

## Accepted Manuscript

Identification of a source in a fractional wave equation from a boundary measurement

K. Šišková, M. Slodička

PII: S0377-0427(18)30565-X  
DOI: <https://doi.org/10.1016/j.cam.2018.09.020>  
Reference: CAM 11915

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 5 December 2017  
Revised date : 5 September 2018

Please cite this article as: K. Šišková, M. Slodička, Identification of a source in a fractional wave equation from a boundary measurement, *Journal of Computational and Applied Mathematics* (2018), <https://doi.org/10.1016/j.cam.2018.09.020>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.





Available online at [www.sciencedirect.com](http://www.sciencedirect.com)



Journal of Computational and Applied Mathematics 00 (2018) 1–17



# Identification of a source in a fractional wave equation from a boundary measurement.

K. Šišková, M. Slodička

Department of Mathematical Analysis, research group of Numerical Analysis and Mathematical Modeling (NaM<sup>2</sup>),  
Ghent University, Gent 9000, Belgium

## Abstract

We deal with an inverse source problem in a time-fractional wave equation. The time-dependent source term is reconstructed using the additional non-invasive measurement in the form of integral over a part of the boundary. We look for the solution and the source term obeying the variational formulation and the equation obtained from applying the measurement on the equation. Using the Rothe method the existence of the solution is proved. The uniqueness is shown in appropriate spaces and some numerical experiments are presented.

© 2017 Published by Elsevier Ltd.

**Keywords:** time-fractional wave equation, inverse source problem, boundary measurement, reconstruction, convergence, time discretization

## 1. Introduction

In this article, we are interested in the following fractional wave equation accompanied with standard initial condition and the Neumann boundary condition

$$\begin{cases} (g_{2-\beta} * \partial_t u(x))(t) - \Delta u(x, t) = h(t)f(x) + F(x, t), & x \in \Omega, t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ \partial_\nu u(x, 0) = v_0(x), & x \in \Omega, \\ -\nabla u(x, t) \cdot \nu = \gamma(x, t), & x \in \Gamma, t \in (0, T), \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^d$  is bounded with Lipschitz boundary  $\Gamma$  (cf. [1]), the symbol  $\nu$  denotes the outer normal vector assigned to the boundary,  $T > 0$ ,  $g_{2-\beta}$  is the Riemann-Liouville kernel given by

$$g_{2-\beta}(t) = \frac{t^{1-\beta}}{\Gamma(2-\beta)}, \quad t > 0, \quad 1 < \beta < 2,$$

and  $*$  is a convolution on the positive half line

$$(k * v)(t) = \int_0^t k(t-s)v(s) \, ds.$$

The convolution term

$$\partial_t^\beta u(x, t) = (g_{2-\beta} * \partial_t u(x))(t)$$

Email addresses: [katarina.siskova@ugent.be](mailto:katarina.siskova@ugent.be) (K. Šišková), [marian.slodicka@ugent.be](mailto:marian.slodicka@ugent.be) (M. Slodička)  
URL: <http://cage.ugent.be/~ms> (M. Slodička)

contained in the equation is also known as the Caputo fractional derivative, here of the order  $\beta \in (1, 2)$ . It is worth to notice that if we assumed  $\beta = 1$  we would get in (1.1) a classical diffusion equation, and for  $\beta = 2$  it would be a wave equation. To the equation with order between 1 and 2 it is often referred in the literature as to fractional wave equation.

The *Inverse Source Problem* (ISP) we are interested in here consists of identifying a couple  $(u(x, t), h(t))$  obeying (1.1) and

$$\int_{\Gamma} u(x, t) \omega(x) dS = m(t), \quad t \in [0, T], \quad (1.2)$$

where  $\omega$  is a solely **space**-dependent function, many times chosen to have a compact support in  $\Gamma$ . This type of measurement is often called non-invasive as opposed to the measurements which take place inside the considered domain.

The fractional wave equation is, for instance, used to model diffusive waves in viscoelastic materials (cf. [2, 3]). The well-posedness of the direct problem for a fractional diffusion-wave equation has been studied in [4]. In [5] a fundamental solution of Cauchy problem is expressed using the Laplace transform. More about the direct fractional wave problem can be found in [6, 7, 8].

There are many papers studying ISPs in hyperbolic or parabolic settings. The recognition of the space-dependent source is studied in [9, 10, 11, 12, 13, 14, 15, 16]. In case of solely time-dependent source term we refer to [9, 17, 18, 19]. The inverse source problems in fractional differential equation has been studied more in the last years. Many of those articles are interested in the fractional diffusion equation, for instance, see [20, 21, 22, 23, 24, 25, 26]. In [27, 28] a time dependent source is identified regarding a fractional integro-differential wave equation with the zero Dirichlet boundary condition, a main tool used there is the Banach fixed point theorem. In [27] the additional measurement in form of the trace at point  $x_0 \in \Omega$  is used for reconstruction, and in [28] two integrals over the inside of the domain is used to identify the time-dependent source and convolution kernel. **In [29] the recognition of the time-dependent part of a source is studied for the fractional diffusion-wave equation with the distributions on the right hand side of the equation and in the initial conditions; they considered the domain  $\mathbb{R}^n \times (0, T]$  and the measurement in the form  $(u(\cdot, t), \varphi_0(\cdot))$  which stands for the value of an distributional solution  $u$  on given test function  $\varphi_0$  for every  $t \in [0, T]$ .**

In [30], we have dealt with the similar equation but the measurement was taken over a subset of  $\Omega$ . The added value of this paper relies on using the non-invasive measurement in the form of the integral over the part of the boundary. The approach, we take, consists of reformulating the inverse problem into the direct one. That means we will look for the couple  $(u, h)$  which will be a solution of two coupled equations. This will demand the estimates for the Laplacian of  $u$  on the boundary, which was not necessary in [30].

This paper is organized as follows. In the short second section, we introduce some notation and reformulate our problem. In the third section, we introduce the time-discretization, prove some useful a priori estimates, introduce the Rothe functions and at the end we prove the existence of a solution. The next section deals with the uniqueness of the solution in appropriate spaces. In the end some numerical experiments are presented including a suggestion for the treatment of noisy data.

## 2. Reformulation of the problem

First, we introduce some notation, which will be used across the whole article. Let  $(\cdot, \cdot)$  be the standard inner product of  $L^2(\Omega)$ , and  $\|\cdot\|$  be its induced norm. We use similar notation for the boundary space, that is  $(\cdot, \cdot)_{\Gamma}$ ,  $L^2(\Gamma)$  and  $\|\cdot\|_{\Gamma}$ . Let  $X$  be a Banach space with the norm  $\|\cdot\|_X$ . By  $C([0, T], X)$  we denote the Banach space of abstract continuous function  $v: [0, T] \rightarrow X$  endowed with the norm  $\max_{t \in [0, T]} \|v(t)\|_X$ . The space  $L^p((0, T), X)$  is a set of  $p$ -integrable functions  $v$  with  $p > 1$  furnished with the norm  $\left( \int_0^T \|v(t)\|_X^p dt \right)^{\frac{1}{p}}$ , cf. [31]. The generic positive constants depending only on the data will be denoted as  $C, \varepsilon$  and  $C_{\varepsilon}$ , where  $\varepsilon$  is a small one, and  $C_{\varepsilon} = C(\frac{1}{\varepsilon})$  is a large one. Different values of those constants in the same discussion are allowed.

Without loss of generality, we may assume that  $F = 0$  and  $\gamma = 0$ . This follows from the superposition principle, which is valid for all linear systems. Then the solution of (1.1) can be written as  $u = v + w$ , where

$$\begin{cases} (g_{2-\beta} * \partial_t v)(x) - \Delta v(x, t) = F(x, t), & x \in \Omega, t \in (0, T), \\ v(x, 0) = u_0(x), & x \in \Omega, \\ \partial_t v(x, 0) = v_0(x), & x \in \Omega, \\ -\nabla v(x, t) \cdot \nu = \gamma(x, t), & (x, t) \in \Gamma \times (0, T), \end{cases} \quad (2.1)$$

and

$$\begin{cases} (g_{2-\beta} * \partial_t w(x))(t) - \Delta w(x, t) = h(t)f(x), & x \in \Omega, t \in (0, T), \\ w(x, 0) = 0, & x \in \Omega, \\ \partial_t w(x, 0) = 0, & x \in \Omega, \\ -\nabla w(x, t) \cdot \nu = 0, & (x, t) \in \Gamma \times (0, T). \end{cases} \quad (2.2)$$

Thus, instead of  $(u, h)$  the new couple  $(w, h)$  has to be found and measurement needs to be modified to

$$\int_{\Gamma} w(x, t) \omega(x) dS = m(t) - \int_{\Gamma} v(x, t) \omega(x) dS =: \tilde{m}(t), \quad t \in [0, T]. \quad (2.3)$$

From now on, we will denote the new sought couple  $(w, h)$  and the measurement function  $\tilde{m}$  again by  $(u, h)$  and  $m$ , respectively.

Next, we reformulate our problem into two coupled equations using the measurement and the variational formulation of (2.2). Taking the first equation of (2.2) and multiplying it by  $\varphi$  and integrating over the boundary  $\Gamma$  we get

$$(g_{2-\beta} * m'')(t) - (\Delta u(t), \omega)_{\Gamma} = h(t) (f, \omega)_{\Gamma}. \quad (\text{MP})$$

if we assume that  $(f, \omega)_{\Gamma} \neq 0$ , we may eliminate  $h$  in the following manner

$$h(t) = \frac{(g_{2-\beta} * m'')(t) - (\Delta u(t), \omega)_{\Gamma}}{(f, \omega)_{\Gamma}}. \quad (2.4)$$

By multiplying the first equation of (2.2) by  $\varphi \in H^1(\Omega)$  integrating over  $\Omega$  and using the Green theorem we obtain the weak formulation, thus, it holds

$$((g_{2-\beta} * \partial_t u)(t), \varphi) + (\nabla u(t), \nabla \varphi) = h(t) (f, \varphi). \quad (\text{P})$$

for any  $\varphi \in H^1(\Omega)$ , a.a.  $t \in [0, T]$ . Hence, in the reformulated inverse source problem, we are interested in finding a couple  $(u, h)$  which solves the equations (P) and (MP) with  $u(0) = 0$ ,  $\partial_t u(0) = 0$ .

### 3. Existence

In this section, we prove the existence of a solution of our problem using the well-known Rothe method ([32, 33, 34, 35]). The method is based on the time-discretization and defining the approximate solution along time lines. The existence of those solutions is proven, and then some a priori estimates are done for them. At the end we define the Rothe functions and with help of the estimates the convergence to the solution is acquired.

We divide the interval  $[0, T]$  into  $n$  equidistant (for the simplicity of notation) pieces, for  $n \in \mathbb{N}$ , and define a time step as  $\tau = \frac{T}{n}$ , for  $i = 1, \dots, n$  then for any function  $z$  we write

$$z_i = z(t_i), \quad \delta z_i = \frac{z_i - z_{i-1}}{\tau}, \quad \delta^2 z_i = \frac{\delta z_i - \delta z_{i-1}}{\tau}.$$

The discrete convolution is defined as

$$(K * v)_i := \sum_{k=1}^i K_{i+1-k} v_k \tau,$$

where we avoid the value of the kernel  $K$  in zero by shifting the index by one. The difference of discrete convolution is then

$$\delta(K * v)_i = \frac{(K * v)_i - (K * v)_{i-1}}{\tau} = K_1 v_i + \sum_{k=1}^{i-1} \delta K_{i+1-k} v_k \tau, \quad i \geq 1, \quad (3.1)$$

where

$$(K * v)_0 := 0.$$

We can also rewrite (3.1) as

$$\delta(K * v)_i = K_i v_0 + \sum_{k=1}^i \delta v_k K_{i+1-k} \tau = K_i v_0 + (K * \delta v)_i, \quad i \geq 1. \quad (3.2)$$

With those definitions, we approximate the solution of (P), (MP) on the  $i$ -th time-layer, for  $i \geq 1$ , by  $(u_i, h_i)$  which solves

$$((g_{2-\beta} * \delta^2 u)_i, \varphi) + (\nabla u_i, \nabla \varphi) = h_i(f, \varphi), \quad (DPi)$$

for  $\varphi \in H^1(\Omega)$ , with  $\delta u_0 := 0$  and

$$(g_{2-\beta} * m'')_i - (\Delta u_{i-1}, \omega)_\Gamma = h_i(f, \omega)_\Gamma. \quad (DMPi)$$

Next, we define set

$$V = \{\varphi : \Omega \rightarrow \mathbb{R}; \|\varphi\| + \|\nabla \varphi\| + \|\Delta \varphi\| + \|\nabla \Delta \varphi\| < \infty\}$$

which equipped with the norm  $\|\varphi\|_V = (\|\varphi\|^2 + \|\nabla \varphi\|^2 + \|\Delta \varphi\|^2 + \|\nabla \Delta \varphi\|^2)^{\frac{1}{2}}$  is Hilbert space compactly embedded in  $L^2(\Omega)$ . Since there occurs  $\Delta u_i$  in (DMPi), we need to control it on the boundary which leads us in looking for the solution in the space  $V$ . The following lemma handles the existence of the unique couple  $(u_i, h_i)$  on every time slice.

**Lemma 3.1.** *Let  $f \in H^1(\Omega)$ ,  $\omega \in L^2(\Gamma)$  with  $(f, \omega)_\Gamma \neq 0$  and  $n \in C^1([0, T])$ . Then for each  $i \in \{1, \dots, n\}$ , there exists a unique couple  $(u_i, h_i) \in V \times \mathbb{R}$  solving (DPi) and (DMPi) for every  $\varphi \in H^1(\Omega)$ .*

*Proof.* Assuming  $(f, \omega)_\Gamma \neq 0$  and  $u_i \in V$ , we can write

$$h_i = \frac{(g_{2-\beta} * m'')_i - (\Delta u_{i-1}, \omega)_\Gamma}{(f, \omega)_\Gamma} \in \mathbb{R}.$$

The equation (DPi) can be rewritten, such that all  $u_k$ 's with  $k \leq i-1$  are placed on the right hand side of the equation, so we get

$$\begin{aligned} \frac{1}{\tau} g_{2-\beta}(\tau)(u_i, \varphi) + (\nabla u_i, \nabla \varphi) &= h_i(f, \varphi) - \sum_{k=1}^{i-1} g_{2-\beta}(t_{i+1-k}) (\delta^2 u_k, \varphi) \tau \\ &\quad + \frac{1}{\tau} g_{2-\beta}(\tau)(u_{i-1}, \varphi) + g_{2-\beta}(\tau)(\delta u_{i-1}, \varphi). \end{aligned} \quad (3.3)$$

When  $u_1, \dots, u_{i-1} \in L^2(\Omega)$ , then, with the assumptions on  $f$ ,  $u_0$  and  $v_0$ , the right hand side (r.h.s.) of the equation can be seen as a linear bounded functional on  $H^1(\Omega)$ , moreover, left hand side (l.h.s.) of the equation is a bounded bilinear form

$$B[u_i, \varphi] := \frac{1}{\tau} g_{2-\beta}(\tau)(u_i, \varphi) + (\nabla u_i, \nabla \varphi),$$

on  $H^1(\Omega) \times H^1(\Omega)$  with  $B[\varphi, \varphi] \geq C \|\varphi\|_{H^1(\Omega)}^2$ . Using the Lax-Milgram lemma iteratively, we can conclude that there exist unique  $u_i \in H^1(\Omega)$  solving (DPi). Now, we want to prove that  $u_i \in V$ . Looking again at the equation (DPi), the term  $(\nabla u_i, \nabla \varphi)$  can be understood as a realization of a linear bounded functional on  $H^1(\Omega)$ . From the Hahn-Banach theorem there exists an extension of that functional on  $L^2(\Omega)$  with the same norm. The Riesz theorem says that this extension can be represented uniquely by a function from  $L^2(\Omega)$ , we denote this function as  $-\Delta u_i$ . We may write

$$-(\Delta u_i, \varphi) = h_i(f, \varphi) - ((g_{2-\beta} * \delta^2 u)_i, \varphi), \quad (3.4)$$

for every  $\varphi \in H^1(\Omega)$ , so,

$$-\Delta u_i = h_i f - (g_{2-\beta} * \delta^2 u)_i \in L^2(\Omega),$$

and using the assumptions of the lemma and applying the gradient on this equality leads to

$$-\nabla \Delta u_i = h_i \nabla f - (g_{2-\beta} * \nabla \delta^2 u)_i \in L^2(\Omega). \quad (3.5)$$

□

Now, we can work properly with  $(u_i, h_i)$ . Our next aim is to gain some estimates of them. The crucial tool for it is the next lemma.

**Lemma 3.2.** *Let  $\{v_i\}_{i \in \mathbb{N}}$  and  $\{K_i\}_{i \in \mathbb{N}}$  be sequences of real numbers. Assume that  $K$  decreases, i.e.  $K_i \leq K_{i-1}$  for any  $i$ . Then*

$$2\delta (K * v)_i v_i \geq \delta (K * v^2)_i + K_i v_i^2, \quad i \in \mathbb{N}.$$

We can see that the lemma deals with estimation of some terms containing the discrete convolution. The proof of it can be found in [36]. With this we can approach to making some a priori estimates. In next, we will use the following notation

$$(g_{2-\beta} * \|u\|^2)_j = \sum_{k=1}^j g_{2-\beta}(t_{j+1-k}) \|u_k\|^2 \tau.$$

**Lemma 3.3.** *Under the assumptions of Lemma 3.1 there exists a positive constant  $C$  (independent of  $n$ ) such that*

$$(g_{2-\beta} * \|\delta u\|^2)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\delta u_i\|^2 \tau + \sum_{i=1}^j \|\delta u_i\|^2 \tau + \|u_j\|_{H^1(\Omega)}^2 + \sum_{i=1}^j \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C \sum_{i=1}^j h_i^2 \tau,$$

for every  $j \in 1, \dots, n$ ,  $n \in \mathbb{N}$ .

*Proof.* Let  $\varphi = \delta u_i$  in (DPi), using the equality (3.2) we get that

$$(\delta (g_{2-\beta} * \delta u)_i, \delta u_i) + (\nabla u_i, \nabla \delta u_i) = h_i (f, \delta u_i).$$

Multiplying the equality by  $\tau$  and summing it up for  $i = 1, \dots, j$  we obtain

$$\sum_{i=1}^j (\delta (g_{2-\beta} * \delta u)_i, \delta u_i) \tau + \sum_{i=1}^j (\nabla u_i, \nabla u_i - \nabla u_{i-1}) = \sum_{i=1}^j h_i (f, \delta u_i) \tau.$$

Next, we use Lemma 3.2 for the first term on the l.h.s. and the Abel summation

$$2 \sum_{i=1}^j (\nabla u_i, \nabla u_i - \nabla u_{i-1}) = \|\nabla u_j\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^j \|\nabla u_i - \nabla u_{i-1}\|^2$$

for the second term. Moreover, the Young inequality is used on the r.h.s., hence, we get

$$\begin{aligned} & \frac{1}{2} (g_{2-\beta} * \|\delta u\|^2)_j + \frac{1}{4} \sum_{i=1}^j g_{2-\beta}(t_i) \|\delta u_i\|^2 \tau + \frac{g_{2-\beta}(T)}{4} \sum_{i=1}^j \|\delta u_i\|^2 \tau + \frac{1}{2} \|\nabla u_j\|^2 + \frac{1}{2} \sum_{i=1}^j \|\nabla u_i - \nabla u_{i-1}\|^2 \\ & \leq C_\varepsilon \sum_{i=1}^j h_i^2 \tau + \varepsilon \sum_{i=1}^j \|\delta u_i\|^2 \tau. \end{aligned}$$

Choosing suitable  $\varepsilon > 0$  we derive

$$(g_{2-\beta} * \|\delta u\|^2)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\delta u_i\|^2 \tau + \sum_{i=1}^j \|\delta u_i\|^2 \tau + \|\nabla u_j\|^2 + \sum_{i=1}^j \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C \sum_{i=1}^j h_i^2 \tau.$$

□

**Lemma 3.4.** *Under the assumptions of Lemma 3.1 there exists a positive constant  $C$  (independent of  $n$ ) such that*

$$(g_{2-\beta} * \|\nabla \delta u\|^2)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau + \sum_{i=1}^j \|\nabla \delta u_i\|^2 \tau + \|\Delta u_j\|^2 + \sum_{i=1}^j \|\Delta u_i - \Delta u_{i-1}\|^2 \leq C \sum_{i=1}^j h_i^2 \tau,$$

for every  $j \in 1, \dots, n$ ,  $n \in \mathbb{N}$ .

*Proof.* To gain the estimate from the lemma, we start with the equation (3.4) from the proof of Lemma 3.1. We set  $\varphi = -\Delta\delta u_i$ , which is justified since  $u_i \in \mathbf{V}$ , for  $0 \leq i \leq j$ ,

$$((g_{2-\beta} * \delta^2 u)_i, -\Delta\delta u_i) + (\Delta u_i, \Delta\delta u_i) = h_i(f, -\Delta\delta u_i).$$

This can be rewritten as

$$(\delta(g_{2-\beta} * \nabla\delta u)_i, \nabla\delta u_i) + (\Delta u_i, \Delta\delta u_i) = h_i(\nabla f, \nabla\delta u_i),$$

by multiplying by  $\tau$  and summing up for  $1 \leq i \leq j$ , we get

$$\sum_{i=1}^j (\delta(g_{2-\beta} * \nabla\delta u)_i, \nabla\delta u_i) \tau + \sum_{i=1}^j (\Delta u_i, \Delta\delta u_i) \tau = \sum_{i=1}^j h_i(\nabla f, \nabla\delta u_i) \tau.$$

This can be estimated in the similar way as in the previous lemma, with the help of Lemma 3.2, the Abel summation, the Cauchy and Young inequalities, we obtain

$$\begin{aligned} & (g_{2-\beta} * \|\nabla\delta u\|^2)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\nabla\delta u_i\|^2 \tau + \sum_{i=1}^j \|\nabla\delta u_i\|^2 \tau + \|\Delta u_j\|^2 + \sum_{i=1}^j \|\Delta u_i - \Delta u_{i-1}\|^2 \\ & \leq C_\varepsilon \sum_{i=1}^j h_i^2 \tau + \varepsilon \sum_{i=1}^j \|\nabla\delta u_i\|^2 \tau. \end{aligned}$$

The estimate from the lemma is acquired by choosing an appropriate  $\varepsilon > 0$ .  $\square$

**Lemma 3.5.** Under the assumptions of Lemma 3.1, if moreover  $f \in H^2(\Omega)$  and  $\nabla f \cdot \mathbf{v} = 0$  on  $\Gamma$  then there exists a positive constant  $C$  (independent of  $n$ ) such that

$$(g_{2-\beta} * \|\Delta\delta u\|^2)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\Delta\delta u_i\|^2 \tau + \sum_{i=1}^j \|\Delta\delta u_i\|^2 \tau + \|\nabla\Delta u_j\|^2 + \sum_{i=1}^j \|\nabla\Delta u_i - \nabla\Delta u_{i-1}\|^2 \leq C \sum_{i=1}^j h_i^2 \tau,$$

for every  $j \in 1, \dots, n$ ,  $n \in \mathbb{N}$ .

*Proof.* Starting from (3.5), we multiply the equality by  $-\nabla\delta\Delta u_i$ , integrate over the domain  $\Omega$  and get

$$((g_{2-\beta} * \nabla\delta^2 u)_i, -\nabla\delta\Delta u_i) + (\nabla\Delta u_i, \nabla\delta\Delta u_i) = h_i(\nabla f, -\nabla\delta\Delta u_i).$$

The equalities for  $1 \leq i \leq j$ ,  $j \in 1, \dots, n$  are multiplied by  $\tau$  and summed up to obtain

$$\sum_{i=1}^j ((g_{2-\beta} * \nabla\delta^2 u)_i, -\nabla\delta\Delta u_i) \tau + \sum_{i=1}^j (\nabla\Delta u_i, \nabla\delta\Delta u_i) \tau = \sum_{i=1}^j h_i(\nabla f, -\nabla\delta\Delta u_i) \tau.$$

Next, the first term on the l.h.s. is rewritten using the Green theorem and (3.2), then Lemma 3.2 is applied, for the second term on the l.h.s. the Abel summation is used. For the r.h.s of the equality, first, the Green theorem is applied, and then the Cauchy and Young inequality are used to acquire

$$\begin{aligned} & (g_{2-\beta} * \|\delta\Delta u\|^2)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\delta\Delta u_i\|^2 \tau + \sum_{i=1}^j \|\delta\Delta u_i\|^2 \tau + \|\nabla\Delta u_j\|^2 + \sum_{i=1}^j \|\nabla\Delta u_i - \nabla\Delta u_{i-1}\|^2 \\ & \leq C_\varepsilon \sum_{i=1}^j h_i^2 \tau + \varepsilon \sum_{i=1}^j \|\delta\Delta u_i\|^2 \tau. \end{aligned}$$

Choosing the appropriate  $\varepsilon > 0$  leads us to the estimate from the lemma.  $\square$

**Lemma 3.6.** Under the assumptions of Lemma 3.1, if moreover  $f \in H^2(\Omega)$  and  $\nabla f \cdot \mathbf{v} = 0$  on  $\Gamma$  there exists a positive constants  $C$  (independent of  $n$ ) such that

$$\begin{aligned}
 (i) \quad & \max_{0 \leq i \leq n} \left( g_{2-\beta} * \|\delta u\|^2 \right)_i + \sum_{i=1}^n g_{2-\beta}(t_i) \|\delta u_i\|^2 \tau + \sum_{i=1}^n \|\delta u_i\|^2 \tau + \max_{0 \leq i \leq n} \|u_i\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \|\nabla u_i - \nabla u_{i-1}\|^2 \\
 & + \max_{0 \leq i \leq n} \left( g_{2-\beta} * \|\nabla \delta u\|^2 \right)_i + \sum_{i=1}^n g_{2-\beta}(t_i) \|\nabla \delta u_i\|^2 \tau + \sum_{i=1}^n \|\nabla \delta u_i\|^2 \tau + \max_{0 \leq i \leq n} \|\Delta u_i\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \|\Delta u_i - \Delta u_{i-1}\|^2 \\
 & + \max_{0 \leq i \leq n} \left( g_{2-\beta} * \|\Delta \delta u\|^2 \right)_i + \sum_{i=1}^n g_{2-\beta}(t_i) \|\Delta \delta u_i\|^2 \tau + \sum_{i=1}^n \|\Delta \delta u_i\|^2 \tau + \sum_{i=1}^n \|\nabla \Delta u_i - \nabla \Delta u_{i-1}\|^2 \leq C. \\
 (ii) \quad & \max_{0 \leq i \leq n} |h_i| \leq C
 \end{aligned}$$

*Proof.* Starting from the equation (DMPi), we can estimate

$$|h_i| = C(1 + \|\Delta u_{i-1}\|_\Gamma) \leq C(1 + \|\Delta u_{i-1}\| + \|\nabla \Delta u_{i-1}\|), \quad (3.6)$$

where the second inequality comes from the trace theorem. By summing all estimates from Lemma 3.3- Lemma 3.5 up and using (3.6), we are prepared to use the discrete Grönwall lemma to obtain the inequality (i) and consequently also (ii).  $\square$

In following set of a priori estimates, we will work with a difference of the discretized equations; therefore, we additionally need to define

$$h_0 = 0. \quad (3.7)$$

**Lemma 3.7.** *Under the assumptions of Lemma 3.1 there exists a positive constant C (independent of n) such that*

$$\begin{aligned}
 & \left( g_{2-\beta} * \|\delta^2 u\|^2 \right)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\delta^2 u_i\|^2 \tau + \sum_{i=1}^j \|\delta^2 u_i\|^2 \tau \\
 & + \|\delta^2 u_j\|_{H^1(\Omega)}^2 + \sum_{i=1}^j \|\nabla \delta u_i - \nabla \delta u_{i-1}\|^2 \leq C \sum_{i=1}^j |\delta h_i| \|\delta^2 u_i\| \tau,
 \end{aligned}$$

for every  $j \in 1, \dots, n$ ,  $n \in \mathbb{N}$ .

*Proof.* Subtracting equation (DPi) for  $i-1$  from the one for  $i$  and dividing by  $\tau$  gives us

$$(\delta(g_{2-\beta} * \delta^2 u)_i, \varphi) + (\nabla \delta u_i, \nabla \varphi) = \delta h_i(f, \varphi).$$

Notice that for  $i = 1$  the above difference is the equation itself as the  $(g_{2-\beta} * \delta^2 u)_0 = 0$ ,  $u_0 = 0$  and  $h_0 = 0$ . We set  $\varphi = \delta^2 u_i \tau$  and sum up equations for  $1 \leq i \leq j$ . By using Lemma 3.2, the Abel summation and Cauchy inequality, we gain the estimate from the lemma for  $j \in \{1, \dots, n\}$ .  $\square$

**Lemma 3.8.** *Under the assumptions of Lemma 3.1 there exists a positive constant C (independent of n) such that*

$$\begin{aligned}
 & \left( g_{2-\beta} * \|\nabla \delta^2 u\|^2 \right)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\nabla \delta^2 u_i\|^2 \tau + \sum_{i=1}^j \|\nabla \delta^2 u_i\|^2 \tau \\
 & + \|\Delta \delta u_j\|^2 + \sum_{i=1}^j \|\Delta \delta u_i - \Delta \delta u_{i-1}\|^2 \leq C \sum_{i=1}^j |\delta h_i| \|\nabla \delta^2 u_i\| \tau
 \end{aligned}$$

for every  $j \in 1, \dots, n$ ,  $n \in \mathbb{N}$ .

*Proof.* Similarly as in the previous lemma, we make an difference, now, for  $\Delta u_i$ , using (3.4) we get that

$$(\delta(g_{2-\beta} * \nabla \delta^2 u)_i, \varphi) - (\Delta \delta u_i, \varphi) = \delta h_i(f, \varphi).$$

Setting  $\varphi = -\Delta \delta^2 u_i$ , using the Green theorem and summing up for  $1 \leq i \leq j$ , we obtain

$$\sum_{i=1}^j (\delta(g_{2-\beta} * \nabla \delta^2 u)_i, \nabla \delta^2 u_i) \tau + \sum_{i=1}^j (\Delta \delta u_i, \Delta \delta^2 u_i) \tau = \sum_{i=1}^j \delta h_i(\nabla f, \nabla \delta^2 u_i) \tau.$$

Using Lemma 3.2, the Abel summation and Cauchy inequality leads us to the estimate in the lemma.  $\square$



**Lemma 3.9.** Under the assumptions of Lemma 3.1, if moreover  $f \in H^2(\Omega)$  and  $\nabla f \cdot \nu = 0$  on  $\Gamma$  there exists a positive constant  $C$  (independent of  $n$ ) such that

$$\begin{aligned} & \left( g_{2-\beta} * \|\Delta \delta^2 u\|^2 \right)_j + \sum_{i=1}^j g_{2-\beta}(t_i) \|\Delta \delta^2 u_i\|^2 \tau + \sum_{i=1}^j \|\Delta \delta^2 u_i\|^2 \tau \\ & + \|\nabla \Delta \delta u_j\|^2 + \sum_{i=1}^j \|\nabla \Delta \delta u_i - \nabla \Delta \delta u_{i-1}\|^2 \leq C \sum_{i=1}^j |\delta h_i| \|\Delta \delta^2 u_i\| \tau, \end{aligned}$$

for every  $j \in 1, \dots, n$ ,  $n \in \mathbb{N}$ .

*Proof.* First, we make an difference from (3.5) to get

$$\delta (g_{2-\beta} * \nabla \delta^2 u)_i - \nabla \Delta \delta u_i = \delta h_i \nabla f,$$

then we multiply by  $-\nabla \Delta \delta^2 u_i \tau$  and integrate over  $\Omega$  to obtain

$$(\delta (g_{2-\beta} * \nabla \delta^2 u)_i, \nabla \Delta \delta^2 u_i) \tau + (\nabla \Delta \delta u_i, \nabla \Delta \delta^2 u_i) \tau = \delta h_i (\nabla f, \nabla \Delta \delta^2 u_i) \tau$$

Using the Green theorem for the first term on the l.h.s. and for the term on the r.h.s, and then summing up for  $1 \leq i \leq j$ , we get

$$\sum_{i=1}^j (\delta (g_{2-\beta} * \Delta \delta^2 u)_i, \Delta \delta^2 u_i) \tau + \sum_{i=1}^j (\nabla \Delta \delta u_i, \nabla \Delta \delta^2 u_i) \tau - \sum_{i=1}^j \delta h_i (\Delta f, \Delta \delta^2 u_i) \tau.$$

We acquire the estimate from the lemma by the same manner as we did in the last step of the proof of Lemma 3.8.  $\square$

**Lemma 3.10.** Under the assumptions of Lemma 3.1, if moreover  $m \in C^3([0, T])$ ,  $f \in H^2(\Omega)$  and  $\nabla f \cdot \nu = 0$  on  $\Gamma$  then there exists a positive constant  $C$  (independent of  $n$ ) such that

$$\begin{aligned} (i) \quad & \max_{0 \leq i \leq n} \left( g_{2-\beta} * \|\delta^2 u\|^2 \right)_i + \sum_{i=1}^n g_{2-\beta}(t_i) \|\delta^2 u_i\|^2 \tau + \sum_{i=1}^n \|\delta^2 u_i\|^2 \tau + \max_{0 \leq i \leq n} \|\delta u_i\|_{H^1(\Omega)}^2 + \sum_{i=1}^n \|\nabla \delta u_i - \nabla \delta u_{i-1}\|^2 \\ & + \max_{0 \leq i \leq n} \left( g_{2-\beta} * \|\nabla \delta^2 u\|^2 \right)_i + \sum_{i=1}^n g_{2-\beta}(t_i) \|\nabla \delta^2 u_i\|^2 \tau + \sum_{i=1}^n \|\nabla \delta^2 u_i\|^2 \tau + \max_{0 \leq i \leq n} \|\Delta \delta u_i\|_{H^1(\Omega)}^2 \\ & + \max_{0 \leq i \leq n} \left( g_{2-\beta} * \|\Delta \delta^2 u\|^2 \right)_i + \sum_{i=1}^n g_{2-\beta}(t_i) \|\Delta \delta^2 u_i\|^2 \tau + \sum_{i=1}^n \|\Delta \delta^2 u_i\|^2 \tau + \sum_{i=1}^n \|\Delta \delta u_i - \Delta \delta u_{i-1}\|_{H^1(\Omega)}^2 \leq C \\ (ii) \quad & |\delta h_i| \leq C (1 + g_{2-\beta}(t_i)). \end{aligned}$$

*Proof.* First, we estimate the difference of  $h_i$ , from (3.7) and (DMPi) we get

$$\delta (g_{2-\beta} * m'')_i - (\Delta \delta u_{i-1}, \nu)_\Gamma = \delta h_i (f, \omega)_\Gamma,$$

eliminating  $\delta h_i$  from it and estimating the absolute value of it using the trace theorem and (3.2) gives us

$$|\delta h_i| \leq C (g_{2-\beta}(t_i) m_0'' + (g_{2-\beta} * |\delta m''|)_i + \|\Delta \delta u_{i-1}\|_\Gamma) \leq C (1 + g_{2-\beta}(t_i) + \|\Delta \delta u_{i-1}\| + \|\nabla \Delta \delta u_{i-1}\|).$$

Next, we sum all the results from Lemma 3.7 to Lemma 3.9 up, and on the r.h.s we can use the above estimate and the Young inequality to obtain

$$\begin{aligned} & \sum_{i=1}^j |\delta h_i| (\|\delta^2 u_i\| + \|\nabla \delta^2 u\| + \|\Delta \delta^2 u_i\|) \tau \\ & \leq \sum_{i=1}^j (1 + g_{2-\beta}(t_i) + \|\Delta \delta u_{i-1}\| + \|\nabla \Delta \delta u_{i-1}\|) (\|\delta^2 u_i\| + \|\nabla \delta^2 u\| + \|\Delta \delta^2 u_i\|) \tau \\ & \leq \sum_{i=1}^j g_{2-\beta}(t_i) (\|\delta^2 u_i\| + \|\nabla \delta^2 u\| + \|\Delta \delta^2 u_i\|) \tau \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^j (1 + \|\Delta \delta u_{i-1}\| + \|\nabla \Delta \delta u_{i-1}\|) (\|\delta^2 u_i\| + \|\nabla \delta^2 u_i\| + \|\Delta \delta^2 u_i\|) \tau \\
& \leq C_\varepsilon \sum_{i=1}^j g_{2-\beta}(t_i) \tau + \varepsilon \sum_{i=1}^j g_{2-\beta}(t_i) (\|\delta^2 u_i\|^2 + \|\nabla \delta^2 u_i\|^2 + \|\Delta \delta^2 u_i\|^2) \tau \\
& \quad + C_\varepsilon \sum_{i=1}^j (1 + \|\Delta \delta u_{i-1}\|^2 + \|\nabla \Delta \delta u_{i-1}\|^2) \tau \\
& \quad + \varepsilon \sum_{i=1}^j (\|\delta^2 u_i\|^2 + \|\nabla \delta^2 u_i\|^2 + \|\Delta \delta^2 u_i\|^2) \tau.
\end{aligned}$$

Now, we choose the appropriate  $\varepsilon > 0$  on the r.h.s and move the terms to the l.h.s. Then we are prepared to use the discrete Grönwall lemma to get (i), consequently, we obtain also (ii).  $\square$

Next step is to introduce functions which helps us to define the approximate solution on the whole time frame. There are known also as the Rothe functions; we define them as  $u_n, \bar{u}_n, \tilde{u}_n : [0, \tau] \rightarrow L^2(\Omega)$  with

$$\begin{aligned}
u_n : t & \mapsto \begin{cases} 0, & t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i, & t \in (t_{i-1}, t_i], \quad 1 \leq i \leq n, \end{cases} \\
\bar{u}_n : t & \mapsto \begin{cases} 0, & t = 0 \\ u_i, & t \in (t_{i-1}, t_i], \quad 1 \leq i \leq n, \end{cases} \\
\tilde{u}_n : t & \mapsto \begin{cases} 0, & t \in [0, \tau] \\ \bar{u}_n(t - \tau), & t \in (t_{i-1}, t_i], \quad 2 \leq i \leq n, \end{cases}
\end{aligned}$$

and  $v_n, \bar{v}_n : [0, T] \rightarrow L^2(\Omega)$  are defined in the similar manner.

$$\begin{aligned}
v_n : t & \mapsto \begin{cases} 0, & t = 0 \\ \delta u_{i-1} + (t - t_{i-1})\delta^2 u_i, & t \in (t_{i-1}, t_i], \quad 1 \leq i \leq n, \end{cases} \\
\bar{v}_n : t & \mapsto \begin{cases} 0, & t = 0 \\ \delta u_i, & t \in (t_{i-1}, t_i], \quad 1 \leq i \leq n, \end{cases}
\end{aligned}$$

in the same way we define also functions  $\tilde{v}_n, \bar{h}_n, g_{2-\beta_n}, \bar{m}''_n$ . As we told before, with those definitions we can extend the discretized solution to the whole interval  $[0, T]$ , so we rewrite (DPi) and (DMPi) to

$$(\overline{g_{2-\beta_n}} * \partial_t v_n)(t_i, \varphi) + (\nabla \tilde{u}_n(t_i), \nabla \varphi) = \bar{h}_n(t)(f, \varphi), \quad (\text{DP})$$

and

$$(\overline{g_{2-\beta_n}} * \bar{m}''_n)(t_i) + (\nabla \tilde{u}_n(t_i), \omega)_\Gamma = \bar{h}_n(t)(f, \omega)_\Gamma, \quad (\text{DMP})$$

for  $t \in (t_{i-1}, t_i]$ . With the above definition and all the estimates we have, we may proceed to the existence theorem. We will prove that the subsequences of Rothe functions converge to functions  $u$  and  $h$ , and that (DP) and (DMP) converge to (P) and (MP), respectively, so the functions  $u$  and  $h$  are then a solution of our problem.

**Theorem 3.1.** [existence of a solution] Let  $\omega \in L^2(\Gamma)$ ,  $f \in H^2(\Omega)$  with  $\nabla f \cdot \nu = 0$  on  $\Gamma$ ,  $(f, \omega)_\Gamma \neq 0$  and  $m \in C^3([0, T])$ , moreover, assume (3.7),  $m(0) = 0$  and  $m'(0) = 0$ .

Then there exists a solution  $(u, h)$  to the (P), (MP) obeying  $u \in C([0, T], V)$  with  $\partial_t u \in C([0, T], L^2(\Omega)) \cap L^2((0, T), V)$ ,  $u|_\Gamma \in L^2((0, T), H^1(\Omega))$ ,  $\partial_{tt} \Delta u \in L^2((0, T), L^2(\Omega))$  and  $h \in C([0, T])$ .

*Proof.* Based on the estimate (ii) of Lemma 3.10, we get

$$|h'_n(t)| = |\delta h_i| \leq C t_i^{1-\beta} + C \leq C t^{1-\beta} + C,$$

for  $t \in (t_{i-1}, t_i]$ . Then for  $t, s \in [0, T]$ , such that  $|t - s| \leq \varepsilon$ , for  $\varepsilon > 0$ , it holds

$$|h_n(t) - h_n(s)| \leq \left| \int_s^t |h'_n(r)| dr \right| \leq C \left| \int_s^t (r^{1-\beta} + 1) dr \right| = C \frac{|t^{2-\beta} - s^{2-\beta}|}{2-\beta} + \varepsilon C = C(\varepsilon^{2-\beta} + \varepsilon)$$

which means that sequence  $\{h_n\}$  is uniformly equi-continuous. The equi-boundedness of the sequence is obtained from the estimate (ii) of Lemma 3.6. The Arzelà-Ascoli theorem [37, Theorem 11.28] gives us the existence of  $h \in C([0, T])$  to which the subsequence  $\{h_{n_k}\}$  (from now on denoted as  $\{h_n\}$ ) converges in  $C([0, T])$ .

From Lemma 3.6(i), we obtain the estimate of the Rothe functions  $u_n, \bar{u}_n$

$$\max_{0 \leq t \leq T} \|\bar{u}_n(t)\|_V^2 + \max_{0 \leq t \leq T} \|\partial_t u_n(t)\|^2 \leq C.$$

Since  $V \subseteq L^2(\Omega)$ , we can use [35, Lemma 1.3.13] which says that there exist  $u \in C([0, T], L^2(\Omega)) \cap L^\infty((0, T), V)$  such that  $\partial_t u \in L^2((0, T), L^2(\Omega))$  and subsequences  $\{u_{n_k}\}_{k \in \mathbb{N}}$ ,  $\{\bar{u}_{n_k}\}_{k \in \mathbb{N}}$  (from now on indexed by  $n$ , for the sake of simplicity) for which it holds

$$\begin{cases} u_n \rightarrow u, & \text{in } C([0, T], L^2(\Omega)) \end{cases} \quad (3.8a)$$

$$\begin{cases} u_n(t) \rightarrow u(t), & \text{in } V, \quad \forall t \in [0, T] \end{cases} \quad (3.8b)$$

$$\begin{cases} \bar{u}_n(t) \rightarrow u(t), & \text{in } V, \quad \forall t \in [0, T] \end{cases} \quad (3.8c)$$

$$\begin{cases} \partial_t u_n \rightarrow \partial_t u, & \text{in } L^2((0, T), L^2(\Omega)). \end{cases} \quad (3.8d)$$

Moreover, for  $\partial_t u$  we have the estimate  $\max_{t \in [0, T]} \|\partial_t u(t)\|_V \leq C$  and from Lemma 3.10(i). The estimate gives us the boundedness of  $\partial_t u$  in the reflexive Banach space  $L^2((0, T), V)$ . Therefore, for a subsequence of  $\{\partial_t u_n\}$ , we get that

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{in } L^2((0, T), V),$$

and, consequently,

$$u(t) - u(s) = \int_s^t \partial_t u(r) \, dr \Rightarrow \|u(t) - u(s)\|_V \leq |t - s| \left( \int_0^T \|\partial_t u(r)\|_V^2 \, dr \right)^{\frac{1}{2}} \leq C |t - s|^{\frac{1}{2}},$$

so we have  $u \in C([0, T], V)$ .

Furthermore, from Lemma 3.10(i) we gain

$$\max_{0 \leq t \leq T} \|\bar{v}_n(t)\|_V^2 + \int_0^T \|\partial_t v_n(t)\|^2 \leq C,$$

using the same lemma as above, we are obtaining  $v \in C([0, T], L^2(\Omega)) \cap L^\infty((0, T), V)$  with  $\partial_t v \in L^2((0, T), L^2(\Omega))$  and subsequences  $\{v_{n_k}\}_{k \in \mathbb{N}}$ ,  $\{\bar{v}_{n_k}\}_{k \in \mathbb{N}}$  (from now on indexed by  $n$ ) such that

$$\begin{cases} v_n \rightarrow v, & \text{in } C([0, T], L^2(\Omega)) \end{cases} \quad (3.9a)$$

$$\begin{cases} v_n(t) \rightarrow v(t), & \text{in } V, \quad \forall t \in [0, T] \end{cases} \quad (3.9b)$$

$$\begin{cases} \bar{v}_n(t) \rightarrow v(t), & \text{in } V, \quad \forall t \in [0, T] \end{cases} \quad (3.9c)$$

$$\begin{cases} \partial_t v_n \rightarrow \partial_t v, & \text{in } L^2((0, T), L^2(\Omega)). \end{cases} \quad (3.9d)$$

To see the connection between  $u$  and  $v$  we start with the equality

$$(u_n(t) - u_0, \varphi) = \int_0^t (v_n(s), \varphi) \, ds \quad \text{for } \varphi \in L^2(\Omega),$$

by passing the limit for  $n \rightarrow \infty$ , it is obtained

$$(u(t) - u_0, \varphi) = \int_0^t (v(s), \varphi) \, ds \quad \text{for } \varphi \in L^2(\Omega).$$

From this we see that  $v(t) = \partial_t u(t)$  in  $L^2(\Omega)$  for a.a.  $t \in [0, T]$ .

Next, from Lemma 3.10(i) we have the estimate  $\sum_{i=1}^n \|\Delta \delta^2 u_i\|^2 \tau \leq C$  that can be rewritten as

$$\int_0^T \|\Delta \partial_t v_n(t)\|^2 \, dt \leq C,$$

and that together with the reflexivity of space  $L^2((0, T), L^2(\Omega))$  imply the weak convergence of a subsequence of  $\{\Delta \partial_t v_n\}$  (indexed again by  $n$ ) to  $z \in L^2((0, T), L^2(\Omega))$ . Since it holds

$$\int_0^T (\Delta \partial_t v_n(t), \varphi) dt = \int_0^T (\partial_t v_n(t), \Delta \varphi) dt,$$

for every  $\varphi \in C_0^\infty(\Omega)$ . By passing to the limit  $n \rightarrow \infty$ , we obtain

$$\int_0^T (z(t), \varphi) dt = \int_0^T (\partial_t v(t), \Delta \varphi) dt = \int_0^T (\Delta \partial_t v(t), \varphi) dt,$$

for all  $\varphi \in C_0^\infty(\Omega)$ , so  $\Delta \partial_t u = z \in L^2((0, T), L^2(\Omega))$ . Analogously we get similar result for  $\nabla \partial_t u$ .

The rest of the proof will consist of proving the convergence of (DMP) and (DP) to (MP) and (P), respectively. First,

$$\begin{aligned} \left| \left( \overline{g_{2-\beta_n}} * \overline{m''_n} \right) (t_i) - \left( \overline{g_{2-\beta_n}} * \overline{m''_n} \right) (t) \right| &\leq \left| \int_0^{t_i} \overline{g_{2-\beta_n}}(t_i - s) \overline{m''_n}(s) ds \right| \\ &+ \left| \int_0^t \left( \overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s) \right) \overline{m''_n}(s) ds \right| \\ &\leq C \int_t^{t_i} \overline{g_{2-\beta_n}}(t_i - s) ds + C \int_0^t \left| \overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s) \right| ds. \end{aligned}$$

As  $\overline{g_{2-\beta_n}} \rightarrow g_{2-\beta}$  in  $(0, T)$  pointwise, the Lebesgue dominated theorem gives

$$\left( \overline{g_{2-\beta_n}} * \overline{m''_n} \right) (t_i) \rightarrow (g_{2-\beta} * m'')(t).$$

Next, notice that since  $\max_{0 \leq i \leq n} \|\Delta \tilde{u}_i\|_{H^1(\Omega)}^2 \leq C$ , we get

$$\int_0^T \|\tilde{u}_n(t) - \bar{u}_n(t)\|_\Gamma dt \leq \int_0^T \|\tilde{u}_n(t) - \bar{u}_n(t)\|_{H^1(\Omega)} dt = \mathcal{O}(\tau).$$

Thanks to Lemma 3.10 (ii), it also holds that

$$\int_0^T \|\bar{h}_n(t) - h_n(t)\| dt = \mathcal{O}(\tau).$$

We next integrate (DMP) for  $\xi \in [0, T]$ , and, thanks to the above facts and the convergences we got, we can pass to the limit  $n \rightarrow \infty$ . Then by the differentiation with respect to  $\xi$  we obtain (MP).

The problematic term in (DP) is the first one on the l.h.s. of (3.10), several estimates need to be done to be able to pass the limit. We start with

$$\begin{aligned} &\left| \int_0^\xi \left( \left( \overline{g_{2-\beta_n}} * \partial_t v_n \right) (t_i) - \left( \overline{g_{2-\beta_n}} * \partial_t v_n \right) (t), \varphi \right) dt \right| \\ &\leq \int_0^\xi \left| \int_t^{t_i} \overline{g_{2-\beta_n}}(t_i - s) \partial_t v_n(s), \varphi ds \right| dt + \int_0^\xi \left| \int_0^t \left( \overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s) \right) \partial_t v_n(s), \varphi ds \right| dt \\ &\leq \int_0^\xi \int_t^{t_i} \overline{g_{2-\beta_n}}(t_i - s) \|\partial_t v_n(s)\| \|\varphi\| ds dt + \int_0^\xi \int_0^t \left| \overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s) \right| \|\partial_t v_n(s)\| \|\varphi\| ds dt. \end{aligned} \quad (3.10)$$

We use Hölder inequality and Lemma 3.10 for the first term on the r.h.s. to get

$$\begin{aligned} \int_0^\xi \int_t^{t_i} \overline{g_{2-\beta_n}}(t_i - s) \|\partial_t v_n(s)\| \|\varphi\| ds dt &\leq \|\varphi\| \int_0^\xi \sqrt{\int_t^{t_i} \overline{g_{2-\beta_n}}(t_i - s) ds} \sqrt{\int_t^{t_i} \overline{g_{2-\beta_n}}(t_i - s) \|\partial_t v_n(s)\|^2 ds} dt \\ &\leq \|\varphi\| \sqrt{\tau^{2-\beta}} \int_0^\xi \sqrt{\int_0^{t_i} \overline{g_{2-\beta_n}}(t_i - s) \|\partial_t v_n(s)\|^2 ds} ds \\ &\stackrel{\text{Lemma 3.10}}{\leq} C \|\varphi\| \sqrt{\tau^{2-\beta}}. \end{aligned}$$

The second term in (3.10) is estimated after switching the order of integration and using Hölder's inequality as follows

$$\begin{aligned}
 & \int_0^\xi \int_0^t |\overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s)| \|\partial_t v_n(s)\| \|\varphi\| \, ds \, dt \\
 & \leq \|\varphi\| \int_0^\xi \int_s^\xi |\overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s)| \|\partial_t v_n(s)\| \, dt \, ds \\
 & \leq \|\varphi\| \int_0^\xi \|\partial_t v_n(s)\| \int_s^\xi |\overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s)| \, dt \, ds \\
 & \leq \|\varphi\| \sqrt{\int_0^\xi \|\partial_t v_n(s)\|^2 \, ds} \sqrt{\int_0^\xi \left( \int_s^\xi |\overline{g_{2-\beta_n}}(t_i - s) - \overline{g_{2-\beta_n}}(t - s)| \, dt \right)^2 \, ds} \leq C \|\varphi\|.
 \end{aligned}$$

The fact that  $\overline{g_{2-\beta_n}} \rightarrow g_{2-\beta}$  in  $(0, T)$  pointwise enables using of Lebesgue's convergence theorem and brings

$$\lim_{n \rightarrow \infty} \left| \int_0^\xi ((\overline{g_{2-\beta_n}} * \partial_t v_n)(t_i) - (\overline{g_{2-\beta_n}} * \partial_t v_n)(t), \varphi) \, dt \right| = 0.$$

Next, we apply the Cauchy, Hölder and Young inequalities to get

$$\left| \int_0^\xi ((\overline{g_{2-\beta_n}} - g_{2-\beta}) * (\partial_t v_n, \varphi))(t) \, dt \right| \leq \int_0^\xi |\overline{g_{2-\beta_n}}(t) - g_{2-\beta}(t)| \, dt \sqrt{\int_0^\xi \|\partial_t v_n(t)\|^2 \, dt} \sqrt{\int_0^\xi \|\varphi\|^2 \, dt} \leq C \|\varphi\|,$$

and by using Lebesgue's convergence theorem, we acquire

$$\lim_{n \rightarrow \infty} \left| \int_0^\xi ((\overline{g_{2-\beta_n}} - g_{2-\beta}) * (\partial_t v_n, \varphi))(t) \, dt \right| = 0.$$

Furthermore,

$$\left| \int_0^\xi (g_{2-\beta} * (\partial_t v_n, \varphi))(t) \, dt \right| \leq \int_0^\xi g_{2-\beta}(t) \, dt \sqrt{\int_0^\xi \|\partial_t v_n(t)\|^2 \, dt} \sqrt{\int_0^\xi \|\varphi\|^2 \, dt} \leq C \|\partial_t v_n\|_{L^2((0,T), L^2(\Omega))} \|\varphi\|,$$

which means that the estimated term can be seen as the linear bounded functional on  $L^2((0, T), L^2(\Omega))$ . Using (3.9d), we arrive to

$$\lim_{n \rightarrow \infty} \int_0^\xi (g_{2-\beta} * (\partial_t v_n, \varphi))(t) \, dt = \int_0^\xi (g_{2-\beta} * (\partial_t v, \varphi))(t) \, dt.$$

In the last step we integrate (DP) over  $(0, \xi)$  and pass to the limit  $n \rightarrow \infty$  to obtain

$$\int_0^\xi ((g_{2-\beta} * \partial_t u)(t), \varphi) \, dt + \int_0^\xi (\nabla u(t), \nabla \varphi) \, dt = \int_0^\xi h(t)(f, \varphi) \, dt, \quad (3.11)$$

where we used the estimates, convergences and relations above. Differentiation of the equality (3.11) with respect to  $\xi$  brings (P).  $\square$

#### 4. Uniqueness

In this section we will prove the uniqueness of the solution in the appropriate spaces. The basic tool without which we could not prove it is the next lemma. It is a continuous version of Lemma 3.2. The proof of the lemma can be found in [36].

**Lemma 4.1.** *Let  $H$  be a real Hilbert space with a scalar product  $(\cdot, \cdot)_H$  and corresponding norm  $\|\cdot\|_H$ . Assume  $T > 0$ ,  $g \in L^1(0, T)$ ,  $g' \in L^{1,loc}(0, T)$ ,  $g' \leq 0$ ,  $g \geq 0$ . If  $v : [0, T] \rightarrow H$  such that  $v(0) \in H$ ,  $v \in H^1((0, T), H)$ , then*

$$\int_0^\xi \left( \frac{d}{dt} (g * v)(t), v(t) \right)_H \, dt \geq \frac{1}{2} (g * \|v\|_H^2)(\xi) + \frac{1}{2} \int_0^\xi g(t) \|v(t)\|_H^2 \, dt \geq \frac{g(T)}{2} \int_0^\xi \|v(t)\|_H^2 \, dt.$$

With this lemma we can proceed to the uniqueness theorem.

**Theorem 4.1.** [uniqueness] Let  $f \in H^2(\Omega)$  with  $\nabla f \cdot \nu = 0$  on  $\Gamma$ ,  $\omega \in L^2(\Gamma)$ ,  $(f, \omega)_\Gamma \neq 0$ ,  $m \in C^2([0, T])$ . Then there exists at most one solution  $(u, h)$  to the (P), (MP) obeying  $u \in C([0, T], \mathbf{V})$ ,  $\partial_t u \in C([0, T], L^2(\Omega)) \cap L^2((0, T), H^1(\Omega))$ ,  $\partial_t \Delta u \in L^2((0, T), L^2(\Omega))$ ,  $\partial_{tt} u \in L^2((0, T), H^1(\Omega))$ ,  $\partial_{tt} \Delta u \in L^2((0, T), L^2(\Omega))$  and  $h \in C([0, T])$ .

*Proof.* We prove this in the classical way by contradiction. Let there be two solutions  $(u_1, h_1)$ ,  $(u_2, h_2)$  of the (P), (MP) belonging to the spaces written in the theorem. We define  $u = u_1 - u_2$  and  $h = h_1 - h_2$  which then obey

$$(\Delta u(t), \omega)_\Gamma = h(t) (f, \omega)_\Gamma, \quad (4.1)$$

and

$$((g_{2-\beta} * \partial_{tt} u)(t), \varphi) + (\nabla u(t), \nabla \varphi) = h(t) (f, \varphi), \quad (4.2)$$

for every  $\varphi \in H^1(\Omega)$ , a.a.  $t \in [0, T]$  and  $u(0) = 0$ ,  $\partial_t u(0) = 0$ . We can eliminate  $h$  from (4.1) and using the trace theorem estimate as

$$|h(t)| \leq \left| \frac{(\Delta u(t), \omega)_\Gamma}{(f, \omega)_\Gamma} \right| \leq C (\|\Delta u(t)\| + \|\nabla \Delta u(t)\|).$$

We put  $\varphi = \partial_t u(t)$  in (4.2), integrate over  $(0, \xi)$  with  $\xi \in (0, T]$  and for the first term on the l.h.s. use the relationship  $(g_{2-\beta} * \partial_{tt} u)(t) = \partial_t (g_{2-\beta} * \partial_t u)(t)$ , which is true since  $\partial_t u = 0$ , to obtain that

$$\int_0^\xi (\partial_t (g_{2-\beta} * \partial_t u)(t), \partial_t u(t)) dt + \frac{1}{2} \|\nabla u(\xi)\|^2 = \int_0^\xi h(t) (f, \partial_t u(t)) dt.$$

Using Lemma 4.1, the Cauchy, the Young inequalities and choosing the appropriate  $\varepsilon$  lead us to the estimate

$$\int_0^\xi \|\partial_t u(t)\|^2 dt + \|\nabla u(\xi)\|^2 \leq C \int_0^\xi |h(t)|^2 dt,$$

similar to the one in the Lemma 3.3. Thanks to the assumption from the theorem we may use the Green identity  $(\nabla u(t), \nabla \varphi) = -(\Delta u(t), \varphi)$  in (4.2), then analogously as in Lemma 3.4 and 3.5, we get that

$$\int_0^\xi \|\partial_t \nabla u(t)\|^2 dt + \|\Delta u(\xi)\|^2 \leq C \int_0^\xi |h(t)|^2 dt,$$

and

$$\int_0^\xi \|\partial_t \Delta u(t)\|^2 dt + \|\nabla \Delta u(\xi)\|^2 \leq C \int_0^\xi |h(t)|^2 dt.$$

Summing the last three estimates together and using estimate  $\|u(\xi)\|^2 \leq \int_0^\xi \|\partial_t u(t)\|^2 dt$ , we obtain

$$\|u(\xi)\|^2 + \|\nabla u(\xi)\|^2 + \|\Delta u(\xi)\|^2 + \|\nabla \Delta u(\xi)\|^2 \leq C \int_0^\xi (\|\Delta u(t)\|^2 + \|\nabla \Delta u(t)\|^2) dt.$$

The Grönwall's argument is applied to get

$$\|u(\xi)\|^2 + \|\nabla u(\xi)\|^2 + \|\Delta u(\xi)\|^2 + \|\nabla \Delta u(\xi)\|^2 \leq 0, \quad (4.3)$$

which is true for any  $\xi \in [0, T]$ . This imply that  $u = 0$  a.e. in  $\Omega \times [0, T]$ , and then also  $h = 0$  a.e. in  $[0, T]$ .  $\square$

## 5. Numerical Experiments

In the section two numerical experiments are presented. The first one is a demonstration of the algorithm arising from the above time discretization. On the  $i$ -th time layer  $h_i$  is calculated from (DMPi) and  $u_i$  from (DPi), then we move to the next time level. In the second experiment we propose a way how to deal with the data containing some percentage of noise.

Both experiments have the following setting. We assume  $(x, y) \in \Omega = (0, \pi) \times (0, \pi)$ ,  $T = 3$  and  $\beta = 1.3$ , next

$$f(x, y) = \cos x + \cos y,$$

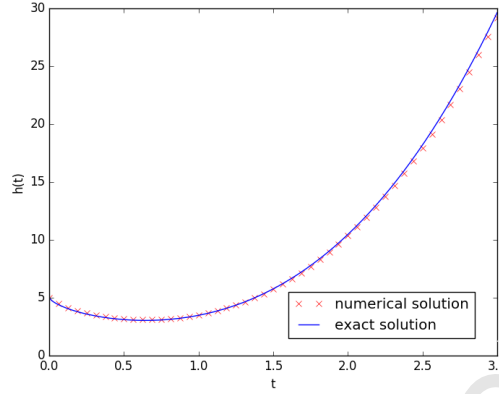


Figure 1: Reconstruction of  $h$  together with exact  $h$  for  $\tau = 0.015625$  and  $\omega$

and the equation is accompanied with the initial and boundary condition

$$\begin{aligned} u_0(x, y) &= 5(\cos x + \cos y), \\ v_0(x, y) &= 0, \\ -\nabla u(x, y, t) \cdot \mathbf{n} &= 0. \end{aligned}$$

The measurement function takes form

$$m(t) = \int_{\Gamma} u(x, y, t) \omega(x, y) dS = \left(\frac{\pi}{2} + 1\right) (t^3 - 2t^2 + 5),$$

where

$$\omega(x, y) = \begin{cases} 1, & |y - \frac{\pi}{4}| \leq \frac{\pi}{4}, x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It can be easily showed that the functions

$$\begin{aligned} u(x, y, t) &= \frac{(t^3 - 2t^2 + 5)(\cos x + \cos y)}{6}, \\ h(t) &= \frac{(t^3 - 2t^2 + 5)}{(2 - \beta)(3 - \beta)\Gamma(2 - \beta)} t^{3-\beta} - \frac{4}{(2 - \beta)\Gamma(2 - \beta)} t^{2-\beta} + t^3 - 2t^2 + 5 \end{aligned}$$

are the solution of the inverse problem with the above settings. We implement the algorithm in Python using the finite element library DOLFIN from the FEniCS Project [38]. The domain is divided into 50 cells in each  $x$ - and  $y$ - direction. In each time step the Lagrange basis function are used which leads to the system with 10201 degrees of freedom.

### 5.1. Exact data

Using the above settings, we calculate the approximate solution for several values of time step  $\tau$ . On Figure 1 the reconstruction of  $h$ . The development of relative error of  $h$  and  $u$  in time can be seen on Figure 2. The decay of maximal relative error of  $h$  and  $u$  for various values of  $\tau$  is shown on Figure 3. The graph of the solid line in 3 is given by  $0.9899 \log_2 \tau - 0.1965$  for the error of  $h$  and  $1.0121 \log_2 \tau + 0.6473$  for the error of  $u$ .

### 5.2. Noisy data

In this experiment, we model a noisy measurement in the following way

$$m_{\epsilon}(t) = m(t) + \delta m_{max},$$

where  $\delta$  is the Gaussian distributed noise with mean and standard deviation equal to 0 and 1, respectively,  $m_{max}$  is the maximum value of measurement  $m$  and  $\epsilon$  is a scale representing the amount of noise.

Since our algorithm requires the continuous second derivative of the measurement, and it can be hardly expected in the case of noisy data, we need to apply some kind of smoothing on the data. We use the least square method to obtain a function of the form

$$m_{app}(t) = at^3 + bt^2 + ct + d$$

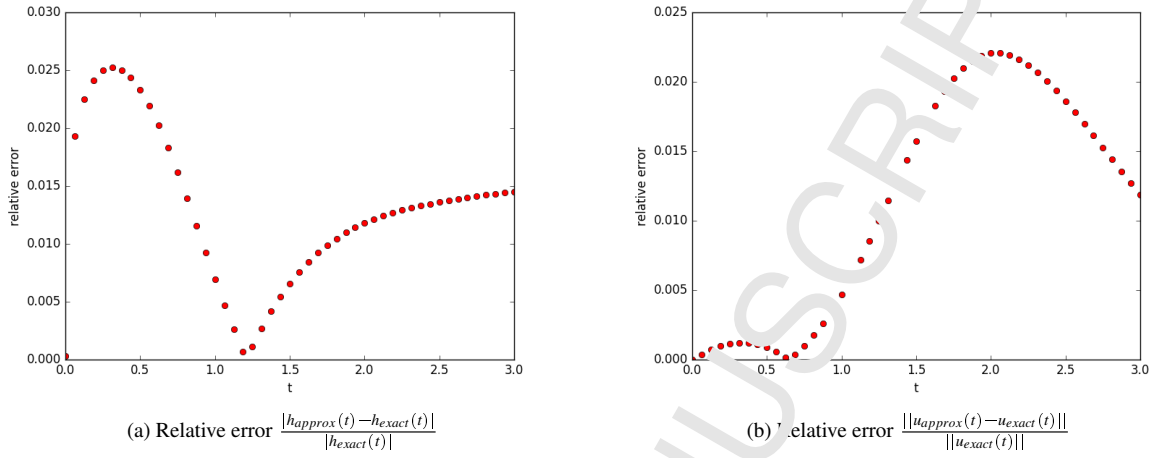
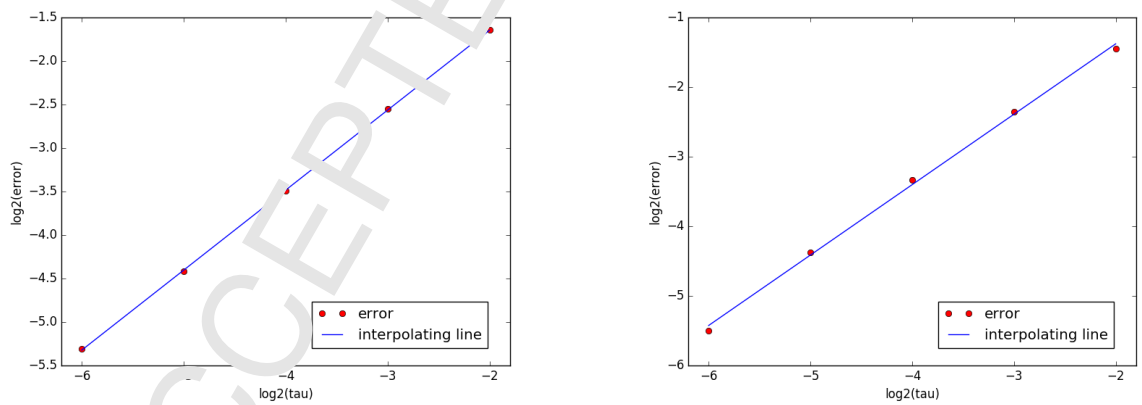


Figure 2: Relative error in time  $\tau = 0.01$ ,  $\omega = 0.25$  for  $\omega$

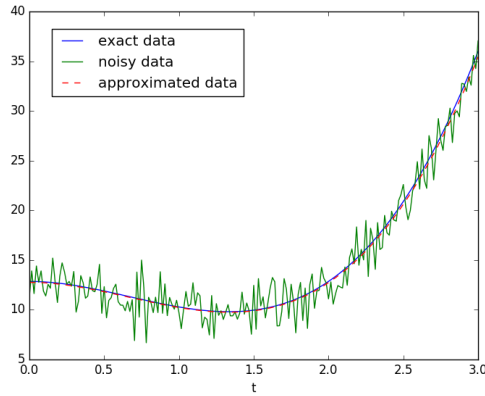


(a) Logarithm of maximal relative error in time of  $h$  for different values of  $\tau$ . Slope of the line is 0.9899.

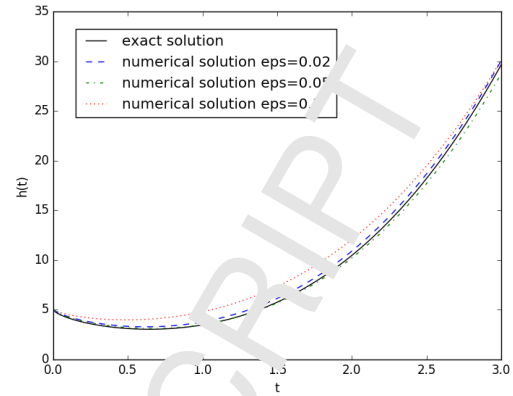
(b) Logarithm of maximal relative error in time of  $u$  for different values of  $\tau$ . Slope of the line is 1.0121.

Figure 3: Decay of maximal relative error for  $\omega$



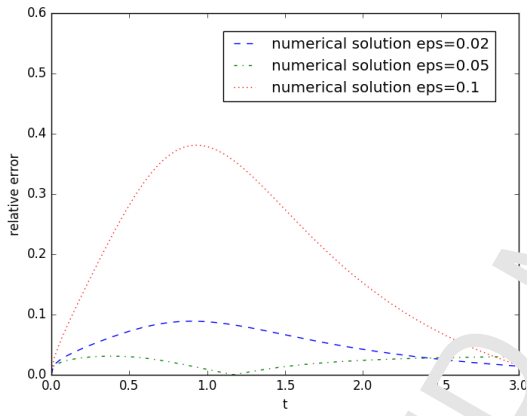


(a) Exact and noisy data for  $\epsilon = 0.05$ . Approximating curve has the form  $m_{app}(t) = 2.529t^3 - 5.0772t^2 + 0.0143t + 12.7607$

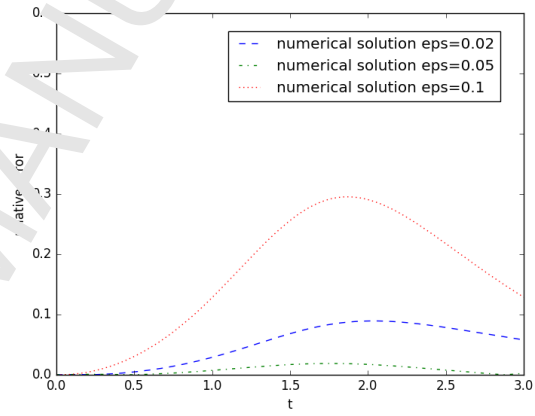


(b) Reconstruction of  $h$  together with exact  $h$  for  $\tau = 0.015625$  and for the different amount of noise

Figure 4: Measurement and reconstruction



(a) Relative error  $\frac{|h_{approx}(t) - h_{exact}(t)|}{|h_{exact}(t)|}$



(b) Relative error  $\frac{|u_{approx}(t) - u_{exact}(t)|}{|u_{exact}(t)|}$

Figure 5: Relative error in time for  $\tau = 0.015625$  and the various amount of noise

which is smooth enough. This function is then used instead of  $m$  in the algorithm. We use the same setting as in the previous experiment. On the Figure 4 we can see reconstruction of source term for the several various amount of noise, and on the Figure 5 the corresponding relative error in time can be seen for  $h$  and  $u$ .

## 6. Conclusion

We considered the fractional wave equation. We were interested in the reconstruction of the source term from the non-invasive measurement on the boundary. We reformulated the problem in to two coupled equations for which we were able to prove the existence of the solution using the Rothe method, and the uniqueness was addressed in the appropriate spaces. The results were supported by the numerical experiments in the last section.

## Acknowledgment

K. Šišková was financed by the BOF project no. 01D23414, Ghent University, Belgium.

## References

- [1] A. Kufner, O. John, S. Fučík, Function Spaces, Monographs and textbooks on mechanics of solids and fluids, Noordhoff International Publishing, Leyden, 1977.

- [2] F. Mainardi, Fractional diffusive waves in viscoelastic solids, in: J.L. Wegner, F.R. Norwood (Eds.), *Nonlinear Waves in Solids*, ASME/AMR, Fairfield, (1995) 93–97.
- [3] F. Mainardi, P. Paradisi, A model of diffusive waves in viscoelasticity based on fractional calculus, in: *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, Vol. 5, 1997, pp. 4961–4966 vol.5.
- [4] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *Journal of Mathematical Analysis and Applications* 382 (1) (2011) 426–447.
- [5] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, *Applied Mathematics Letters* 9 (1996) 23–28.
- [6] Y. Luchko, F. Mainardi, Cauchy and signaling problems for the time-fractional diffusion-wave equation, *ASME. J. Vib. Acoust.* 5 (136) (2014) 050904–050904–7.
- [7] F. Mainardi, The time fractional diffusion-wave equation, *Radiophysics and Quantum Electronics* 38 (1995) 13–24.
- [8] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons and Fractals* 7 (9) (1996) 1461–1477.
- [9] A. Prilepko, D. Orlovsky, I. Vasin, *Methods for solving inverse problems in mathematical physics*, Vol. 222 of Monographs and textbooks in pure and applied mathematics, Marcel Dekker, Inc., New York-Basel, 2000.
- [10] W. Rundell, Determination of an unknown non-homogeneous term in a linear partial differential equation from overspecified boundary data, *Applicable Analysis* 10 (1980) 231–242.
- [11] M. Yamamoto, Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method, *Inverse Problems* 11 (2) (1995) 481.
- [12] V. Isakov, *Inverse source problems.*, Mathematical Surveys and Monographs, 34. Providence, RI: American Mathematical Society (AMS), 1990.
- [13] A. Hasanov, Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: Weak solution approach, *J. Math. Anal. Appl.* 330 (2007) 766–779.
- [14] T. Johansson, D. Lesnic, A variational method for identifying a spacewise dependent heat source, *IMA J. Appl. Math.* 72 (2007) 748–760.
- [15] K. Van Bockstal, M. Slodička, Recovery of a space-dependent vector source in thermoelastic systems, *Inverse Problems in Science and Engineering* 23 (6) (2015) 956–968.
- [16] S. O. Hussein, D. Lesnic, Determination of forcing functions in the wave equation. part i: the space-dependent case, *Journal of Engineering Mathematics* 96 (1) (2016) 115–133.
- [17] S. O. Hussein, D. Lesnic, Determination of forcing functions in the wave equation. part ii: the time-dependent case, *Journal of Engineering Mathematics* 96 (1) (2016) 135–153.
- [18] A. Hasanov, B. Pektaş, Identification of an unknown time-dependent heat source from overspecified Dirichlet boundary data by conjugate gradient method, *Computers & Mathematics with Applications* 65 (1) (2015) 42–57.
- [19] M. Slodička, Determination of a solely time-dependent source in a semilinear parabolic problem by means of boundary measurements, *JCAM* 289 (2015) 433–440.
- [20] B. Jin, W. Rundell, An inverse problem for a one-dimensional time-fractional diffusion problem, *Inverse Problems* 28 (7) (2012) 075010.
- [21] M. Kirane, S. A. Malik, M. A. Al-Gwaiz, An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions, *Mathematical Methods in the Applied Sciences* 36 (9) (2013) 1056–1069.
- [22] X.-M. Yang, Z.-L. Deng, A point source identification problem for a time fractional diffusion equation, *Advances in Mathematical Physics* 2013, article ID 485273.
- [23] S. Tatar, S. Ulusoy, An inverse source problem for a one-dimensional space-time fractional diffusion equation, *Applicable Analysis* 94 (11) (2015) 2233–2244.
- [24] T. Wei, Z. Zhang, Reconstruction of a time-dependent source term in a time-fractional diffusion equation, *Engineering Analysis with Boundary Elements* 37 (1) (2013) 23–31.
- [25] K. Fujishiro, Y. Kian, Determination of time dependent factors of coefficients in fractional diffusion equations, *ArXiv e-prints* arXiv: 1501.01945.
- [26] H. Lopushanska, A problem with an integral boundary for a time fractional diffusion equation and an inverse problem, *Fractional Differential Calculus* 6 (1) (2016) 133–145.
- [27] B. Wu, S. Wu, Existence and uniqueness of an inverse source problem for a fractional integrodifferential equation, *Computers & Mathematics with Applications* 68 (10) (2014) 1123–1136.
- [28] H. Wang, B. Wu, On the well-posedness of determination of two coefficients in a fractional integrodifferential equation, *Chinese Annals of Mathematics, Series B* 35 (3) (2014) 447–468.
- [29] A. Lopushansky, H. Lopushanska, Inverse source cauchy problem for a time fractional diffusion-wave equation with distributions, *Journal of Differential Equations* 2017 (2017) 1–14.
- [30] K. Šišková, M. Slodička, Recognition of a time-dependent source in a time-fractional wave equation, *Applied Numerical Mathematics* 121 (Supplement C) (2017) 1–7. doi:https://doi.org/10.1016/j.apnum.2017.06.005. URL <http://www.sciencedirect.com/science/article/pii/S0168927417301411>
- [31] H. Gajewski, K. Gröger, K. Zajączkowski, *Nichtlineare Operatorgleichungen und Operatorindifferentialgleichungen.*, Mathematische Lehrbücher und Monographien. II. Abteilung. Band 38. Berlin: Akademie-Verlag, 1974.
- [32] E. Rothe, Zweidimensionale parabolische randwertaufgaben als grenzfall eindimensionaler randwertaufgaben, *Mathematische Annalen* 102 (1) (1930) 650–70.
- [33] O. Ladyzhenskaya, On the solution of a mixed problem for hyperbolic equations, *Izv. Akad. Nauk SSSR Ser. Mat.* 15 (6) (1951) 545–562.
- [34] O. Ladyzhenskaya, The solution in the large of the first boundary-value problem for quasi linear parabolic equations, *Dokl. Akad. Nauk SSSR Ser. Mat.* 107 (1966) 636–639.
- [35] J. Kačur, *Method of Rothe in evolution equations*, Teubner-Texte zur Mathematik, 1985.
- [36] M. Slodička, K. Šišková, An inverse source problem in a semilinear time-fractional diffusion equation, *Computers and Mathematics with Applications* 72 (2016) 1655–1669.
- [37] W. Rudin, *Real and complex analysis*, McGraw-Hill, 1987.
- [38] A. Logg, K.-A. Mardal, G. Wells, *Automated Solution of Differential Equations by the Finite Element Method*, Springer, 2012.