)} + \left\|\widehat{f_{ex}}\right\|_{L^2(\mathbb{R}^2)} + EB\pi\right)\right].
\end{aligned}$$

3 Numerical experiment

In this section, we deal with the following Cauchy problem of the inhomogeneous Helmholtz equation.

$$\Delta u(x,y,z) + u(x,y,z) = S(x,y,z), (x,y) \in \mathbb{R}^2, z \in (0;1], \tag{3.1}$$

$$u_z(x,y,0) = f_{ex}(x,y), (x,y) \in \mathbb{R}^2, \tag{3.2}$$

$$u(x,y,0) = g_{ex}(x,y), (x,y) \in \mathbb{R}^2, \tag{3.3}$$

where $f_{ex}, g_{ex} \in L^2(\mathbb{R}^2)$, $S \in L^2(\mathbb{R}^2 \times (0,1))$ are given by $f_{ex}(x,y) = g_{ex}(x,y) = e^{-\frac{1}{4}\left(x^2+y^2\right)}$ and $S(x,y,z) = z.e^{-\frac{1}{4}\left(x^2+y^2\right)}$, then we have

$$\widehat{f_{ex}}(\xi) = \widehat{g_{ex}}(\xi) = \frac{e^{-\xi_1^2}}{\sqrt{2.\frac{1}{4}}}\frac{e^{-\xi_2^2}}{\sqrt{2.\frac{1}{4}}} = 2e^{-\left(\xi_1^2+\xi_2^2\right)} = 2e^{-|\xi|^2}, \tag{3.4}$$

$$\widehat{S}(\xi,z) = 2ze^{-\left(\xi_1^2+\xi_2^2\right)} = 2ze^{-|\xi|^2},$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2$, $\xi \in \mathbb{R}^2$ and $k_{ex} = 1, z \in (0,1]$.

On the other words, we consider the measured data $(f_\varepsilon, g_\varepsilon, k_\varepsilon)$ satisfying

$$\begin{aligned} f_\varepsilon(x,y) &= \left(\frac{\varepsilon\cdot \mathrm{rand}(\cdot)}{\sqrt{2\pi}}+1\right) f_{ex}(x,y),\\ g_\varepsilon(x,y) &= \left(\frac{\varepsilon\cdot \mathrm{rand}(\cdot)}{\sqrt{2\pi}}+1\right) g_{ex}(x,y),\\ k_\varepsilon &= k_{ex} + \varepsilon\cdot \mathrm{rand}(\cdot), \end{aligned} \tag{3.5a}$$

where $\varepsilon \in [0;1]$, the term $\mathrm{rand}(\cdot)$ is the random noise which is uniformly determined on $[-1,1]$.

From (3.4) and (3.5a), we get

$$\begin{aligned} \|f_\varepsilon - f_{ex}\|_{L^2(\mathbb{R}^2)} &= \|g_\varepsilon - g_{ex}\|_{L^2(\mathbb{R}^2)}\\ &= \left\|\widehat{f_\varepsilon} - \widehat{f_{ex}}\right\|_{L^2(\mathbb{R}^2)}\\ &= \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{2\left(\varepsilon\cdot \mathrm{rand}(\cdot)\right)^2}{\pi} e^{-2\left(\xi_1^2+\xi_2^2\right)} d\xi_1 d\xi_2\right]^{\frac{1}{2}} \le \varepsilon,\\ |k_\varepsilon - k_{ex}| &\le \varepsilon. \end{aligned}$$

Morever, from (3.4), we have the following exact solution of problem (3.1)-(3.3)

$$\begin{aligned} &\widehat{u}^{ex}_{(f_{ex},g_{ex},k_{ex})}(\xi,z)\\ =\;& 2e^{-|\xi|^2}\left[\cosh(z\eta_{ex}) + \frac{\sinh(z\eta_{ex})}{\eta_{ex}} + \int_0^z \frac{t.\sinh\left((z-t)\,\eta_{ex}\right)}{\eta_{ex}}dt\right] \times \chi_{\mathbb{R}^2\setminus B_{k_{ex}}(O)}(\xi)\\ &+2e^{-|\xi|^2}\left[\cos(z\theta_{ex}) + \frac{\sin(z\theta_{ex})}{\theta_{ex}} + \int_0^z \frac{t.\sin\left((z-t)\,\eta_{ex}\right)}{\eta_{ex}}dt\right] \times \chi_{B_{k_{ex}}(O)}(\xi). \end{aligned}$$

Let $\alpha=\varepsilon$, $b=2$ and $\beta_\varepsilon = \sqrt{\frac{1}{2^2}\ln^2\left(\frac{4}{\sqrt{\varepsilon}}\right)+k_\varepsilon^2}$, we get the following regularized solution

$$\begin{aligned} &\widehat{u}^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\xi,z)\\ =\;& 2e^{-|\xi|^2}\left(1-\frac{\varepsilon}{4}e^{2\eta_\varepsilon}\right)\left(\frac{\varepsilon\cdot \mathrm{rand}(\cdot)}{\sqrt{2\pi}}+1\right)\left(\cosh(z\eta_\varepsilon) + \frac{\sinh(z\eta_\varepsilon)}{\eta_\varepsilon}\right) \times \chi_{V^{\beta_\varepsilon}_{k_\varepsilon}}(\xi)\\ &+2e^{-|\xi|^2}\left(1-\frac{\varepsilon}{4}e^{2\eta_\varepsilon}\right)\int_0^z \frac{t.\sinh\left((z-t)\,\eta_\varepsilon\right)}{\eta_\varepsilon}dt \times \chi_{V^{\beta_\varepsilon}_{k_\varepsilon}}(\xi)\\ &+2e^{-|\xi|^2}\left[\left(\frac{\varepsilon\cdot \mathrm{rand}(\cdot)}{\sqrt{2\pi}}+1\right)\left(\cos(z\theta_\varepsilon) + \frac{\sin(z\theta_\varepsilon)}{\theta_\varepsilon}\right) + \int_0^z \frac{t.\sin\left((z-t)\,\theta_\varepsilon\right)}{\theta_\varepsilon}dt\right] \times \chi_{B_{k_\varepsilon}(O)}(\xi), \end{aligned}$$

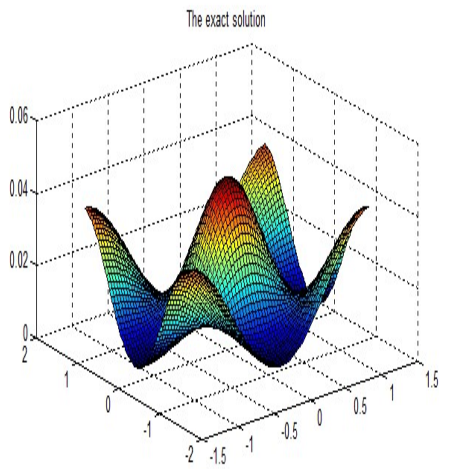
where $V_{k_\varepsilon}^{\beta_\varepsilon} = \{\xi \in \mathbb{R}^2 | k_\varepsilon < |\xi| < \beta_\varepsilon\}$, $B_{k_\varepsilon}(O) = \{\xi \in \mathbb{R}^2 | \, |\xi| < k_\varepsilon\}$ and $B_{k_{ex}}(O) = \{\xi \in \mathbb{R}^2 | \, |\xi| < k_{ex}\}$.

Let ε be $\varepsilon_1 = 10^{-1}, \varepsilon_2 = 10^{-2}, \varepsilon_3 = 10^{-3}, \varepsilon_4 = 10^{-4}$, respectively and $z \in \{0; 0.5; 1\}$. The error estimates between the exact solution and the regularized solutions is given through the following table

	$\left\| u^{\varepsilon,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}(\cdot,z) - u^{ex}_{(f_{ex},g_{ex},k_{ex})}(\cdot,z) \right\|_{L^2(\mathbb{R}^2)}$			
z	$\varepsilon_1 = 10^{-1}$	$\varepsilon_2 = 10^{-2}$	$\varepsilon_3 = 10^{-3}$	$\varepsilon_4 = 10^{-4}$
0	5.1025×10^{-1}	6.3125×10^{-2}	6.3567×10^{-3}	5.3154×10^{-3}
0.5	6.4789×10^{-1}	7.2456×10^{-2}	8.1253×10^{-3}	6.3458×10^{-3}
1	8.2459×10^{-1}	5.1344×10^{-1}	9.6926×10^{-2}	8.3158×10^{-3}

Table 1: The error between the exact solution and the regularized solution.

By applying FFT technique, we reconstruct discrete points of the exact solution $u^{ex}_{(f_{ex},g_{ex},k_{ex})}$ and the regularized solutions $u^{\varepsilon_i,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$, $i = 1, 2, 4$, corresponding to $z = 0.5$ then plot the graphs of these solutions in Figure 1 - Figure 2.



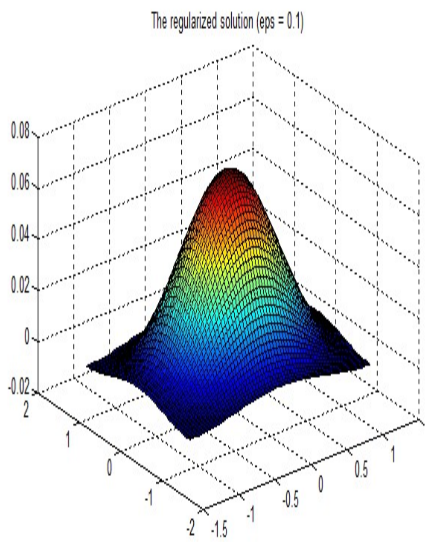


Figure 1: The exact solution $u^{ex}_{(f_{ex},g_{ex},k_{ex})}$ and the regularized solution $u^{\varepsilon_1,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$.

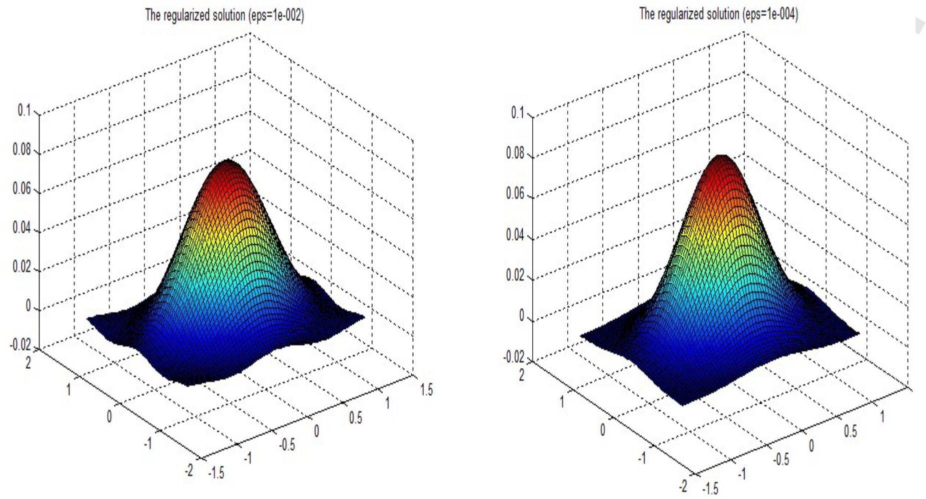


Figure 2: The regularized solutions $u^{\varepsilon_2,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$ and $u^{\varepsilon_4,b}_{(f_\varepsilon,g_\varepsilon,k_\varepsilon)}$.

4 Conclusion

In this article, we propose a regularization method based on the cut-off method for solving a three dimensional Cauchy problem for inhomogeneous Helmholtz equation with inhomogenous Cauchy conditions given at $z = 0$, associated with perturbed wave number k in an infinite rectangular parallelepiped. As a result, the Holder type of the error estimate between the exact solution and its regularized solution is also obtained. Eventually, the numerical experiment is carried out to illustrate the efficiency of our method. In the future, we will consider the problem (1.1) - (1.3) with the global and local Lipschitz source S.

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References

[1] D. E. Beskos, (1997), *Boundary element method in dynamic analysis: Part II (1986 - 1996)*, Applied Mechanics Review 50 (1997), 149-197.

[2] J. T. Chen and F. C. Wong, (1998), *Dual formulation of multiple reciprocity method for the acoustic mode of a cavity with a thin partition*, Journal of Sound and Vibration 217 (1998), 75-95.

[3] H. W. Cheng, J. Huang and T. J. Leiterman, (2006), *An adaptive fast solver for the modifed Helmholtz equation in two dimensions,* J. Comput. Phys. 211 (2006), 616-637.

[4] X. -L. Feng, C. -L. Fu, H. Cheng, (2011), *A regularization method for solving the Cauchy problem for the Helmholtz equation*, Applied Mathematical Modelling 35 (2011) 3301-3315.

[5] W. S. Hall, X. Q. Mao, (1995), *Boundary element investigation of irregular frequencies in electromagnetic scattering*, Eng. Anal. Bound. Elem. 16 (1995), 245-252.

[6] I. Harari, P. E. Barbone, M. Slavutin and R. Shalom, (1998), *Boundary infinite elements for the Helmholtz equation in exterior domains*, Int. J. Numer. Meth. Eng. 41 (1998), 1105-1131.

[7] P. T. Hieu, P. H. Quan, (2015), *On regularization and error estimates for the Cauchy problem of the modified inhomogeneous Helmholtz equation,* J. Inverse Ill-Posed Probl. (2015), 1-12.

[8] B. Jin, Y. Zheng, (2005), *Boundary knot method for the Cauchy problem associated with the inhomogeneous Helmholtz equation,* Engineering Analysis with Boundary Elements 15 (2005), 925-925.

[9] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic and X. Wen, (2003), *Conjugate gradient-boudary element solution to the Cauchy problem for Helmholtz-type equation*, Comput. Mech. 31 (2003), 367-377.

[10] L. Marin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic and X. Wen, (2004), *BEM solution for the Cauchy problem associated with Helmholtz-type equation by the Landweber method*, Eng. Anal. Boundary Elem. 28 (2004), 1025-1034.

[11] L. Marin and D. Lesnic, (2007), *The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations,* Comput. Structures 83 (2007), 267-278.

[12] T. H. Nguyen, P. V. Tri, H. Vu and N. V. Hoa, (2012), *A new method for solving the modified Helmholtz equation,* Bulletin of Mathematical Analysis and Applications 4 (2012), 6-17.

[13] W. H. Press et al, (1996), *Numerical recipes in Fortran 90*, 2nd ed, Cambridge University, Press New York.

[14] Z. Qian, C. -L. Fu and X. -T. Xiong, (2008), *Two regularization methods for a Cauchy problem for the Laplace equation*, J. Math. Anal. Appl. 338 (2008), 479-489.

[15] H. H. Qin, T. Wei (2009), *Modified regularization method for the Cauchy problem of the Helmholtz equation*, Applied Mathematical Modelling 33 (2009), 2334-2348.

[16] P. H. Quan, P. T. Hieu, (2011), *A new regularization method for the Cauchy problem of the Helmholtz equation with nonhomogeneous Cauchy data*, Acta Mathematica Vietnamica 36 (2011), 419-430.

[17] R. Shi, T. Wei and H. H. Qin, (2009), *A fourth-order modified method for the Cauchy problem of the modified Helmholtz equation*, Numer. Math. Theory Methods Appl. 2 (2009), 326-340.

[18] N. H. Tuan, V. A. Khoa, D. M. Khang and T. Thanh, (2014), *Reconstruction of the electric field of the Helmholtz equation in three dimensions,* eprint arXiv:1408.1932 (2014).

[19] T. Q. Viet, N. H. Tuan, N. V. Thinh and D. D. Trong, (2014), *A general lter regularization method to solve the three dimensional Cauchy problem for inhomogeneous Helmholtz-type equations: theory and numerical simulation,* Applied Mathematical Modelling 38 (2014), 17-18.

[20] T. Wei, Y. Hon and L. Ling, (2007), *Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators*, Eng. Anal. Boundary Elem. 31 (2007), 373-385.