



# Legendre multi-Galerkin methods for Fredholm integral equations with weakly singular kernel and the corresponding eigenvalue problem

Bijaya Laxmi Panigrahi<sup>a,\*</sup>, Moumita Mandal<sup>b</sup>, Gnaneshwar Nelakanti<sup>c</sup>

<sup>a</sup> Department of Mathematics, Sambalpur University, Odisha 768019, India

<sup>b</sup> Department of Mathematics, VIT University, Vellore 632014, India

<sup>c</sup> Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, India

## ARTICLE INFO

### Article history:

Received 12 February 2017

Received in revised form 23 August 2017

### Keywords:

Legendre spectral methods

Multi-Galerkin methods

Fredholm integral equations

Eigenvalue problem

Weakly singular kernel

## ABSTRACT

In this paper, we consider Legendre multi-Galerkin methods to solve Fredholm integral equations of the second kind with weakly singular kernel and the corresponding eigenvalue problem. We obtain the convergence rates for the approximated solution and iterated solution in weakly singular Fredholm integral equations of the second kind in both  $L^2$  and infinity-norm. We also establish error bounds of approximated eigenelements with exact eigenelements in the eigenvalue problem of a compact integral operator with weakly singular kernel in both infinity and  $L^2$ -norm. Numerical examples are presented to prove the theoretical estimates.

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## 1. Introduction

Consider the following integral operator defined on the Banach space  $\mathbb{X} = L^2[a, b]$  or  $C[a, b]$  by

$$\mathcal{K}u(s) = \int_a^b k(s, t)u(t) dt, \quad s \in [a, b], \quad (1.1)$$

where the kernel  $k(s, t)$  is of the form

$$k(s, t) = \begin{cases} m(s, t)|s - t|^{\alpha-1}, & \text{if } 0 < \alpha < 1, \\ m(s, t)\log(|s - t|), & \text{if } \alpha = 1, \end{cases}$$

and  $m(s, t) \in C^1([a, b] \times [a, b])$ . Then  $\mathcal{K}$  is a compact linear integral operator from  $\mathbb{X}$  to  $\mathbb{X}$  (see, [1] and [2]).

We are interested to solve the following Fredholm integral equations of the second kind

$$(\mathcal{I} - \mathcal{K})u = f,$$

where  $\mathcal{K}$  is an integral operator as defined in (1.1),  $f$  is a given function,  $\mathcal{I}$  be the identity operator defined on  $\mathbb{X}$  and  $u \in \mathbb{X}$  is an unknown to be determined.

We are also interested to solve the corresponding eigenvalue problem of a compact integral operator

$$\mathcal{K}u = \lambda u,$$

\* Corresponding author.

E-mail addresses: [blpanigrahi@suniv.ac.in](mailto:blpanigrahi@suniv.ac.in) (B.L. Panigrahi), [abmoumita001@gmail.com](mailto:abmoumita001@gmail.com) (M. Mandal), [gnanesh@maths.iitkgp.ernet.in](mailto:gnanesh@maths.iitkgp.ernet.in) (G. Nelakanti).

to find the eigenvector  $u \in \mathbb{X}$  and the eigenvalue  $\lambda \in \mathbb{C} \setminus \{0\}$ .

The integral equations with weakly singular kernels of algebraic and logarithmic type cover many practical applications in mathematical physics. For last some years, it becomes a difficult task to solve the Fredholm integral equations with weakly singular kernel and the corresponding eigenvalue problem. The regularity properties of the solution for weakly singular integral equations of the second kind were studied by many authors [3–9]. Firstly, Rice [10] introduced nonlinear spline approximation methods to solve integral equations. Many authors extend this idea to solve integral equations with weakly singular kernel by using graded mesh techniques instead of using uniform partition. In [11], Graham discussed the iterated Galerkin method for second kind integral equations with singularities, and derived global convergence estimates by using splines on arbitrary quasiuniform meshes as approximating subspaces. In all the above methods, authors use piecewise based methods to solve the integral equations with weakly singular kernel.

However, to obtain more accurate approximate solutions using piecewise polynomial basis function, we must need to divide the interval more finely. Hence, in such cases, we need to solve a large system of equations and large matrix eigenvalue problem. This is computationally very much expensive. This motivates us to use global polynomials rather than piecewise polynomials. Spectral methods have developed rapidly in the past two decades and this method applied successfully in many fields. In the spectral methods, various orthogonal systems of infinitely differentiable global functions have been taken. Different global functions in different problems, generate different spectral approximations. For instance trigonometric polynomials for periodic problems and Legendre, Chebyshev polynomials for non-periodic problems, Laguerre polynomials for problems on half line and Hermite polynomials for problems on the whole line.

In [12], Legendre projection methods have been used to solve the eigenvalue problem of a compact integral operator with smooth kernel and in [13], Legendre spectral Galerkin methods have been discussed for weakly singular Fredholm integral equations of the second kind and for the corresponding eigenvalue problem. However, there have been considerable research at improving the accuracy of numerical solutions of various projection methods. In [14], the author first introduces the multi-projection methods for linear compact operator equations. In [15], Legendre multi-projection methods are being used to solve the eigenvalue problem of a compact integral operator with smooth kernel and showed that the proposed multi-projection methods exhibit the superconvergence results over the projection methods. This method also has been applied to solve nonlinear integral equations [16,17]. In this paper, we are interested to use Legendre multi-Galerkin method to solve the Fredholm integral equations with weakly singular kernel and the corresponding eigenvalue problem.

We organize this paper as follows. In Section 2, we discuss the Legendre multi-Galerkin method for the weakly singular Fredholm integral equations and the corresponding eigenvalue problem. In Section 3, we discuss on the convergence rates. The convergence rates for Fredholm integral equations and the corresponding eigenvalue problem have been discussed in Sections 3.1 and 3.2, respectively. In Section 4 we present numerical results. Throughout the paper, we assume that  $c$  is a generic constant.

## 2. Legendre multi-Galerkin method

Consider the following integral operator  $\mathcal{K}$  defined on  $\mathbb{X} = L^2[-1, 1]$  or  $C[-1, 1]$  by

$$\mathcal{K}u(s) = \int_{-1}^1 k(s, t)u(t) dt, \quad s \in [-1, 1], \quad (2.1)$$

where the kernel  $k(s, t)$  is of the form

$$k(s, t) = \begin{cases} m(s, t)|s - t|^{\alpha-1}, & \text{if } 0 < \alpha < 1, \\ m(s, t) \log(|s - t|), & \text{if } \alpha = 1, \end{cases} \quad (2.2)$$

and  $m(s, t) \in C^1([-1, 1] \times [-1, 1])$ . The kernels of this type are common in integral equations resulting from solving the boundary value problems of partial differential equations. Then  $\mathcal{K}$  is a compact linear operator on  $C[-1, 1]$  and  $L^2[-1, 1]$ , for  $\frac{1}{2} < \alpha \leq 1$  (cf., [1] and [2]). We are interested to solve the following Fredholm integral equations of the second kind

$$(\mathcal{I} - \mathcal{K})u = f, \quad (2.3)$$

to find the solution  $u \in \mathbb{X}$ , and to solve the corresponding eigenvalue problem

$$\mathcal{K}u = \lambda u, \quad (2.4)$$

to find the eigenvector  $u \in \mathbb{X}$  and the eigenvalue  $\lambda \in \mathbb{C} \setminus \{0\}$ . Assume  $\lambda \neq 0$  be the eigenvalue of  $\mathcal{K}$  with algebraic multiplicity  $m$  and ascent  $\ell$ . Let  $\Gamma \subset \rho(\mathcal{K})$  be a simple closed rectifiable curve such that  $\sigma(\mathcal{K}) \cap \text{int } \Gamma = \{\lambda\}$ ,  $0 \notin \text{int } \Gamma$ , where  $\text{int } \Gamma$  denotes the interior of  $\Gamma$ .

To discuss the Legendre multi-Galerkin method, first we will approximate the space  $\mathbb{X}$  by a finite dimensional space  $\mathbb{X}_n$ . Let  $\mathbb{X}_n = \text{span}\{\phi_0, \phi_1, \dots, \phi_n\}$  be the sequence of Legendre polynomial subspaces of  $\mathbb{X}$  of degree  $\leq n$ , where  $\{\phi_0, \phi_1, \dots, \phi_n\}$  forms an orthonormal basis for  $\mathbb{X}_n$ . Here  $\phi_i$ 's are given by

$$\phi_i(s) = \sqrt{\frac{2i+1}{2}} L_i(s), \quad i = 0, 1, \dots, n,$$

where  $L_i$ 's are the Legendre polynomials of degree  $\leq i$ . These Legendre polynomials can be generated by the following three-term recurrence relation

$$L_0(s) = 1, \quad L_1(s) = s, \quad s \in [-1, 1],$$

and

$$(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s), \quad i = 1, 2, 3, \dots, n-1.$$

Let  $\mathcal{P}_n^G : \mathbb{X} \rightarrow \mathbb{X}_n$  be the orthogonal projection defined by

$$\mathcal{P}_n^G u = \sum_{j=0}^n \langle u, \phi_j \rangle \phi_j, \quad u \in \mathbb{X}, \quad (2.5)$$

where  $\langle u, \phi_j \rangle = \int_{-1}^1 u(t) \phi_j(t) dt$ .

We quote the following lemma from [18] and [19], which gives the properties of the orthogonal projection operator  $\mathcal{P}_n^G$ .

**Lemma 2.1** ([18] and [19]). Let  $\mathcal{P}_n^G : \mathbb{X} \rightarrow \mathbb{X}_n$  denote the orthogonal projection defined by (2.5). Then the projection  $\mathcal{P}_n^G$  satisfies the following properties:

(i)  $\|\mathcal{P}_n^G u\|_{L^2} \leq c_1 \|u\|_\infty$ , where  $c_1$  is a constant independent of  $n$ .

(ii) There exists a constant  $c > 0$  such that for any  $n \in \mathbb{N}$  and  $u \in \mathbb{X}$ ,

$$\|\mathcal{P}_n^G u - u\|_{L^2} \leq c \inf_{\chi \in \mathbb{X}_n} \|u - \chi\|_{L^2} \rightarrow 0, \quad (2.6)$$

as  $n \rightarrow \infty$ .

(iii) For any  $u \in \mathcal{C}^r[-1, 1]$ , the following holds,

$$\|u - \mathcal{P}_n^G u\|_{L^2} \leq c n^{-r} \|u^{(r)}\|_{L^2}, \quad (2.7)$$

$$\|u - \mathcal{P}_n^G u\|_\infty \leq c n^{\frac{3}{4}-r} \|u^{(r)}\|_{L^2}, \quad (2.8)$$

$$\|u - \mathcal{P}_n^G u\|_\infty \leq c n^{\frac{1}{2}-r} V(u^{(r)}), \quad (2.9)$$

where  $c$  is a constant independent of  $n$  and  $V(u^{(r)})$  denotes the total variation of  $u^{(r)}$ .

Now, we will discuss on the Legendre multi-Galerkin method for Fredholm integral equations of the second kind and about the process of finding the approximate solution  $u_n^M$ .

With the projection  $\mathcal{P}_n^G$ , the Legendre multi-Galerkin operator  $\mathcal{K}_n^{M,G}$  is defined by

$$\mathcal{K}_n^{M,G} = \mathcal{P}_n^G \mathcal{K} \mathcal{P}_n^G + \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) + (\mathcal{I} - \mathcal{P}_n^G) \mathcal{K} \mathcal{P}_n^G. \quad (2.10)$$

By using the Legendre multi-Galerkin operator  $\mathcal{K}_n^{M,G}$ , the Legendre multi-Galerkin method for the Fredholm integral equations of the second kind is

$$u_n^M - \mathcal{K}_n^{M,G} u_n^M = f, \quad (2.11)$$

where  $u_n^M \in \mathbb{X}_n$  is the approximation of  $u$ . We define the iterated solution as  $\tilde{u}_n^M = \mathcal{K}_n^{M,G} u_n^M + f$ . To solve (2.11), applying  $\mathcal{P}_n^G$  and  $\mathcal{I} - \mathcal{P}_n^G$  to the equation yields

$$\mathcal{P}_n^G u_n^M - \mathcal{P}_n^G \mathcal{K} \mathcal{P}_n^G u_n^M - \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) u_n^M = \mathcal{P}_n^G f, \quad (2.12)$$

and

$$(\mathcal{I} - \mathcal{P}_n^G) u_n^M - (\mathcal{I} - \mathcal{P}_n^G) \mathcal{K} \mathcal{P}_n^G u_n^M = (\mathcal{I} - \mathcal{P}_n^G) f, \quad (2.13)$$

respectively. Substituting Eq. (2.13) into Eq. (2.12), we get

$$\mathcal{P}_n^G u_n^M - [\mathcal{P}_n^G \mathcal{K} + \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) \mathcal{K}] \mathcal{P}_n^G u_n^M = \mathcal{P}_n^G f + \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) f.$$

This means that, we can seek  $u_n^{M,1} = \mathcal{P}_n^G u_n^M \in \mathbb{X}_n$  from the following equation

$$[\mathcal{I} - \mathcal{S}_n \mathcal{K}] u_n^{M,1} = \mathcal{S}_n f,$$

where  $\mathcal{S}_n = \mathcal{P}_n^G + \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G)$  and then obtain  $u_n^M = u_n^{M,1} + u_n^{M,2}$  with

$$u_n^{M,2} = (\mathcal{I} - \mathcal{P}_n^G) (\mathcal{K} u_n^{M,1} + f),$$

by using Eq. (2.13).

Next, we will discuss on the Legendre multi-Galerkin method for the corresponding eigenvalue problem and about the process of finding the approximate eigenvector  $u_n^M$ .

The Legendre multi-Galerkin method for the eigenvalue problem is to find  $u_n^M \in \mathbb{X}_n$  and  $\lambda_n^M \in \mathbb{C} \setminus \{0\}$  such that

$$\mathcal{K}_n^{M,G} u_n^M = \lambda_n^M u_n^M, \quad (2.14)$$

where  $\lambda_n^M$  is the eigenvalue which approximates  $\lambda$  and  $u_n^M$  is the approximate eigenvector. Define iterated approximate eigenvector as  $\tilde{u}_n^M = \frac{1}{\lambda_n^M} \mathcal{K} u_n^M$ .

To solve (2.14), applying  $\mathcal{P}_n^G$  and  $\mathcal{I} - \mathcal{P}_n^G$  to the equation yields

$$\mathcal{P}_n^G \mathcal{K} \mathcal{P}_n^G u_n^M + \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) u_n^M = \lambda_n^M \mathcal{P}_n^G u_n^M, \quad (2.15)$$

and

$$(\mathcal{I} - \mathcal{P}_n^G) u_n^M = \frac{1}{\lambda_n^M} (\mathcal{I} - \mathcal{P}_n^G) \mathcal{K} \mathcal{P}_n^G u_n^M, \quad (2.16)$$

respectively. Substituting (2.16) into (2.15) yields

$$\mathcal{P}_n^G \mathcal{K} \mathcal{P}_n^G u_n^M + \frac{1}{\lambda_n^M} \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) \mathcal{K} \mathcal{P}_n^G u_n^M = \lambda_n^M \mathcal{P}_n^G u_n^M, \quad (2.17)$$

which is equivalent to solving the following eigenvalue problem

$$\begin{bmatrix} \mathcal{P}_n^G \mathcal{K} \mathcal{P}_n^G & \mathcal{P}_n^G \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) \mathcal{K} \mathcal{P}_n^G \\ \mathcal{I} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \psi_n^1 \\ \psi_n^2 \end{bmatrix} = \lambda_n^M \begin{bmatrix} \psi_n^1 \\ \psi_n^2 \end{bmatrix} \quad (2.18)$$

where  $\mathcal{I}$  and  $\mathcal{O}$  are  $n \times n$  identity and zero matrices respectively, and  $\psi_n^1 = \mathcal{P}_n^G u_n$  and  $\psi_n^2 = \frac{\mathcal{P}_n^G u_n^M}{\lambda_n^M}$ . By solving the eigenvalue problem (2.18), we find  $\lambda_n$  and  $\psi_n^1 = u_n^1$ , and using (2.16), we find  $u_n^{M,2} = (\mathcal{I} - \mathcal{P}_n^G) u_n^{M,1} = \frac{1}{\lambda_n^M} (\mathcal{I} - \mathcal{P}_n^G) \mathcal{K} u_n^{M,1}$ . Thus the eigenvector is given by  $u_n^M = u_n^{M,1} + u_n^{M,2}$ .

To discuss the convergence of Legendre multi-Galerkin operator  $\mathcal{K}_n^{M,G}$  with the integral operator  $\mathcal{K}$  having the weakly singular kernel particularly algebraic and logarithmic type, we need the following result from [13].

**Lemma 2.2** ([13]). *Let the kernel  $k(s, t)$  be the algebraic kernel  $m(s, t)|s - t|^{\alpha-1}$  for  $0 < \alpha < 1$  or logarithmic kernel  $m(s, t) \log|s - t|$  for  $\alpha = 1$ , where  $m(s, t) \in C^1([-1, 1] \times [-1, 1])$ , then for each  $s \in [-1, 1]$ , there exists a polynomial  $u_s$  of degree less than or equal to  $n$  such that*

$$\|k_s(\cdot) - u_s\|_{L^1} = \begin{cases} \mathcal{O}(n^{-\alpha}), & \text{for } 0 < \alpha < 1, \\ \mathcal{O}((1/n) \log n), & \text{for } \alpha = 1. \end{cases}$$

Now, we are ready to prove that the Legendre multi-Galerkin operator  $\mathcal{K}_n^{M,G}$  converges to  $\mathcal{K}$  in some sense.

**Theorem 2.3.** *The multi-Galerkin operator  $\mathcal{K}_n^{M,G}$  is norm-convergent to  $\mathcal{K}$  in infinity and  $L^2$ -norm both for algebraic and logarithmic kernel.*

**Proof.** Using the fact that  $\|\mathcal{P}_n^G\|_\infty \leq c \log n$  (cf., Page-147, [20]), we obtain

$$\begin{aligned} \|(\mathcal{K} - \mathcal{K}_n^{M,G})u\|_\infty &= \|(\mathcal{I} - \mathcal{P}_n^G) \mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) u\|_\infty \\ &\leq (1 + \|\mathcal{P}_n^G\|_\infty) \|\mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) u\|_\infty \\ &\leq (1 + c \log n) \|\mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) u\|_\infty. \end{aligned} \quad (2.19)$$

Since  $\mathcal{P}_n^G$  be the orthogonal projection from the space  $\mathbb{X}$  into  $\mathbb{X}_n$ , then we have

$$\langle v, (\mathcal{I} - \mathcal{P}_n^G) u \rangle = 0, \quad \forall v \in \mathbb{X}_n. \quad (2.20)$$

Now for each  $s \in [-1, 1]$  and  $u_s \in \mathbb{X}_n$ , we obtain

$$\begin{aligned} \|\mathcal{K} (\mathcal{I} - \mathcal{P}_n^G) u\|_\infty &= \sup_{s \in [-1, 1]} \left| \int_{-1}^1 k(s, t) (\mathcal{I} - \mathcal{P}_n^G) u(t) dt \right| \\ &= \sup_{s \in [-1, 1]} \langle k_s - u_s, (\mathcal{I} - \mathcal{P}_n^G) u \rangle \\ &\leq \|k_s - u_s\|_{L^1} \|(\mathcal{I} - \mathcal{P}_n^G) u\|_\infty \\ &\leq \|k_s - u_s\|_{L^1} (1 + \|\mathcal{P}_n^G\|_\infty) \|u\|_\infty \\ &\leq (1 + c \log n) \|k_s - u_s\|_{L^1} \|u\|_\infty. \end{aligned} \quad (2.21)$$

Using the estimate (2.21) in Eq. (2.19) with the Lemma 2.2, we get

$$\|(\kappa_n^{M,G} - \kappa)\|_\infty \leq \begin{cases} c n^{-\alpha} (1 + c \log n)^2, & \text{for } 0 < \alpha < 1, \\ c n^{-1} (1 + c \log n)^2 \log n, & \text{for } \alpha = 1. \end{cases}$$

It is now easy to show that,

$$\|(\kappa_n^{M,G} - \kappa)\|_\infty \rightarrow 0, \text{ as } n \rightarrow \infty,$$

by using L'Hospital's rule twice and thrice for algebraic kernel and logarithmic kernel, respectively. Thus,  $\kappa_n^{M,G}$  is norm-convergent to  $\kappa$  in infinity norm.

Next, we need to prove  $\kappa_n^{M,G}$  is norm convergent to  $\kappa$  in  $L^2$ -norm. To do this, we consider

$$\begin{aligned} \|(\kappa - \kappa_n^{M,G})u\|_{L^2} &= \|(\mathcal{I} - \mathcal{P}_n^G)\kappa(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \\ &\leq \|(\mathcal{I} - \mathcal{P}_n^G)\kappa\|_{L^2} \|(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \\ &\leq \|(\mathcal{I} - \mathcal{P}_n^G)\kappa\|_{L^2} (1 + \|\mathcal{P}_n^G\|_{L^2}) \|u\|_{L^2} \end{aligned}$$

By using  $\|\mathcal{P}_n^G\|_{L^2} \leq p_1$  in the above estimate, we obtain

$$\|(\kappa - \kappa_n^{M,G})\|_{L^2} \leq (1 + p_1) \|(\mathcal{I} - \mathcal{P}_n^G)\kappa\|_{L^2}.$$

Since  $\mathcal{P}_n^G$  converges to the identity operator pointwise from Lemma 2.1 and  $\kappa$  is a compact operator, for  $\frac{1}{2} < \alpha \leq 1$ , it follows that  $\|(\mathcal{I} - \mathcal{P}_n^G)\kappa\| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,

$$\|(\kappa_n^{M,G} - \kappa)\|_{L^2} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

This completes the proof.  $\square$

To discuss the existence and convergence rates for solution of Legendre multi-Galerkin method for Fredholm integral equations of second kind, we quote the following lemmas from [20].

**Lemma 2.4** ([20]). Let  $\mathbb{X}$  be a Banach space and  $\mathcal{T}, \mathcal{T}_n \in \mathbb{BL}(\mathbb{X})$ . If  $\mathcal{T}_n$  is norm convergent to  $\mathcal{T}$  or  $\mathcal{T}_n$  is  $\nu$ -convergent to  $\mathcal{T}$  and  $(\mathcal{I} - \mathcal{T})^{-1}$  exists and bounded on  $\mathbb{X}$ , then  $(\mathcal{I} - \mathcal{T}_n)^{-1}$  exists and uniformly bounded on  $\mathbb{X}$  for sufficiently large  $n$ .

**Theorem 2.5.** Then for sufficiently large  $n$ , the operators  $(\mathcal{I} - \kappa_n^{M,G})^{-1}$  are invertible on both  $L^2$  and infinity norm and there exist constants  $M_1$  and  $M_2$  independent of  $n$  such that  $\|(\mathcal{I} - \kappa_n^{M,G})^{-1}\|_{L^2} \leq M_1$  and  $\|(\mathcal{I} - \kappa_n^{M,G})^{-1}\|_\infty \leq M_2$ .

**Proof.** Proof follows by combining Lemma 2.4 and Theorem 2.3.  $\square$

To obtain convergence rates for approximated eigenelements with exact eigenelements in Legendre multi-Galerkin method, we discuss some basic framework for this. Since  $\kappa_n^{M,G}$  is norm-convergent to  $\kappa$  in both  $L^2$  and infinity-norm, the spectrum of  $\kappa_n^{M,G}$  inside  $\Gamma$  consists of  $m$  eigenvalues say  $\lambda_{n,1}^M, \lambda_{n,2}^M, \dots, \lambda_{n,m}^M$  counted accordingly to their algebraic multiplicities inside  $\Gamma$  (cf., [21] and [22]). Let

$$\hat{\lambda}_n^M = \frac{\lambda_{n,1}^M + \lambda_{n,2}^M + \dots + \lambda_{n,m}^M}{m},$$

denote their arithmetic mean and we approximate  $\lambda$  by  $\hat{\lambda}_n^M$ . Let  $\mathcal{P}^S$  and  $\mathcal{P}_n^{S,G}$  be the spectral projections of  $\kappa$  and  $\kappa_n^{M,G}$ , respectively, associated with their corresponding spectra inside  $\Gamma$ .

To discuss the closeness of eigenvectors of the original operator  $\kappa$  and those of the approximate operator  $\kappa_n^{M,G}$ , we recall the concept of gap between the spectral subspaces. For nonzero closed subspaces  $\mathbb{Y}_1$  and  $\mathbb{Y}_2$  of  $\mathbb{X}$ , and for  $p = L^2$  or  $\infty$ , let

$$\delta_p(\mathbb{Y}_1, \mathbb{Y}_2) = \sup\{\text{dist}_p(y, \mathbb{Y}_2) : y \in \mathbb{Y}_1, \|y\|_p = 1\},$$

then

$$\hat{\delta}_p(\mathbb{Y}_1, \mathbb{Y}_2) = \max\{\delta_p(\mathbb{Y}_1, \mathbb{Y}_2), \delta_p(\mathbb{Y}_2, \mathbb{Y}_1)\},$$

denotes the gap between the spectral subspaces in  $L^2$  norm ( $p = L^2$ ) and infinity norm ( $p = \infty$ ).

We quote the following lemmas, which give the error bounds for the eigenelements.

**Lemma 2.6** ([23] and [22]). Let  $\lambda$  be the eigenvalue of  $\kappa$  with algebraic multiplicity  $m$  and ascent  $l$  and  $\kappa_n^{M,G}$  is  $\nu$ -convergent to  $\kappa$  in  $L^2$ -norm. Let  $\mathcal{R}(\mathcal{P}^S)$  and  $\mathcal{R}(\mathcal{P}_n^{S,G})$  be the ranges of  $\mathcal{P}^S$  and  $\mathcal{P}_n^{S,G}$ , respectively. Then for  $n$  sufficiently large enough and  $p = L^2$  or  $\infty$ ,

$$\begin{aligned} \hat{\delta}_p(\mathcal{R}(\mathcal{P}_n^{S,G}), \mathcal{R}(\mathcal{P}^S)) &\leq c \sup\{\|(\kappa_n^{M,G} - \kappa)u\|_p : u \in \mathcal{R}(\mathcal{P}^S)\}, \\ \delta_{L^2}(\mathcal{R}(\mathcal{P}_n^{S,G}), \mathcal{R}(\mathcal{P}^S)) &\leq c \sup\{\|\kappa(\kappa_n^{M,G} - \kappa)u\|_{L^2} : u \in \mathcal{R}(\mathcal{P}_n^{S,G})\}. \end{aligned}$$

In particular, for any  $u_n^M \in \mathcal{R}(\mathcal{P}_n^{S,G})$ ,  $\tilde{u}_n^M = \mathcal{K}u_n^M$ , we have

$$\begin{aligned}\|u_n^M - \mathcal{P}_n^S u_n^M\|_p &\leq c \sup\{\|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_p : u \in \mathcal{R}(\mathcal{P}^S)\}, \\ \|\tilde{u}_n^M - \mathcal{P}_n^S \tilde{u}_n^M\|_{L^2} &\leq c \sup\{\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} : u \in \mathcal{R}(\mathcal{P}_n^{S,G})\}.\end{aligned}$$

**Lemma 2.7** ([23] and [22]). Let  $A_n = \mathcal{P}_n^{S,G}|_{\mathcal{R}(\mathcal{P}^S)} : \mathcal{R}(\mathcal{P}^S) \rightarrow \mathcal{R}(\mathcal{P}_n^{S,G})$ . Then

$$\begin{aligned}|\lambda - \hat{\lambda}_n^G| &\leq c \sup\{\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} : u \in \mathcal{R}(\mathcal{P}^S)\}, \\ |\lambda - \lambda_{n,j}^G|^\ell &\leq \alpha_n \sup\{\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} : u \in \mathcal{R}(\mathcal{P}^S)\},\end{aligned}$$

where

$$\alpha_n = \sum_{k=0}^{l-1} \|\lambda \mathcal{I}|_{\mathcal{R}(\mathcal{P}^S)} - A_n^{-1} \mathcal{P}_n^G \mathcal{K} A_n\|^{l-1-k} \|\lambda \mathcal{I}|_{\mathcal{R}(\mathcal{P}^S)} - \mathcal{K}|_{\mathcal{R}(\mathcal{P}^S)}\|^k,$$

is a bounded constant independent of  $n$ .

### 3. Convergence rates

In this section, we first evaluate the convergence rates for Legendre multi-Galerkin operator with the integral operator having the weakly singular kernel particularly of algebraic and logarithmic type. We then evaluate the convergence rates for approximate and iterated approximate solution in Fredholm integral equations of the second kind and also the error bounds of approximate eigenlements in the corresponding eigenvalue problem, which have been described in Sections 3.1 and 3.2, respectively. Now, we will proceed towards to evaluate the convergence rates for Legendre multi-Galerkin operator  $\mathcal{K}_n^{M,G}$  with the given integral operator  $\mathcal{K}$ .

**Theorem 3.1.** Let  $\mathcal{K}$  be a compact integral operator with a weakly singular kernel  $k(s, t)$  is of the form (2.2) and  $\mathcal{P}_n^G$  be the orthogonal projection defined by (2.5). Then the following holds

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \leq \sqrt{2} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty = \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1, \end{cases}$$

where  $c$  is a constant independent of  $n$ .

**Proof.** Now using the estimate (2.20) for each  $s \in [-1, 1]$  and  $u_s \in \mathbb{X}_n$ , we get

$$\begin{aligned}|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u(s)| &= \left| \int_{-1}^1 k(s, t)(\mathcal{I} - \mathcal{P}_n^G)u(t) dt \right| \\ &= |\langle k_s(\cdot), (\mathcal{I} - \mathcal{P}_n^G)u \rangle| \\ &= |\langle k_s(\cdot) - u_s, (\mathcal{I} - \mathcal{P}_n^G)u \rangle| \\ &\leq \|k_s(\cdot) - u_s\|_{L^1} \|(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty.\end{aligned}\tag{3.1}$$

By using Lemma 2.2 and the estimate (2.9) in the above equation, we obtain

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty \leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log(n) V(u^{(r)}), & \text{for } \alpha = 1. \end{cases}$$

Since  $\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \leq \sqrt{2} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty$ , we obtain the required result. This completes the proof.  $\square$

**Theorem 3.2.** Let  $\mathcal{K}_n^{M,G}$  be the Legendre multi-Galerkin operator defined by (2.10). If  $\mathcal{K}_n^{M,G}$  is norm-convergent to  $\mathcal{K}$  in both  $L^2$  and infinity norm, then the following holds:

$$\begin{aligned}\|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} &\leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1, \end{cases} \\ \|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_\infty &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{1}{2}} \log(n) V(u^{(r)}), & \text{for } \alpha = 1. \end{cases}\end{aligned}$$

**Proof.** By using the Legendre multi-Galerkin operator defined in (2.10) and the property (i) of Lemma 2.1, we obtain

$$\|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} = \|(\mathcal{I} - \mathcal{P}_n^G)\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \leq c \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty.$$

Now using the error bounds of [Theorem 3.1](#) in the above equation, we obtain

$$\|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} \leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log(n) V(u^{(r)}), & \text{for } \alpha = 1. \end{cases}$$

This proves the first inequality. Now to prove the second, we use the fact that  $\|\mathcal{P}_n^G\|_\infty \leq c \log n$  (cf., Page-147, [\[20\]](#)). Then we obtain

$$\begin{aligned} \|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_\infty &= \|(\mathcal{P}_n^G - \mathcal{I})\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty \\ &\leq (1 + \|\mathcal{P}_n^G\|_\infty)\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty \leq (1 + c \log n)\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_\infty. \end{aligned}$$

Now using the error bounds of [Theorem 3.1](#) in the above equation, we obtain

$$\|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_\infty \leq \begin{cases} c(1 + c \log n)n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n)n^{-r-\frac{1}{2}} \log(n) V(u^{(r)}), & \text{for } \alpha = 1. \end{cases}$$

This completes the proof.  $\square$

**Theorem 3.3.** *The following hold.*

$$\begin{aligned} \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} &\leq \sqrt{2}\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_\infty \\ &\leq \begin{cases} c(1 + c \log n)n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2 V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

**Proof.** By using the estimate [\(2.20\)](#) for each  $s \in [-1, 1]$  and  $u_s \in \mathbb{X}_n$ , we get

$$\begin{aligned} |\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u(s)| &= \left| \int_{-1}^1 k(s, t)(\mathcal{K}_n^{M,G} - \mathcal{K})u(t) dt \right| \\ &= |\langle k_s, (\mathcal{K}_n^{M,G} - \mathcal{K})u \rangle| \\ &= |\langle k_s - u_s, (\mathcal{K}_n^{M,G} - \mathcal{K})u \rangle| \\ &\leq \|k_s - u_s\|_{L^1} \|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_\infty. \end{aligned} \tag{3.2}$$

By using the error bounds of [Theorem 3.2](#) and [Lemma 2.2](#) in the above equation, we obtain

$$\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_\infty \leq \begin{cases} c(1 + c \log n)n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2 V(u^{(r)}), & \text{for } \alpha = 1. \end{cases}$$

Now combining the estimate  $\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} \leq \sqrt{2}\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_\infty$  with the above, completes the proof.  $\square$

### 3.1. Fredholm integral equations

In this subsection, we evaluate the error bounds of the approximated and iterated approximate solutions for the Fredholm integral equations of the second kind by using Legendre multi-Galerkin method both in  $L^2$  and infinity-norm.

**Theorem 3.4.** *Let  $u_n^M$  be the Legendre multi-Galerkin solution defined in Eq. [\(2.11\)](#) and  $\tilde{u}_n^M = \mathcal{K}u_n^M + f$  be the iterated Legendre multi-Galerkin solution. Then the followings hold.*

$$\begin{aligned} \|u - u_n^M\|_{L^2} &\leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } \frac{1}{2} < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1, \end{cases} \\ \|u - \tilde{u}_n^M\|_{L^2} &\leq \begin{cases} c(1 + c \log n)n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}), & \text{for } \frac{1}{2} < \alpha < 1, \\ c(1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2 V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

**Proof.** Using [Theorem 2.5](#), we obtain

$$\begin{aligned} \|u_n^M - u\|_{L^2} &= \|(\mathcal{I} - \mathcal{K}_n^{M,G})^{-1}f - (\mathcal{I} - \mathcal{K})^{-1}f\|_{L^2} \\ &= \|(\mathcal{I} - \mathcal{K}_n^{M,G})^{-1}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} \\ &\leq \|(\mathcal{I} - \mathcal{K}_n^{M,G})^{-1}\|_{L^2} \|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} \\ &\leq M_1 \|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2}. \end{aligned} \tag{3.3}$$

By using Theorem 3.2, we prove the first estimate. We have  $\tilde{u}_n^M = \mathcal{K}u_n^M + f$ . Then

$$\begin{aligned}\|\tilde{u}_n^M - u\|_{L^2} &= \|\mathcal{K}(u - u_n^M)\|_{L^2} \\ &= \|\mathcal{K}[(\mathcal{I} - \mathcal{K})^{-1} - (\mathcal{I} - \mathcal{K}_n^{M,G})^{-1}]f\|_{L^2} \\ &= \|\mathcal{K}(\mathcal{I} - \mathcal{K})^{-1}(\mathcal{K} - \mathcal{K}_n^{M,G})(\mathcal{I} - \mathcal{K}_n^{M,G})^{-1}f\|_{L^2} \\ &= \|(\mathcal{I} - \mathcal{K})^{-1}\|_{L^2} \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})u_n^M\|_{L^2}.\end{aligned}$$

Since  $(\mathcal{I} - \mathcal{K})^{-1}$  exists and uniformly bounded, it follows that

$$\begin{aligned}\|\tilde{u}_n^M - u\|_{L^2} &\leq c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})(u_n^M - u + u)\|_{L^2} \\ &\leq c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})(u_n^M - u)\|_{L^2} + c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})u\|_{L^2} \\ &= c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})\|_{L^2} \|u_n^M - u\|_{L^2} + c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})u\|_{L^2}.\end{aligned}\quad (3.4)$$

By using the estimate (3.2), we obtain

$$\begin{aligned}\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} &\leq \sqrt{2} \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{\infty} \\ &\leq \sqrt{2} \|k_s - u_s\|_{L^1} \|(\mathcal{I} - \mathcal{P}_n^G)\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty} \\ &\leq \sqrt{2}(1 + c \log n) \|k_s - u_s\|_{L^1} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty}.\end{aligned}\quad (3.5)$$

Since  $\sup_{s \in [-1, 1]} \int_{-1}^1 |k(s, t)|^2 dt \leq M < \infty$ ,  $\frac{1}{2} < \alpha < 1$ , hence by using Cauchy–Schwarz inequality with  $\mathcal{P}_n^G$  is uniformly bounded, we obtain

$$\begin{aligned}\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty} &= \sup_{s \in [-1, 1]} \left| \int_{-1}^1 k(s, t)(\mathcal{I} - \mathcal{P}_n^G)u(t) dt \right| \\ &\leq \sqrt{M} \|(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \leq c\sqrt{M} \|u\|_{L^2}.\end{aligned}\quad (3.6)$$

Hence combining the estimates (3.5) and (3.6), we have

$$\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})\|_{L^2} \leq c\sqrt{2M}(1 + c \log n) \|k_s - u_s\|_{L^1}.\quad (3.7)$$

By using Eq. (3.7) in Eq. (3.4), we obtain

$$\|\tilde{u}_n^M - u\|_{L^2} \leq c\sqrt{2M}(1 + c \log n) \|k_s - u_s\|_{L^1} \|u_n^M - u\|_{L^2} + c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})u\|_{L^2}.$$

By using Theorems 3.2, 3.3 with the estimate (3.3) with the above estimate, we get the second estimate. This completes the proof.  $\square$

**Theorem 3.5.** Let  $u_n^M$  be the Legendre multi-Galerkin solution defined in Eq. (2.11) and  $\tilde{u}_n^M = \mathcal{K}u_n^M + f$  be the iterated Legendre multi-Galerkin solution. Then the followings hold.

$$\begin{aligned}\|u - u_n^M\|_{\infty} &\leq \begin{cases} c(1 + c \log n)n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n)n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1, \end{cases} \\ \|u - \tilde{u}_n^M\|_{\infty} &\leq \begin{cases} c(1 + c \log n)n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2 V(u^{(r)}), & \text{for } \alpha = 1. \end{cases}\end{aligned}$$

**Proof.** We have

$$\begin{aligned}\|u_n^M - u\|_{\infty} &= \|(\mathcal{I} - \mathcal{K}_n^{M,G})^{-1}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{\infty} \\ &\leq \|(\mathcal{I} - \mathcal{K}_n^{M,G})^{-1}\|_{\infty} \|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{\infty} \leq M_2 \|(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{\infty}.\end{aligned}\quad (3.8)$$

By using Theorem 2.5 with Theorem 3.2, we prove the first estimate. We have  $\tilde{u}_n^M = \mathcal{K}u_n^M + f$ . Then by proceeding in the similar way as last theorem, we get

$$\|\tilde{u}_n^M - u\|_{\infty} \leq c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})\|_{\infty} \|u_n^M - u\|_{\infty} + c \|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^{M,G})u\|_{\infty}.\quad (3.9)$$

From estimates (3.1) and (3.2), we obtain

$$\begin{aligned}\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{\infty} &\leq \|k_s - u_s\|_{L^1} \|(\mathcal{I} - \mathcal{P}_n^G)\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty} \\ &\leq (1 + c \log n) \|k_s - u_s\|_{L^1} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty} \\ &\leq (1 + c \log n)^2 (\|k_s - u_s\|_{L^1})^2 \|u\|_{\infty}.\end{aligned}\quad (3.10)$$

This implies

$$\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})\|_\infty \leq (1 + c \log n)^2 (\|k_s - u_s\|_{L^1})^2. \quad (3.11)$$

Hence using the Theorem 3.3, the first estimate and the estimate (3.11) in the estimate (3.9), we prove the second estimate. This completes the proof.  $\square$

**Remark 3.6.** From Theorems 3.4 and 3.5, we observe that the approximate solution converges with the order  $\mathcal{O}(n^{\frac{1}{2}-r-\alpha})$  and the iterated solution converges with the order  $\mathcal{O}((1 + c \log n)n^{\frac{1}{2}-r-2\alpha})$  in  $L^2$ -norm for algebraic kernel. However, in infinity norm the approximate solution converges with the order  $\mathcal{O}((1 + c \log n)n^{\frac{1}{2}-r-\alpha})$  and the iterated solution converges with the order  $\mathcal{O}((1 + c \log n)n^{\frac{1}{2}-r-2\alpha})$ . This shows that the iterated solutions give superconvergence results over the approximate solutions for algebraic kernel.

**Remark 3.7.** From Theorems 3.4 and 3.5, we observe that the approximate solution converges with the order  $\mathcal{O}(n^{-r-\frac{1}{2}} \log n)$  and the iterated solution converges with the order  $\mathcal{O}((1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2)$  in  $L^2$ -norm for logarithmic kernel. However, in infinity norm the approximate solution converges with the order  $\mathcal{O}((1 + c \log n)n^{-r-\frac{1}{2}} \log n)$  and the iterated solution converges with the order  $\mathcal{O}((1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2)$ . This shows that the iterated solutions give superconvergence results over the approximate solutions for logarithmic kernel.

### 3.2. Eigenvalue problem

In this subsection, we evaluate the error bounds for approximate eigenelements with exact eigenelements by using Legendre multi-Galerkin methods for the eigenvalue problem of a compact integral operator  $\mathcal{K}$  with the particular algebraic and logarithmic kernel.

**Theorem 3.8.** Let  $\mathcal{K}$  be a compact linear integral operator with algebraic kernel  $k(s, t) = m(s, t)|s - t|^{\alpha-1}$  for  $0 < \alpha < 1$  or logarithmic kernel  $k(s, t) = m(s, t) \log|s - t|$  for  $\alpha = 1$ , where  $m(s, t) \in C^1([-1, 1] \times [-1, 1])$ . Let  $\mathcal{P}^S$  and  $\mathcal{P}_n^{S,G}$  be the spectral projections of  $\mathcal{K}$  and  $\mathcal{K}_n^{M,G}$ , associated with their spectra inside  $\Gamma$ , respectively. Let  $\mathcal{R}(\mathcal{P}^S)$  and  $\mathcal{R}(\mathcal{P}_n^{S,G})$  be the ranges of the spectral projections  $\mathcal{P}^S$  and  $\mathcal{P}_n^{S,G}$ , respectively. Then

$$\begin{aligned} \hat{\delta}_{L^2}(\mathcal{R}(\mathcal{P}_n^{S,M}), \mathcal{R}(\mathcal{P}^S)) &\leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \\ \hat{\delta}_\infty(\mathcal{R}(\mathcal{P}_n^{S,M}), \mathcal{R}(\mathcal{P}^S)) &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

In particular, for any  $u_n^M \in \mathcal{R}(\mathcal{P}_n^{S,G})$ , we have

$$\begin{aligned} \|u_n^M - \mathcal{P}^S u_n^M\|_{L^2} &\leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \\ \|u_n^M - \mathcal{P}^S u_n^M\|_\infty &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

where  $c$  is a constant independent of  $n$ .

**Proof.** Since  $\mathcal{R}(\mathcal{P}^S)$  is invariant under  $\mathcal{K}$ , we have for any  $v \in \mathcal{R}(\mathcal{P}^S)$ ,  $\mathcal{K}v \in \mathcal{R}(\mathcal{P}^S)$ . Hence we have

$$\begin{aligned} \|(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{K}|_{\mathcal{R}(\mathcal{P}^S)}\| &= \sup\{\|(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{K}v\| : v \in \mathcal{R}(\mathcal{P}^S)\} \\ &= \sup\{\|(\mathcal{K}_n^{M,G} - \mathcal{K})u\| : u = \mathcal{K}v \in \mathcal{R}(\mathcal{P}^S)\}. \end{aligned} \quad (3.12)$$

Using Theorem 3.2 with Lemma 2.6, and the estimate (3.12), we obtain

$$\begin{aligned} \hat{\delta}_{L^2}(\mathcal{R}(\mathcal{P}_n^{S,G}), \mathcal{R}(\mathcal{P}^S)) &\leq c \|(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{K}|_{\mathcal{R}(\mathcal{P}^S)}\|_{L^2} \\ &\leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \\ \hat{\delta}_\infty(\mathcal{R}(\mathcal{P}_n^{S,G}), \mathcal{R}(\mathcal{P}^S)) &\leq c \|(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{K}|_{\mathcal{R}(\mathcal{P}^S)}\|_\infty \\ &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

In particular, for any  $u_n^M \in \mathcal{R}(\mathcal{P}_n^{S,G})$ , we have

$$\begin{aligned} \|u_n^M - \mathcal{P}^S u_n^M\|_{L^2} &\leq c \|(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{K}|_{\mathcal{R}(\mathcal{P}^S)}\|_{L^2} \\ &\leq \begin{cases} c n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \\ \|u_n^M - \mathcal{P}^S u_n^M\|_{\infty} &\leq c \|(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{K}|_{\mathcal{R}(\mathcal{P}^S)}\|_{\infty} \\ &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{1}{2}} \log n V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.9.** Let  $\mathcal{K}$  be a compact linear integral operator with algebraic kernel  $k(s, t) = m(s, t)|s - t|^{\alpha-1}$  for  $0 < \alpha < 1$  or logarithmic kernel  $k(s, t) = m(s, t) \log|s - t|$  for  $\alpha = 1$ , where  $m(s, t) \in C^1([-1, 1] \times [-1, 1])$ . Let  $\mathcal{P}^S$  and  $\mathcal{P}_n^{S,G}$  be the spectral projections of  $\mathcal{K}$  and  $\mathcal{K}_n^{M,G}$ , associated with their spectra inside  $\Gamma$ , respectively. Then

$$\begin{aligned} \delta_{L^2}(\mathcal{K}\mathcal{R}(\mathcal{P}_n^{S,M}), \mathcal{R}(\mathcal{P}^S)) &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}) + cM(1 + c \log n)^2 n^{-2\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{3}{2}} (\log n)^2 V(u^{(r)}) + cM(1 + c \log n)^2 (\log n)^2 n^{-2}, & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

In particular, for any  $u_n^M \in \mathcal{R}(\mathcal{P}_n^{S,G})$ , we have

$$\begin{aligned} \|\mathcal{K}u_n^M - \mathcal{P}^S \mathcal{K}u_n^M\|_{L^2} &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}) + cM(1 + c \log n)^2 n^{-2\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{3}{2}} (\log n)^2 V(u^{(r)}) + cM(1 + c \log n)^2 (\log n)^2 n^{-2}, & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

where  $c$  is a constant independent of  $n$ .

**Proof.** From Lemma 2.6, and according to the analysis of Theorem 2.2 of [24], We have

$$\begin{aligned} \delta_{L^2}(\mathcal{K}\mathcal{R}(\mathcal{P}_n^{S,M}), \mathcal{R}(\mathcal{P}^S)) &\leq c \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{P}_n^{S,G}u_n^M\|_{L^2} \\ &\leq c \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})\mathcal{P}_n^{S,G}\|_{L^2} \\ &\leq c \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})^2\|_{L^2} + \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})|_{\mathcal{R}(\mathcal{P}^S)}\|_{L^2}. \end{aligned} \quad (3.13)$$

From Theorem 3.3, for the second term of the right hand side of the estimate (3.13), we obtain

$$\begin{aligned} \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})|_{\mathcal{R}(\mathcal{P}^S)}\|_{L^2} &= \sup\{\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})u\|_{L^2} : u \in \mathcal{R}(\mathcal{P}^S)\} \\ &\leq \begin{cases} c(1 + c \log n) n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n) n^{-r-\frac{3}{2}} (\log n)^2 V(u^{(r)}), & \text{for } \alpha = 1. \end{cases} \end{aligned} \quad (3.14)$$

By using the estimate (3.1) twice, for the first term of the right hand side of the estimate (3.13), we get

$$\begin{aligned} \|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})^2u\|_{L^2} &= \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \\ &\leq c(1 + c \log n) \|k_s - u_s\|_{L^1} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty} \\ &\leq c(1 + c \log n)^2 (\|k_s - u_s\|_{L^1})^2 \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty}. \end{aligned} \quad (3.15)$$

Since  $\sup_{s \in [-1, 1]} \int_{-1}^1 |k(s, t)|^2 dt \leq M < \infty$ ,  $1/2 < \alpha < 1$ , hence by using Cauchy–Schwarz inequality

$$\begin{aligned} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n^G)u\|_{\infty} &= \sup_{s \in [-1, 1]} \left| \int_{-1}^1 k(s, t)(\mathcal{I} - \mathcal{P}_n^G)u(t) dt \right| \\ &\leq \sqrt{M} \|(\mathcal{I} - \mathcal{P}_n^G)u\|_{L^2} \leq c\sqrt{M} \|u\|_{L^2}. \end{aligned} \quad (3.16)$$

Hence combining the estimates (3.15) and (3.16), we have

$$\|\mathcal{K}(\mathcal{K}_n^{M,G} - \mathcal{K})^2\|_{L^2} \leq c(1 + c \log n)^2 (\|k_s - u_s\|_{L^1})^2. \quad (3.17)$$

Hence from the estimates (3.13), (3.14) and (3.17), we obtain the first inequality.

In particular, for any  $u_n^M \in \mathcal{R}(\mathcal{P}_n^{S,G})$ ,  $\tilde{u}_n^M = \frac{1}{\lambda_n^M} \mathcal{K}u_n^M$ , using Lemma 2.6 and the estimates (3.13), (3.14) and (3.17), we prove the second inequality. This completes the proof.  $\square$

**Theorem 3.10.** Let  $\mathcal{K}$  be a compact integral operator with algebraic kernel  $k(s, t) = m(s, t)|s - t|^{\alpha-1}$  for  $0 < \alpha < 1$  or logarithmic kernel  $k(s, t) = m(s, t)\log|s - t|$  for  $\alpha = 1$ , where  $m(s, t) \in C^1([-1, 1] \times [-1, 1])$ . Assume  $\lambda$  be the eigenvalue of  $\mathcal{K}$  with algebraic multiplicity  $m$  and ascent  $\ell$ . Assume  $\mathcal{R}(\mathcal{P}^S) \subset C^r[-1, 1]$ . Let  $\hat{\lambda}_n^M$  be the arithmetic mean of the eigenvalues  $\lambda_{n,j}^M$ , for  $j = 1, 2, \dots, m$ , then

$$|\lambda - \hat{\lambda}_n^M| = |\lambda - \lambda_{n,j}^M|^\ell \leq \begin{cases} c(1 + c \log n)n^{\frac{1}{2}-r-2\alpha} V(u^{(r)}), & \text{for } 0 < \alpha < 1, \\ c(1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2 V(u^{(r)}), & \text{for } \alpha = 1. \end{cases}$$

where  $c$  is a constant independent of  $n$ .

**Proof.** The proof completes by combining Lemma 2.7, Theorem 3.1 with the estimate (3.14).  $\square$

**Remark 3.11.** From Theorem 3.10, we observe that for algebraic and logarithmic kernels eigenvalue converges with the orders  $\mathcal{O}((1 + c \log n)n^{\frac{1}{2}-r-2\alpha})$  and  $\mathcal{O}((1 + c \log n)n^{-r-\frac{3}{2}}(\log n)^2)$ , respectively. Thus we obtain superconvergence results for eigenvalues.

#### 4. Numerical results

In this section, we present the numerical examples. To solve the problem by using Legendre multi-Galerkin method, we first choose Legendre polynomials as the basis functions of  $\mathbb{X}_n$  as defined in Section 2.

**Fredholm integral equations.** We present two examples to validate the errors of the approximated solutions and the iterated solutions by using Legendre multi-Galerkin method in  $L^2$ -norm and infinity norm. Furthermore, we are comparing the results of Legendre multi-Galerkin method with the earlier results of Legendre Galerkin method.

Let  $u_n^M, \tilde{u}_n^M$  be the approximated and iterated solution of the Fredholm integral equation in the Legendre multi-Galerkin method, whereas  $u_n, \tilde{u}_n$  be the approximated and iterated solution in the Legendre Galerkin method for Examples 1 and 2. For different kernels, and for different values of  $n$ , we compute  $u_n^M, \tilde{u}_n^M, u_n$  and  $\tilde{u}_n$  and evaluate the error bounds with exact solution  $u$ . The computed errors in  $L^2$  and infinity norms for Legendre multi Galerkin and Legendre Galerkin are presented in Tables 1–2 for Example 1 and Tables 3–4 for Example 2.

**Example 1.** We consider the following Fredholm integral equation

$$u(s) - \int_{-1}^1 k(s, t) u(t) dt = f(s),$$

with  $k(s, t) = \frac{2}{3}|s - t|^{-1/4}$  and  $f(s) = (1 - s^2)^{3/4} - \frac{\pi}{2\sqrt{2}}(2 - s^2)$ , where the exact solution is given by  $u(s) = (1 - s^2)^{3/4}$ .

**Example 2.** We consider the Fredholm integral equation with logarithmic kernel

$$u(s) - \int_{-1}^1 \log|s - t| u(t) dt = f(s),$$

with  $f(s) = s - 1/2 \left\{ s^2 \log s + (1 - s^2) \log(1 - s) - s - (1/2) \right\}$ , where the exact solution is given by  $u(s) = s$ .

**Table 1**  
Legendre multi-Galerkin method.

$n$	$\ u - u_n^M\ _{L^2}$	$\ u - \tilde{u}_n^M\ _{L^2}$	$\ u - u_n^M\ _\infty$	$\ u - \tilde{u}_n^M\ _\infty$
2	3.788757e-04	4.066135e-05	5.091029e-04	6.058001e-05
3	3.663376e-04	3.896694e-05	4.897500e-04	5.803607e-05
4	2.560372e-04	2.363914e-05	3.932884e-04	5.162335e-05
5	1.554802e-04	1.912914e-05	3.117839e-04	4.703051e-05
6	0.155086e-04	1.009185e-05	2.468563e-04	3.951442e-05

**Table 2**  
Legendre Galerkin method.

$n$	$\ u - u_n\ _{L^2}$	$\ u - \tilde{u}_n\ _{L^2}$	$\ u - u_n\ _\infty$	$\ u - \tilde{u}_n\ _\infty$
2	3.471299e-02	8.418728e-04	9.881348e-02	2.855917e-04
3	3.469085e-02	8.147233e-04	9.636352e-02	2.816357e-04
4	1.560372e-02	7.363914e-04	6.332884e-02	2.762335e-04
5	1.554802e-02	6.912914e-04	6.017839e-02	2.703051e-04
6	1.155086e-02	6.809185e-04	4.468563e-02	2.651442e-04

**Table 3**

Legendre multi-Galerkin Method.

$n$	$\ u - u_n^M\ _{L^2}$	$\ u - \tilde{u}_n^M\ _{L^2}$	$\ u - u_n^M\ _\infty$	$\ u - \tilde{u}_n^M\ _\infty$
2	4.175304e-04	4.638219e-05	6.814441e-04	8.573753e-05
3	2.915325e-04	3.631351e-05	4.538688e-04	7.461478e-05
4	2.228204e-04	2.610848e-05	3.855898e-04	6.420558e-05
5	1.178572e-04	1.572246e-05	2.251786e-04	5.384819e-05
6	9.77444e-05	1.005381e-05	1.153028e-04	4.349261e-05

**Table 4**

Legendre Galerkin Method.

$n$	$\ u - u_n\ _{L^2}$	$\ u - \tilde{u}_n\ _{L^2}$	$\ u - u_n\ _\infty$	$\ u - \tilde{u}_n\ _\infty$
2	9.942773e-02	1.638219e-03	1.612173e-02	2.535860e-04
3	9.825008e-02	1.631351e-03	1.597225e-02	2.461478e-04
4	9.695885e-02	1.610848e-03	1.435071e-02	2.420558e-04
5	2.178572e-02	1.572246e-03	1.251786e-02	2.384819e-04
6	1.577444e-02	1.505381e-03	1.153028e-02	2.349261e-04

**Table 5**

Legendre Galerkin method.

$n$	$ \lambda - \hat{\lambda}_n $	$\ u_n - \mathcal{P}^S u_n\ _{L^2}$	$\ \tilde{u}_n - \mathcal{P}^S \tilde{u}_n\ _{L^2}$	$\ u_n - \mathcal{P}^S u_n\ _\infty$	$\ \tilde{u}_n - \mathcal{P}^S \tilde{u}_n\ _\infty$
2	1.833756e-02	2.628887e-02	4.793690e-03	3.851345e-02	4.793690e-03
3	6.527387e-03	1.238326e-02	9.554837e-04	1.044636e-02	1.554837e-03
4	1.527745e-03	7.643485e-03	1.555954e-04	9.933627e-03	9.555954e-04
5	6.486037e-04	2.630633e-03	8.461649e-05	4.004851e-03	3.461649e-04
6	2.506907e-04	1.779833e-03	1.467996e-05	1.799540e-03	1.465896e-05

**Table 6**

Legendre multi-Galerkin method.

$n$	$ \lambda - \hat{\lambda}_n^M $	$\ u_n^M - \mathcal{P}^S u_n^M\ _{L^2}$	$\ \tilde{u}_n^M - \mathcal{P}^S \tilde{u}_n^M\ _{L^2}$	$\ u_n^M - \mathcal{P}^S u_n^M\ _\infty$	$\ \tilde{u}_n^M - \mathcal{P}^S \tilde{u}_n^M\ _\infty$
2	2.213178e-03	1.070332e-03	5.789342e-05	3.045712e-03	1.029619e-05
3	2.831588e-04	5.428705e-04	6.209817e-06	9.683237e-04	5.223194e-06
4	1.527387e-04	1.977495e-04	2.776546e-06	1.197652e-04	1.057090e-06
5	2.913363e-05	6.963363e-05	5.601883e-07	6.256914e-05	5.625201e-07
6	6.506907e-06	1.779833e-05	3.330707e-07	2.234986e-05	1.467996e-07

### Eigenvalue Problem.

Consider the eigenvalue problem (2.4) with the integral operator  $\mathcal{K}$  defined in (2.1). We will present two examples to validate the errors of the approximated eigenvalues, eigenvectors and iterated eigenvectors with exact eigenvalues, eigenvectors and iterated eigenvectors by using Legendre multi-Galerkin method in both  $L^2$  and infinity-norm. Furthermore, we are comparing the results of Legendre multi-Galerkin method with the earlier results of Legendre Galerkin method.

For different kernels, and for different values of  $n$ , we compute the eigenvalue  $\hat{\lambda}_n^M$ , eigenvector  $u_n^M$  and iterated eigenvector  $\tilde{u}_n^M$  in the Legendre multi-Galerkin method and also we compute  $\hat{\lambda}_n$ ,  $u_n$  and  $\tilde{u}_n$  in Legendre Galerkin method. The computed errors in both  $L^2$  and infinity norm of the approximated eigenvalues to those of the exact eigenvalues are presented in Tables 5–6 for Example 3 and Tables 7–8 for Example 4. The comparison of Legendre Galerkin and Legendre multi-Galerkin methods also can be seen in Tables 5–8.

**Example 3.** Consider the eigenvalue problem

$$\int_{-1}^1 \log|s - t| u(t) dt = \lambda u(s).$$

Here  $\alpha = 1$ .

**Example 4.** Consider the eigenvalue problem

$$\int_{-1}^1 |s - t|^{-1/4} (st + 1) u(t) dt = \lambda u(s).$$

Here  $\alpha = 3/4$ .

**Table 7**

Legendre Galerkin method.

$n$	$ \lambda - \hat{\lambda}_n $	$\ u_n - \mathcal{P}^S u_n\ _{L^2}$	$\ \tilde{u}_n - \mathcal{P}^S \tilde{u}_n\ _{L^2}$	$\ u_n - \mathcal{P}^S u_n\ _\infty$	$\ \tilde{u}_n - \mathcal{P}^S \tilde{u}_n\ _\infty$
2	5.768082e–02	1.516612e–02	1.552175e–03	8.617840e–02	2.359971e–03
3	4.096182e–03	2.405890e–03	5.399077e–04	2.793753e–02	8.087731e–04
4	3.093007e–03	1.165435e–03	8.177666e–05	7.536897e–03	1.232367e–04
5	2.398562e–04	5.303441e–04	1.800626e–05	2.441239e–03	3.099771e–05
6	9.054147e–05	2.210033e–04	7.579851e–06	8.025752e–04	7.579851e–06

**Table 8**

Legendre multi-Galerkin method.

$n$	$ \lambda - \hat{\lambda}_n^M $	$\ u_n^M - \mathcal{P}^S u_n^M\ _{L^2}$	$\ \tilde{u}_n^M - \mathcal{P}^S \tilde{u}_n^M\ _{L^2}$	$\ u_n^M - \mathcal{P}^S u_n^M\ _\infty$	$\ \tilde{u}_n^M - \mathcal{P}^S \tilde{u}_n^M\ _\infty$
2	3.339131e–03	3.904018e–03	1.479162e–05	2.421124e–03	4.937460e–05
3	2.573616e–04	8.233152e–04	5.728299e–06	6.020894e–04	2.288907e–05
4	6.440104e–05	5.620202e–04	9.670518e–06	4.046618e–04	1.084150e–05
5	3.044933e–05	5.620202e–05	2.678256e–07	1.416143e–04	8.362101e–06
6	9.965386e–06	2.095192e–05	2.325914e–07	9.027752e–05	6.361847e–07

**Remark 4.1.** For solving the Fredholm integral equations of the second kind with weakly singular kernel and the corresponding eigenvalue problem, we use global polynomials of degree  $n$ . The size of the system of equations and the corresponding matrix eigenvalue problem, required to solve is  $(n+1) \times (n+1)$ . We choose  $n = 2, 3, 4, 5, 6$ , means we only solved the matrix of size  $3 \times 3, 4 \times 4, 5 \times 5, 6 \times 6, 7 \times 7$  and also obtained the minimum error bounds.

## Acknowledgment

The authors want to thank the referee for careful reading of the manuscript and to give the valuable suggestions which helped to improve the version of the paper.

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